國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） MATH 103 / GEAI 1215: Linear Algebra I

期末考 January 7, 2019 **Final Examination**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	9 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Find the general solution of the following linear system.

$$\begin{cases} w - 2x + 3y - \ \ z = 2 \\ 2w - 4x + 6y - 1z = 6 \\ 3w - 6x + 9y - 3z = 6 \end{cases}$$

That is, find $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{\beta_1}}, \dots, \vec{\boldsymbol{\beta_k}}$ such that

$$\{\vec{\boldsymbol{p}} + c_1\vec{\boldsymbol{\beta_1}} + \cdots + c_k\vec{\boldsymbol{\beta_k}} : c_1, \dots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

$$A = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 2 & -4 & 6 & -1 & 6 \\ 3 & -6 & 9 & -3 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 0 & 0 & 0 & 1 & \cancel{4}2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & 0 & \cancel{6}4 \\ & & & 1 & \cancel{4}2 \\ & & & & \end{array}\right).$$

$\uparrow \uparrow$

x, y are free.

Let $x = y = 0$. Solve $A\vec{p} = \vec{b}$ and get $\vec{p} = \begin{pmatrix} \cancel{x}4 \\ 0 \\ 0 \\ \cancel{x}2 \end{pmatrix}$.

Let $x=1, y=0$. Solve $A\vec{\beta_1} = \vec{0}$ and get $\vec{\beta_1} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Let $x=0, y=1$. Solve $A\vec{\beta_2} = \vec{0}$ and get $\vec{\beta_2} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

all solutions $= \{ \vec{p} + c_1\vec{\beta_1} + c_2\vec{\beta_2} \mid c_1, c_2 \in \mathbb{R} \}$.

2. [5pt] Suppose $S = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k\}$ is a set of nonzero vectors in $\mathbb{R}^n$ such that $\vec{\boldsymbol{w}}_i \cdot \vec{\boldsymbol{w}}_j = 0$ for any distinct i and j. (That is, any two vectors in S are orthogonal to each other.) Show that S is linearly independent.

See Midterm 2.

3. [3pt] Let $\vec{x} = \begin{bmatrix} 3+i \\ i \\ 2+i \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1-2i \\ 2+3i \\ 1 \end{bmatrix}$. Find the values of the inner products $\langle \vec{x}, \vec{y} \rangle$, $\cancel{\langle \vec{x}, \vec{y} \rangle}$ $\langle \vec{y}, \vec{x} \rangle$ and the norm $|\vec{x}|$ in $\mathbb{C}^3$.

$\langle \vec{x}, \vec{y} \rangle = (3+i)\cdot(1+2i) + i(2-3i) + (2+i)$

$= 1+7i+3+2i+2+i = \underline{\underline{6+10i}}$.

$\langle \vec{y}, \vec{x} \rangle = \overline{\langle \vec{x}, \vec{y} \rangle} = \underline{\underline{6-10i}}$.

$|\vec{x}|^2 = 9+1+1+4+1 = 16$

$\Rightarrow \underline{\underline{|\vec{x}| = \sqrt{16} = 4}}$.

4. [2pt] Let $\vec{u} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $\vec{x}$ and $\vec{y}$ such that $\vec{u} = \vec{x} + \vec{y}$ with $\vec{x} \in \operatorname{span}\{\vec{v}\}$ and $\langle \vec{v}, \vec{y} \rangle = 0$.

$\vec{x}$ = projection of $\vec{u}$ onto span$\{\vec{v}\}$.

$\vec{x} = |\vec{u}| \cdot \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \cdot \dfrac{\vec{v}}{|\vec{v}|}$

$= \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2} \cdot \vec{v} = \dfrac{4\cdot1+2\cdot1+3\cdot1}{1^2+1^2+1^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

$\Rightarrow \begin{cases} \vec{x} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \\ \vec{y} = \vec{u} - \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{cases}$

5. [5pt] Let $V = \operatorname{span}\left\{\begin{bmatrix}1\\-2\\3\\-1\end{bmatrix},\begin{bmatrix}2\\-4\\6\\-1\end{bmatrix},\begin{bmatrix}3\\-6\\9\\-3\end{bmatrix}\right\}$. Find a basis of V and a basis of $V^{\perp}$.

