Sample Solutions for Sample Questions II.

1. Recall Nullspace (A) = Rowspace $(A)^{\perp}$ for any A.

Let $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 1 & 1 \\ 3 & -3 & 2 & 0 \end{pmatrix}$.

Then V = Rowspace (A).

$V^{\perp}$ = Nullspace (A).

To compute $V^{\perp}$, solve $A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 3 & -3 & 2 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

$$\begin{array}{cc} \uparrow & \uparrow \\ y & w \end{array}$$

y, w are free.

Set $z=1$, $w=0$ $\Rightarrow$ $\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Set $z=0$, $w=1$ $\Rightarrow$ $\beta_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix}$

So $\{\beta_1, \beta_2\}$ is a basis of $V^{\perp}$.

$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix} \right\}$ is a basis of V.

2.

$A\vec{x} = \vec{b}$ 無解是因為 $\vec{b} \notin$ Colspace(A).

所以把 $\vec{b}$ 投影到 Colspace(A)，叫 $\vec{b_0}$.

$$\Rightarrow \begin{cases} \vec{b_0} = A(A^TA)^{-1}A^T\vec{b} \\ \vec{x_0} = (A^TA)A^T\vec{b}. \end{cases}$$

用 Sage 得出 (參考 Sample Question 10的程式碼)

$$\Rightarrow \begin{cases} \vec{b_0} = \frac{11}{5}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ \vec{x_0} = \begin{pmatrix} 0 \\ 11/5 \end{pmatrix}. \end{cases}$$

3. $B = A^T$. $B\vec{x} = \vec{b}$ 有無限多解

$\Rightarrow$ $\vec{x_0}$ 取為 $\vec{p}$ 在 Rowspace(B) 上的投影

$\leftarrow$ $B\vec{x} = \vec{b}$ 的任何一個特解 ($B\vec{p} = \vec{b}$).

$\Rightarrow$ $\vec{x_0}$ 是 $\vec{p}$ 在 Colspace(A) 的投影

$$\Rightarrow \vec{x_0} = A(A^TA)^{-1}A^T\vec{p} = A(A^TA)^{-1}B\vec{p} = A(A^TA)^{-1}\vec{b}$$

$$\Rightarrow \vec{x_0} = \begin{pmatrix} -3/5 \\ -3/10 \\ 0 \\ 3/10 \\ 3/5 \end{pmatrix}$$ by Sage.

4.

$$\underbrace{\begin{pmatrix}1&1\\2&1\\3&1\\4&1\\5&1\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}a\\b\end{pmatrix}}_{\vec{x}} = \underbrace{\underbrace{\begin{pmatrix}f(1)\\f(2)\\f(3)\\f(4)\\f(5)\end{pmatrix}}_{\vec{b}} \longleftrightarrow \begin{pmatrix}5\\1\\-1\\1\\5\end{pmatrix}}_{\text{希望距離最短}}$$

$A\vec{x}=\vec{b}$ 無解.

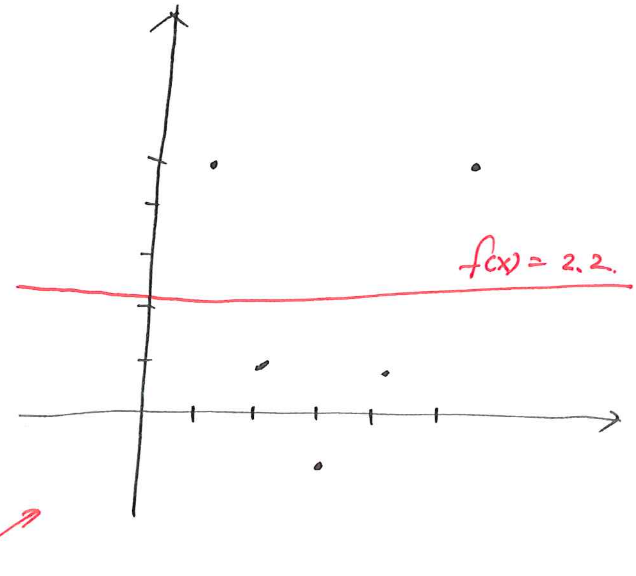
令 $\vec{b}_0$ 為 $\vec{b}$ 在 Colspace (A) 的 projection.

$$\vec{b}_0 = A(A^TA)^{-1}A^T\vec{b} = \frac{11}{5}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

換解 $A\vec{x}_0=\vec{b}_0 \Rightarrow \vec{x}_0 = \begin{pmatrix}0\\11/5\end{pmatrix}$.

$$\Rightarrow \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\11/5\end{pmatrix}$$

$$f(x) = 0\cdot x + 11/5 = 2.2$$



資料不像一條線的時候,

回歸線不太有意義.

