國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

期末考 January 7, 2019 **Final Examination**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	9 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Find the general solution of the following linear system.

$$\begin{cases} w + 3x + 2y + \ \ z = 2 \\ 2w + 6x + 4y + 3z = 9 \\ 3w + 9x + 6y + 3z = 6 \end{cases}$$

That is, find $\overrightarrow{\boldsymbol{p}}$ and $\overrightarrow{\boldsymbol{\beta_1}}, \ldots, \overrightarrow{\boldsymbol{\beta_k}}$ such that

$$\{\overrightarrow{\boldsymbol{p}} + c_1\overrightarrow{\boldsymbol{\beta_1}} + \cdots + c_k\overrightarrow{\boldsymbol{\beta_k}} : c_1, \ldots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 2 \\ 2 & 6 & 4 & 3 & 9 \\ 3 & 9 & 6 & 3 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 2 & 0 & -3 \\ & & & 1 & 5 \\ & & & & \end{array}\right)$$

$\uparrow \uparrow$

x, y are free.

Set $x = y = 0$. Solve for $\vec{p}$ $A\vec{p} = \vec{b}$ $\vec{p} = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 5 \end{pmatrix}$

Set $x = 1, y = 0$. Solve for $A\vec{\beta_1} = \vec{0}$ $\vec{\beta_1} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Set $x = 0, y = 1$. Solve $A\vec{\beta_2} = \vec{0}$ for $\vec{\beta_2} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

all solution $= \left\{ \vec{p} + c_1\vec{\beta_1} + c_2\vec{\beta_2} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$.

2. [5pt] Suppose $S = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k\}$ is a set of nonzero vectors in $\mathbb{R}^n$ such that $\vec{\boldsymbol{w}}_i \cdot \vec{\boldsymbol{w}}_j = 0$ for any distinct i and j. (That is, any two vectors in S are orthogonal to each other.) Show that S is linearly independent.

See Midterm 2.

3. [3pt] Let $\vec{x} = \begin{bmatrix} i \\ 2+i \\ 3+2i \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 3+i \\ 1 \\ 2-i \end{bmatrix}$. Find the values of the inner products $\langle \vec{x}, \vec{y} \rangle$, $\cancel{\langle \vec{x}, \vec{y} \rangle}$ $\langle \vec{y}, \vec{x} \rangle$ and the norm $|\vec{x}|$ in $\mathbb{C}^3$.

$$\langle \vec{x}, \vec{y} \rangle = i \cdot \overline{(3+i)} + (2+i) \cdot \overline{1} + (3+2i) \cdot \overline{(2-i)}$$

$$= i \cdot (3-i) + (2+i) + (3+2i)(2+i)$$

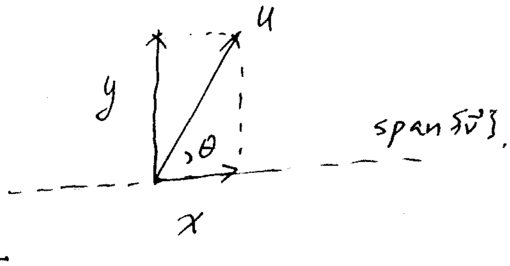
$$= 3i + 1 + 2 + i + 4 + 7i = \underline{\underline{7 + 11i}}$$

$$\langle \vec{y}, \vec{x} \rangle = \overline{\langle \vec{x}, \vec{y} \rangle} = \underline{\underline{7 - 11i}}$$

$$|\vec{x}|^2 = \langle \vec{x}, \vec{x} \rangle = 1 + 4 + 1 + 9 + 4 = 19$$

$$\Rightarrow \underline{\underline{|\vec{x}| = \sqrt{19}}}.$$

4. [2pt] Let $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $\vec{x}$ and $\vec{y}$ such that $\vec{u} = \vec{x} + \vec{y}$ with $\vec{x} \in \operatorname{span}\{\vec{v}\}$ and $\langle \vec{v}, \vec{y} \rangle = 0$.



