國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第二次期中考 November 19, 2018 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **8 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Find the inverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & -1 & -4 \\ 2 & -1 & -1 & -6 \\ 0 & -1 & 0 & -2 \\ 2 & -2 & -3 & -7 \end{bmatrix}.$$

$$(A \mid I) = \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & -4 & 1 & & & \\ 2 & -1 & -1 & -6 & & 1 & & \\ 0 & -1 & 0 & -2 & & & 1 & \\ 2 & -2 & -3 & -7 & & & & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & -4 & 1 & & & \\ 0 & 1 & 1 & 2 & -2 & 1 & & \\ 0 & -1 & 0 & -2 & 0 & & 1 & \\ 0 & 0 & -1 & 1 & -2 & & & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & -4 & 1 & & & \\ 0 & 1 & 1 & 2 & -2 & 1 & & \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & \\ 0 & 0 & -1 & 1 & -2 & & & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & -4 & 1 & & & \\ 0 & 1 & 1 & 2 & -2 & 1 & & \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & \\ 0 & 0 & 0 & 1 & -4 & 1 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & -15 & 4 & 4 & 4 \\ & 1 & 1 & 0 & 6 & -1 & -2 & -2 \\ & & 1 & 0 & -2 & 1 & 1 & \\ & & & 1 & -4 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -17 & 5 & 5 & 4 \\ & 1 & 0 & 0 & 8 & -2 & -3 & -2 \\ & & 1 & 0 & -2 & 1 & 1 & 0 \\ & & & 1 & -4 & 1 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & & & & -9 & 3 & 2 & 2 \\ & 1 & & & 8 & -2 & -3 & -2 \\ & & 1 & & -2 & 1 & 1 & 0 \\ & & & 1 & -4 & 1 & 1 & 1 \end{array}\right), \text{ so } A^{-1} = \begin{pmatrix} -9 & 3 & 2 & 2 \\ 8 & -2 & -3 & -2 \\ -2 & 1 & 1 & 0 \\ -4 & 1 & 1 & 1 \end{pmatrix}$$

2. [2pt] Suppose V is a vector space over $\mathbb{R}$ and S is a nonempty subset of V. What property (or properties) you have to check in order to make sure S is a subspace of V?

① if $\vec{v} \in \cancel{V} S$ and $r \in \mathbb{R}$, then $r\vec{v} \in \cancel{V} S$.

② if $\vec{u}, \vec{v} \in \cancel{V} S$, then $\vec{u} + \vec{v} \in \cancel{V} S$

3. [3pt] For each of V below, write T or F in the box to indicate V is a vector space over $\mathbb{R}$ or not. If your answer is F, provide a brief reason of why V is not a vector space.

(a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x, y \in \mathbb{Z} \right\}$.

$\boxed{F}$ Brief reason if F: 純量乘法不封閉

(b) $V = \{\boldsymbol{X} \in \mathcal{M}_{n \times n} : \boldsymbol{AX} = 0\}$. Here $\mathcal{M}_{n \times n}$ is the set of all $n \times n$ real matrices, and $\boldsymbol{A}$ is a matrix in $\mathcal{M}_{n \times n}$.

$\boxed{T}$ Brief reason if F:

(c) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 0 \right\}$.

$\boxed{T}$ Brief reason if F:

(d) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y + z = 1 \right\}$.

$\boxed{F}$ Brief reason if F: 沒有零元素、
沒有加法反元素
加法不封閉
純量乘法不封閉

4. [2pt] Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors. Write down the definition of that S is linearly independent. (Your answer should be clear in mathematical sense instead of a descriptive sentence in human language.)

If $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for some $c_1, \ldots, c_k \in \mathbb{R}$,

then $c_1 = \cdots = c_k = 0$.

5. [2pt] Find all possible solutions $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ that satisfies

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ 4 & 1 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\uparrow$ c_3 is free.

Let $c_3 = 1 \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix}$

Solutions $= \left\{ t \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$.

6. [1pt] Is the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\}$ linearly independent? Provide your reason.

No. $\frac{1}{3} \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = 0$.

7. [5pt] Let

$$V = \text{span}\left(\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}, \begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}-2\\-5\\-6\end{bmatrix}\right\}\right).$$

Find **a basis** and **the dimension** of V.

$$\begin{pmatrix}1 & 3 & 1 & -2\\ 2 & 6 & 3 & -5\\ 3 & 9 & 3 & -6\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 3 & 1 & -2\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{matrix}\uparrow & & \uparrow\end{matrix}$$

leading variables.

basis $= \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\3\\3\end{pmatrix} \right\rangle$, dimension $= 2$.