Let $A = \begin{pmatrix}1 & -2 & 3 & -1\\ 2 & -4 & 6 & -1\\ 3 & -6 & 9 & -3\end{pmatrix} \longrightarrow \begin{pmatrix}1 & -2 & 3 & -1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$

$\longrightarrow \begin{pmatrix}1 & -2 & 3 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$.

V = Rowspace (A) $\Rightarrow$ basis of $V = \left\{\begin{pmatrix}1\\-2\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$.

$V^{\perp}$ = Nullspace (A)

Solve $\begin{pmatrix}1 & -2 & 3 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}w\\x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\uparrow\ \uparrow$

x, y are free.

Set ~~x~~ $x=1, y=0$. Solve $A\vec{\beta_1} = \vec{0}$ $\Rightarrow$ $\vec{\beta_1} = \begin{pmatrix}2\\1\\0\\0\end{pmatrix}$.

Set $x=0, y=1$. Solve $A\vec{\beta_2} = \vec{0}$ $\Rightarrow$ $\vec{\beta_2} = \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}$

$\Rightarrow$ basis of $V^{\perp} = \left\{\begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}\right\}$.

6. [5pt] Consider the following data:

x	1	2	3	4	5
y_i	1	2	2	3	3

Find a line $f(x) = ax + b$ such that the error

$$\sum_{i=1}^{5}(f(x_i) - y_i)^2$$

is minimized.

[The orthogonal projection of a vector $\vec{v}$ onto the column space of a matrix $\boldsymbol{A}$ is $\boldsymbol{A}(\boldsymbol{A}^\top \boldsymbol{A})^{-1}\boldsymbol{A}^\top \vec{v}$. Your answer can be a formula without computing the final answer, but you have to specify all matrices or vectors occurred in your formula.]

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}$

Solve $A\begin{pmatrix} a \\ b \end{pmatrix} = \vec{v}$.

取 $\begin{pmatrix} a \\ b \end{pmatrix} = (A^T A)^{-1} A^T \vec{v} = \begin{pmatrix} 7/10 \\ 1/2 \end{pmatrix}$

7. [5pt] Let $f : V \to W$ be a homomorphism. Show that $f^{-1}(Y)$ is a subspace of V if Y is a subspace of W.

See version A.

8. [5pt] Let $f : V \to W$ be a homomorphism. Show that f is one-to-one if and only if nullspace$(f) = \{\vec{0}\}$.

See version A.

9. Let

$$\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}10\\1\\0\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}5\\10\\1\end{bmatrix}, \text{ and } \vec{u} = \begin{bmatrix}4\\3\end{bmatrix}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a homomorphism such that

$$f(\vec{v}_1) = f(\vec{v}_2) = f(\vec{v}_3) = \vec{u}.$$

(a) [extra 1pt] Find range(f) in set notation and give a brief reason to your answer. [Hint: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.]

$$\text{range}(f) = \left\{ t\begin{pmatrix}4\\3\end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

基底所有元素都送到 $\vec{u}$.

(b) [extra 2pt] Find the rank of f and the nullity of f.

$$\text{rank} = 1 \xRightarrow{\text{dim thm}} \text{nullity} = 3 - \text{rank} = 2.$$

(c) [extra 2pt] Find a basis of nullspace(f) and give a brief reason to your answer.

$$f(\vec{v}_1 - \vec{v}_2) = \vec{u} - \vec{u} = \vec{0}$$
$$f(\vec{v}_2 - \vec{v}_3) = \vec{u} - \vec{u} = \vec{0}.$$

So $\vec{v}_1 - \vec{v}_2$, $\vec{v}_2 - \vec{v}_3 \in$ nullspace (f).

$$\begin{cases} \text{nullity} = 2. \\ \{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3\} \text{ independent} \end{cases}$$

$\Rightarrow \{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3\}$ is a basis of nullspace(f).

10. [extra 2pt] Recall that the trace of a matrix is the sum of its diagonal entries. That is,

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{i,i}$$

if $A = \left[a_{i,j}\right]$ is an $n \times n$ matrix. Let $\mathcal{M}_{n\times n}$ be the set of all $n \times n$ matrices. Then

$$\begin{aligned} f : \mathcal{M}_{n\times n} &\to \mathbb{R} \\ A &\mapsto \mathrm{tr}(A) \end{aligned}$$

is a homomorphism. (You don't have to check this fact.) Find the rank of f and the nullity of f in terms of n.

See version A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	2	
Total	35 (+7)	