x = projection of $\vec{u}$ onto span$\{\vec{v}\}$.

$$= |\vec{u}| \cdot \cos\theta \cdot \frac{\vec{v}}{|\vec{v}|} = |\vec{u}| \cdot \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$\cancel{\sqrt{3^2 + 4^2 + 2^2}} \quad = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v} = \frac{3 \cdot 1 + 4 \cdot 1 + 2 \cdot 1}{1^2 + 1^2 + 1^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \vec{x} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \\ \vec{y} = \vec{u} - \vec{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \end{cases}.$$

5. [5pt] Let $V = \text{span}\left\{ \begin{bmatrix}1\\3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\6\\4\\3\end{bmatrix}, \begin{bmatrix}3\\9\\6\\3\end{bmatrix} \right\}$. Find a basis of V and a basis of $V^{\perp}$.

Let $A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 4 & 3 \\ 3 & 9 & 6 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

$V = \text{Rowspace}(A)$

$\Rightarrow$ basis of $V = \left\{ \begin{pmatrix}1\\3\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$

$V^{\perp} = \text{Nullspace}(A)$

Solve $\begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

x y are free.

Set $x=1$, $y=0$. Solve $A\vec{\beta_1} = \vec{0} \Rightarrow \vec{\beta_1} = \begin{pmatrix}-3\\1\\0\\0\end{pmatrix}$

Set $x=0$, $y=1$. Solve $A\vec{\beta_2} = \vec{0} \Rightarrow \vec{\beta_2} = \begin{pmatrix}-2\\0\\1\\0\end{pmatrix}.$

$\Rightarrow$ basis of $V^{\perp} = \{\vec{\beta_1}, \vec{\beta_2}\}$

6. [5pt] Consider the following data:

x_i	1	2	3	4	5
y_i	2	2	3	3	4

Find a line $f(x) = ax + b$ such that the error

$$\sum_{i=1}^{5} (f(x_i) - y_i)^2$$

is minimized.

[The orthogonal projection of a vector $\vec{\boldsymbol{v}}$ onto the column space of a matrix $\boldsymbol{A}$ is $\boldsymbol{A}(\boldsymbol{A}^\top \boldsymbol{A})^{-1}\boldsymbol{A}^\top \vec{\boldsymbol{v}}$. Your answer can be a formula without computing the final answer, but you have to specify all matrices or vectors occurred in your formula.]

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \\ 4 \end{pmatrix}$

Goal: Solve $A \begin{pmatrix} a \\ b \end{pmatrix} = \vec{v}$ ← 無解.

取 $\vec{v}_0$ 為 $\vec{v}$ 在 Colspace (A) 的投影.

$\vec{v}_0 = A(A^T A)^{-1} A^T \vec{v}$.

則 $\begin{pmatrix} a \\ b \end{pmatrix} = (A^T A)^{-1} A^T \vec{v} = \begin{pmatrix} 13/10 \\ 1/2 \end{pmatrix}$

7. [5pt] Let $f : V \to W$ be a homomorphism. Show that $f^{-1}(Y)$ is a subspace of V if Y is a subspace of W.

Assumption: $\begin{cases} f \text{ is homo} \\ Y \text{ is a subspace in } W. \end{cases}$

[Method: Suppose $\vec{v}_1, \vec{v}_2 \in \cancel{V}\ f^{-1}(Y)$ and $c_1, c_2 \in \mathbb{R}$.
Show $c_1\vec{v}_1 + c_2\vec{v}_2 \in \cancel{V}\ f^{-1}(Y)$].

Suppose $\vec{v}_1, \vec{v}_2 \in \cancel{V}\ f^{-1}(Y)$.

By definition, $f(\vec{v}_1) \in Y$, $f(\vec{v}_2) \in Y$.