8. Let

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 2 & 6 & 3 & -5 \\ 3 & 9 & 3 & -6 \end{bmatrix}.$$

(a) [2pt] Find **a basis** and **the dimension** of the row space of A.

$$A \longrightarrow \begin{pmatrix} 1 & 3 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis $= \langle (1,3,0,-1), (0,0,1,-1) \rangle$

dimension $= 2$.

(b) [3pt] Find **a basis** and **the dimension** of the null space of A.

Solve $\begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

y, w are free.

Let $y = 1$, $w = 0$.

$$\Rightarrow \beta_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Let $y = 0$, $w = 1$.

$$\Rightarrow \beta_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

basis $= \left\langle \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ dimension $= 2$.

9. [5pt] Suppose $S = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k\}$ is a set of nonzero vectors in $\mathbb{R}^n$ such that $\vec{\boldsymbol{w}}_i \cdot \vec{\boldsymbol{w}}_j = 0$ for any distinct i and j. (That is, any two vectors in S are orthogonal to each other.) Show that S is linearly independent.

Suppose $c_1 \vec{w_1} + \cdots + c_k \vec{w_k} = \vec{0}$ for some $c_1, \ldots, c_k \in \mathbb{R}$.

Let i be a number in $\{1, \ldots, k\}$.

Then

$$\vec{w_i} \cdot (c_1 \vec{w_1} + \cdots + c_k \vec{w_k}) = \vec{0}$$

Since $\vec{w_i} \cdot \vec{w_j} = 0$ for all $j \neq i$,

$$c_i |w_i|^2 = 0.$$

Since $|w_i|^2 \neq 0 \Rightarrow c_i = 0$.

This argument holds for every $i = 1, \ldots, k$,

so $c_1 = c_2 = \cdots = c_k = 0$.

$\Rightarrow S$ is linearly indep.

$= \{\vec{w}_1, \ldots, \vec{w}_k\}$.

10. [5pt] Let S be a set of vectors. Suppose S is linearly independent. Show that $S \cup \{\vec{v}\}$ is linearly independent if and only if the vector $\vec{v}$ is not in span(S).

Claim: $S \cup \{\vec{v}\}$ is indep. $\Leftrightarrow$ $\vec{v} \notin \text{span}(S)$.

"$\Rightarrow$" Let $T = S \cup \{\vec{v}\}$.

If T is indep, then by definition

$\vec{u} \notin \text{span}(T \setminus \{\vec{u}\})$ for every $\vec{u} \in T$.

When $\vec{u} = \vec{v}$.

$\Rightarrow$ $\vec{v} \notin \text{span}(T \setminus \{\vec{v}\}) = \text{span}(S)$.

"$\Leftarrow$" Suppose $c_0\vec{v} + c_1\vec{w}_1 + \ldots + c_k\vec{w}_k = \vec{0}$ for some $c_0, \ldots, c_k \in \mathbb{R}$.

If $c_0 \neq 0$, then

$\vec{v} = -\frac{1}{c_0}(c_1\vec{w}_1 + \ldots + c_k\vec{w}_k) \in \text{span}(S)$,

a contradiction.

Thus, $c_0 = 0$.

Then $c_1\vec{w}_1 + \ldots + c_k\vec{w}_k = \vec{0}$ implies

$c_1 = \ldots = c_k = 0$ since S is indep.

Therefore, $S \cup \{\vec{v}\}$ is linearly indep.

11. [extra 2pt] Let $\mathcal{S}_n$ be the set of all $n \times n$ symmetric matrices. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \boldsymbol{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be two matrices in $\mathcal{S}_3$. Find a basis for

$$V = \{\boldsymbol{X} \in \mathcal{S}_3 : \boldsymbol{A}\boldsymbol{X} = O\}.$$

Suppose $X = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$

Then $AX = 0 \Leftrightarrow \begin{array}{l} a+d+e = 0 \\ b+d+f = 0 \\ e+f+c = 0 \end{array}$

That is
$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$
$$\begin{pmatrix} 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

d, e, f are free.

$\Rightarrow$ basis for the null space

$$= \left\langle \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

So basis of $V = \left\langle \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \right\rangle$

$\Rightarrow$ dimension $= 3$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	35 (+2)	