國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第一次期中考 October 8, 2018 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

1. [1pt] Suppose $S = \{\vec{p_1}, \vec{p_2}, \dots, \vec{p_n}\}$ is a set of n vectors over $\mathbb{R}$. Write down the definition of that "$\vec{v}$ is a *linear combination* of vectors in S."

$$\vec{v} = c_1\vec{p}_1 + c_2\vec{p}_2 + \dots + c_n\vec{p}_n$$

for some $c_1, \dots, c_n \in \mathbb{R}$

2. [1pt] Suppose $\vec{a} = (a_1, \dots, a_n)$ $\cancel{\vec{p} = (p_1, \dots, p_n)}$ and $\vec{b} = (b_1, \dots, b_n)$ $\vec{q} = (q_1, \dots, q_n)$ are two vectors in $\mathbb{R}^n$. Write down the definition of the *inner product* of $\cancel{\vec{p}}$ $\vec{a}$ and $\cancel{\vec{q}}$ $\vec{b}$.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \dots + a_n b_n$$

3. [2pt] Give a linear system of **three equations** in **reduced echelon form** with **three free variables**, and indicate the free variables. [The answer is not unique. You only need to find one.]

$$\begin{aligned} x \qquad\quad + u + v + w &= 0 \\ y \quad + u + v + w &= 0 \\ z + u + v + w &= 0 \end{aligned}$$

free : u, v, w.

4. [2pt] Suppose A is a 5×5 nonsingular matrix. What is the minimum number of nonzero entries on A? [Justify your answer with an example of such A and explain why the number of nonzero entries cannot be fewer.]

minimum = 5 nonzero entries

For example,

$$A = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix}$$

If only 4 nonzero entries, at least one row is all zero $\Rightarrow$ singular.

5. Let

$$\vec{u} = (1, 1, 0, 0,) \text{ and}$$
$$\vec{v} = \left(\sqrt{3}, \sqrt{3}, 1, 1\right).$$

(a) [1pt] Find the length $|\vec{u}|$.

$$|\vec{u}| = \sqrt{1^2 + 1^2 + 0^2 + 0^2} = \underline{\sqrt{2}}$$

(b) [1pt] Find the length $|\vec{v}|$.

$$|\vec{v}| = \sqrt{3 + 3 + 1 + 1} = \sqrt{8} = \underline{2\sqrt{2}}$$

(c) [2pt] Find the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} \cdot \vec{v} = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{2\sqrt{3}}{\sqrt{2} \cdot 2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \underline{\theta = \frac{\pi}{6}}$$

(d) [2pt] Find a vector $\vec{p}$ that is orthogonal to both of $\vec{u}$ and $\vec{v}$. [The answer is not unique. You only need to find one.]

Suppose $\vec{p} = (x, y, z, w)$.

$$\vec{p} \cdot \vec{u} = x + y = 0$$
$$\vec{p} \cdot \vec{v} = \sqrt{3}x + \sqrt{3}y + z + w = 0$$

Any solution is an answer.

For example, $\vec{p} = (0, 0, 1, -1)$.

6. [6pt] Find the general solution of the following linear system.

$$\begin{cases} w + 3x + y - 2z = -1 \\ 2w + 6x + \cancel{2}^{3}y - \cancel{4}_{5}z = -7 \\ 3w + 9x + 3y - 6z = \cancel{-8}\ -3 \end{cases}$$

That is, find $\vec{p}$ and $\vec{\beta_1}, \ldots, \vec{\beta_k}$ such that

$$\{\vec{p} + c_1\vec{\beta_1} + \cdots + c_k\vec{\beta_k} : c_1, \ldots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

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$$\underbrace{\left(\begin{array}{cccc|c} 1 & 3 & 1 & -2 & -1 \\ 2 & 6 & 3 & -\cancel{4}5 & -7 \\ 3 & 9 & 3 & -6 & \cancel{8}\ -3 \end{array}\right)}_{A \quad \vec{b}} \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 1 & -2 & -1 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\uparrow \quad \uparrow$

x, z are free.

- Set $x = z = 0$, solve for $\vec{p}$ with $A\vec{p} = \vec{b}$.

$$\begin{aligned} w &= 4 \\ y &= -5 \end{aligned} \Rightarrow p = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \end{pmatrix}$$

- set $x=1, z=0$, solve for β_1 with $A\vec{\beta_1} = \vec{0}$.

$$\begin{aligned} w + 3 &= 0 \\ y &= 0 \end{aligned} \Rightarrow p = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

- set $x=0, z=1$, solve for β_2 with $A\vec{\beta_2} = \vec{0}$.

$$\begin{aligned} w - 1 &= 0 \\ y - 1 &= 0 \end{aligned} \Rightarrow p = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

general solution $= \left\{ \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\}$.

7. [6pt] Let

$$A = \begin{bmatrix} 1 & 5 & 4 & 3 \\ 2 & 10 & 8 & 7 \\ -1 & -5 & -4 & -3 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is known that $\boldsymbol{R}$ can be obtained from $\boldsymbol{A}$ by performing some row operations. Find a matrix $\boldsymbol{C}$ such that $\boldsymbol{CA} = \boldsymbol{R}$. [The answer is not unique. You only need to find one.]

$$A \xrightarrow[\rho_1 + \rho_3]{-2\rho_1 + \rho_2} \begin{pmatrix} 1 & 5 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-3\rho_2 + \rho_1} \begin{pmatrix} 1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{-3\rho_2+\rho_1}{\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{\rho_1+\rho_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \underset{-2\rho_1+\rho_2}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} A = R$$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} A = R$$

$$\begin{pmatrix} 7 & -3 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} A = R.$$

$$C = \begin{pmatrix} 7 & -3 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

8. [6pt] Find a matrix A whose reduced echelon form is ~~the reduced echelon form of~~

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and every entry of A is nonzero. [The answer is not unique. You only need to find one.]

Key: do any row operations to make entries ~~not~~ nonzero.

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\rho_2+\rho_1} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\rho_1+\rho_3]{\rho_1+\rho_2} \underline{\underline{\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 2 & -4 \\ 1 & 1 & 1 & -1 \end{pmatrix}}}$$

A

9. [extra 2pt] There are three types of 2×2 elementary matrices:

(1) $\rho_i \leftrightarrow \rho_j$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(2) $k\rho_i$: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

(3) $k\rho_i + \rho_j$: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Find four matrices E_1, E_2, E_3, E_4 of type (2) or type (3) such that

$$E_4 E_3 E_2 E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

[This means the first operation (swapping) can be done by only using the second and the third operations.]

Key: use only Type (2) or type(3) to obtain $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\rho_1+\rho_2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{-\rho_2+\rho_1} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\rho_1+\rho_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow{-\rho_1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So

$$\underset{E_4}{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} \underset{E_3}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} \underset{E_2}{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}} \underset{E_1}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$-\rho_1$ $\quad \rho_1+\rho_2$ $\quad -\rho_2+\rho_1$ $\quad \rho_1+\rho_2$ $\quad = \rho_1 \leftrightarrow \rho_2$.

[END]

Page	Points	Score
1	6	
2	6	
3	6	
4	6	
5	6	
6	2	
Total	30 (+2)	