5.

$$\underbrace{\begin{array}{ccc} \color{red}{x^2} & \color{red}{x} & \color{red}{1} \\ \hline 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{array}}_{A} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \end{pmatrix}}_{\vec{b}} \qquad \begin{pmatrix} 5 \\ 1 \\ -1 \\ 1 \\ 5 \end{pmatrix}$$

希望距離最短.

令 $\vec{b}_0$ 為 $\vec{b}$ 在 Colspace(A) 上的投影

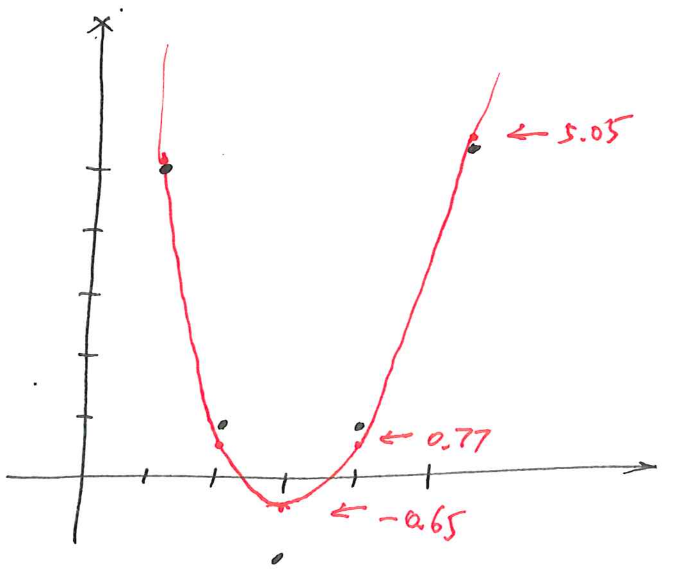
$$\Rightarrow \quad \vec{b}_0 = A(A^TA)^{-1}A^T\vec{b}$$

換解 $A\vec{x}_0 = \vec{b}_0$ $\cancel{\Rightarrow \vec{x}_0 = A(A^TA)}$

$$\Rightarrow \quad \vec{x}_0 = (A^TA)^{-1}A^T\vec{b}.$$

用 Sage:

$$\vec{b}_0 = \frac{1}{35}\begin{pmatrix} 177 \\ 27 \\ -23 \\ 27 \\ 177 \end{pmatrix}$$



much better approximation!

$$\vec{x}_0 = \begin{pmatrix} 10/7 \\ -60/7 \\ 61/5 \end{pmatrix}$$

$$\Rightarrow f(x) = \frac{10}{7}x^2 - \frac{60}{7}x + \frac{61}{5}$$

6. $\sum x_i y_i = \underbrace{y^T x = \vec{x} \circ \vec{y} = \mathrm{diag}(\vec{x}) \cdot \vec{y}}_{\text{either way is fine.}}$.

$\sum x_i = 1^T x$.

$\sum y_i = 1^T y$.

So covariance $= \dfrac{y^T x}{N} - \left(\dfrac{1^T x}{N}\right)\left(\dfrac{1^T y}{N}\right)$

7. If $\vec{\theta}^{(k+t)} = \vec{\theta}^{(k)} + t\vec{x}_i$,

then $\langle \vec{\theta}^{(k+t)}, \vec{x}_i \rangle = \langle \vec{\theta}^{(k)} + t\vec{x}_i , \vec{x}_i \rangle$

$= \underbrace{\langle \vec{\theta}^{(k)}, \vec{x}_i \rangle}_{<0} + t \underbrace{|\vec{x}_i|^2}_{>0}$

When t is large enough, $\langle \vec{\theta}^{(k+t)}, \vec{x}_i \rangle > 0$.

The minimum t to achieve this is

$t = \left\lceil \dfrac{\langle \vec{\theta}^{(k)}, \vec{x}_i \rangle}{|\vec{x}_i|^2} \right\rceil$ ← 上高斯 (無條件進位)

because $\langle \vec{\theta}^{(k)}, \vec{x}_i \rangle + t|\vec{x}_i|^2 > 0$

$\Rightarrow \quad t|\vec{x}_i|^2 > \langle \vec{\theta}^{(k)}, \vec{x}_i \rangle$

$\Rightarrow \quad t > \dfrac{\langle \vec{\theta}^{(k)}, \vec{x}_i \rangle}{|\vec{x}_i|^2}$