Sample Questions 12

1. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal basis. Suppose $\alpha \cup \beta = \mathcal{B}$ and $\alpha \cap \beta = \emptyset$. Show that span$(\beta)$ is the orthogonal complement of span(α).

2. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal basis and $\alpha \subseteq \mathcal{B}$. Suppose

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

   Show that the projection of $\mathbf{y}$ onto the space span(α) is

$$\sum_{\mathbf{v}_k \in \alpha} c_k \mathbf{v}_k.$$

3. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis. Suppose

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

   Show that $c_k = \langle \mathbf{y}, \mathbf{v}_k \rangle = \mathbf{v}_k^\top \mathbf{y}$ for each k. Moreover, let $\mathbf{A}$ be the matrix whose columns are vectors in $\mathcal{B}$. Then $\mathbf{A}^\top \mathbf{y} = \text{Rep}_{\mathcal{B}}(\mathbf{y})$. [You see that orthonormal bases are so nice!]

For the following questions, $\mathcal{M}_{m\times n}$ is the space of all $m \times n$ real matrices, and $\mathcal{S}_n$ is the space of all $n \times n$ real symmetric matrices.

4. Determine whether f is an isomorphism.

   (a) $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} a \\ a \\ a \end{bmatrix}$

   (b) $f : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a+b+c+d \\ a+b+c \\ a+b \\ a \end{bmatrix}$$

   (c) $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^3$

5. Let V be the plane in $\mathbb{R}^3$ defined by the equation $x+y+z=0$. It is known that V can also be written as

$$\left\{ a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

   Find an isomorphism from V to $\mathbb{R}^2$. [Hint: The Rep function.]

6. Find a such that $\mathcal{M}_{m\times n} \equiv \mathbb{R}^a$ and find b such that $\mathcal{S}_n = \mathbb{R}^b$.

7. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$. If a vector $\mathbf{y}$ has the representation

$$\text{Rep}_{\mathcal{B}}(\mathbf{y}) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

   How do you recover $\mathbf{y}$ from $\text{Rep}_{\mathcal{B}}(\mathbf{y})$? Indeed, find a matrix $\mathbf{A}$ such that

$$\mathbf{y} = \mathbf{A}\,\text{Rep}_{\mathcal{B}}(\mathbf{y}).$$

   Conversely, how do you find $\text{Rep}_{\mathcal{B}}(\mathbf{y})$ from $\mathbf{y}$? Find a matrix $\mathbf{B}$ such that

$$\text{Rep}_{\mathcal{B}}(\mathbf{y}) = B\mathbf{y}.$$