Since f is homo,

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2).$$

Since Y is a subspace,

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 \underbrace{f(\vec{v}_1)}_{\in Y} + c_2 \underbrace{f(\vec{v}_2)}_{\in Y} \in Y.$$

Thus, $c_1\vec{v}_1 + c_2\vec{v}_2 \in f^{-1}(Y)$.

Goal: $f^{-1}(Y)$ is a subspace in V.

8. [5pt] Let $f : V \to W$ be a homomorphism. Show that f is one-to-one if and only if nullspace$(f) = \{\vec{0}\}$.

Claim: f is 1-1 $\iff$ nullspace $(f) = \{\vec{0}\}$.

"$\Longrightarrow$" Assumption : $\begin{cases} f \text{ is homo.} \\ f \text{ is } 1\text{-}1. \end{cases}$

Since f is homo, $f(\vec{0}) = \vec{0}$.

Since f is 1-1, $f(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$.

Thus, nullspace $(f) = \{\vec{v} \in V \mid f(\vec{v}) = \vec{0}\}$
$= \{\vec{0}\}$.

Goal: nullspace $(f) = \{\vec{0}\}$.

"$\Longleftarrow$" Assumption : $\begin{cases} f \text{ is homo} \\ \cancel{f} \text{ nullspace}(f) = \{\vec{0}\}. \end{cases}$

Suppose $f(\vec{x}) = f(\vec{y})$.

Since f is homo,

$f(\vec{x} - \vec{y}) = f(\vec{x}) - f(\vec{y}) = \vec{0}$.

Thus, $\vec{x} - \vec{y} \in$ nullspace $(f) = \{\vec{0}\}$.

$\Rightarrow \vec{x} - \vec{y} = \vec{0}$

$\Rightarrow \vec{x} = \vec{y}$.

Goal: f is 1-1.

9. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a homomorphism such that

$$f(\vec{v}_1) = f(\vec{v}_2) = f(\vec{v}_3) = \vec{u}.$$

(a) [extra 1pt] Find range(f) in set notation and give a brief reason to your answer. [Hint: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.]

$\underline{\text{range}(f) = \{ t\vec{u} \mid t \in \mathbb{R} \}}$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis,

$\text{range}(f) = \{ f(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \mid c_1, c_2, c_3 \in \mathbb{R} \} = \{ (c_1+c_2+c_3)\vec{u} \mid c_1, c_2, c_3 \in \mathbb{R} \}$

add space

(b) [extra 2pt] Find the rank of f and the nullity of f.

$\{\vec{u}\}$ is a basis of range(f).

$\underline{\text{rank} = 1}$, $\underline{\text{nullity} = 3 - \text{rank} = 2}$ by dimension thm.

(c) [extra 2pt] Find a basis of nullspace(f) and give a brief reason to your answer.

$f(\vec{v}_1 - \vec{v}_2) = \vec{u} - \vec{u} = \vec{0}$

$f(\vec{v}_2 - \vec{v}_3) = \vec{u} - \vec{u} = \vec{0}$.

$\Rightarrow$ $\vec{v}_1 - \vec{v}_2 \in$ nullspace(f)

$\vec{v}_2 - \vec{v}_3 \in$ nullspace(f).

$\left\{ \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\}$ is a basis of nullspace(f),

10. [extra 2pt] Recall that the trace of a matrix is the sum of its diagonal entries. That is,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{i,i}$$

if $A = [a_{i,j}]$ is an $n \times n$ matrix. Let $\mathcal{M}_{n\times n}$ be the set of all $n \times n$ matrices. Then

$$f : \mathcal{M}_{n\times n} \to \mathbb{R}$$
$$A \mapsto \operatorname{tr}(A)$$

is a homomorphism. (You don't have to check this fact.) Find the rank of f and the nullity of f in terms of n.

$\dim(M_{n\times n}) = n^2$.

$\text{range}(f) = \mathbb{R} \Rightarrow$ rank $= 1$.

by dimension theorem, nullity $= n^2 - 1$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	2	
Total	35 (+7)	