translated by Alessandro Rosa - *released* on May 15th, 2001

FOURTH PART

On the study of singular convergences.

102. In the 2nd and in the 3rd part I have treated the study of the convergence to a point with uniform convergence [$\zeta = \varphi(\zeta)$, $|\,\varphi'(\zeta)\,| < 1$] or the convergence to a periodic cycle [$\zeta = \varphi_p(\zeta)$, $|\,\varphi'_p(\zeta)\,| < 1$].

In this fields of researches, new and various results has come out.
The consideration of the perfect set E′ has ruled these studies and it has enlightened this argument preciously.

I will immediately work on the singular cases: roots points where

$$\zeta = \varphi(\zeta) \qquad \text{for} \qquad |\,\varphi'(\zeta)\,| = 1$$

or of (*for periodic cycles*)

$$\zeta = \varphi_p(\zeta) \qquad \text{for} \qquad |\,\varphi'_p(\zeta)\,| = 1 .$$

This study has been less easy than the study of regular limit points. The results I have obtained are not generals, but, as well as one see them, they allow the reader to imagine this difficult question.

History tells that Leau has already studied the neighbourhood of a point $\zeta = \varphi(\zeta)$, satisfying $|\,\varphi'(\zeta)\,| = 1$. But one may obtain more results than Leau's, on the condition to use the set E′.
Fatou, in its Note of May 21th 1917, has studied the singular cases, related to the fractions in the fundamental circle.
They always refer to the case $\zeta = \varphi(\zeta)$ satisfying $|\,\varphi'(\zeta)\,| = 1$: they are very simple.

I give further some additions to these known results; the reader would realize their simplicity. Moreover, I will give further less easy examples to deal with: anyway they are more general than what we have studied up to present.

Finally I sketch out the general case for those points $\zeta = \varphi(\zeta)$ satisfying $\varphi'(\zeta) = e^{i\theta}$, where θ is real and arbitrary; the method I used is a way to reduce all cases to just two possible hypotheses, so that if one of them occurs, then it excludes the possibility for the other hypothesis to come true in the same example (i.e.: a mutual exclusion).
Up to present, when θ is incommensurable to 2π, I did not manage to find out ([1]) examples of fractions $\varphi(z)$ referring to each one of the two previous hypotheses.

([1]) After the depositing of this Memoir, I managed to exclude one of those properties; this could be the argument of a further Memoir.

Anyway, when θ is commensurable to 2π and $\varphi(z)$ is rational, *only one of these two hypotheses holds* so that each hypothesis excludes the other one.

104. *The study of the points* $\zeta = \varphi(\zeta)$ *where* $\varphi'(\zeta) = e^{i\theta}$, *and* $\zeta = \varphi_p(\zeta)$ *where* $\varphi'_p(\zeta) = e^{i\theta}$, *when* θ *is commensurable to* 2π. I show *that a point belongs always to* E′.

It is necessary to consider $\zeta = \varphi(\zeta)$ so that $|\varphi'(\zeta)| = 1$, because:

1. If $\zeta = \varphi(\zeta)$, where $\varphi'(\zeta) = e^{2\pi i \frac{p}{q}}$, then the substitution $z_q = \varphi_q(z)$ admits a double point $\zeta = \varphi_q(\zeta)$ with $\varphi'_q(\zeta) = 1$; since this substitution $z_q = \varphi_q(z)$ has the same E′ as $z_1 = \varphi(z)$, then the theorem is proved.
2. If $\zeta = \varphi(\zeta)$, where $\varphi'(\zeta) = e^{2\pi i \frac{p'}{q}}$, then the substitution $z_p = \varphi_p(z)$ admits a double point $\zeta = \varphi_p(\zeta)$ with $\varphi'_p(\zeta) = e^{2\pi i \frac{p'}{q}}$; so we come back to the first case.

On the following, I will deal only with the case $\zeta = \varphi(\zeta)$ with $\varphi'(\zeta) = 1$, since the broadenings of the 1) and 2) are very easy.
Suppose ζ to be the origin: $\zeta = 0$; then let

$$\varphi(z) = z + a_n z^n + \ldots (n \geq 2)$$

be the Taylor development of φ.
One then gets

$$\varphi_p(z) = z + pa_n z^n + \ldots (p \geq 1, 2, \ldots, \infty)$$

The functions $\varphi_p(z)$ vanish at the origin.
If the origin does not belong to E′, then the sequence of $\varphi_p(z)$ is *normal* in a small area D surrounding O, and the φ_p are bounded in O like they are bounded in all D too.

One could extract an infinite sequence φ_{p1}, φ_{p2}, …, converging *uniformly* in D to a limit function $f(z)$, which is analytic in D and satisfying $f(O) = 0$, $f'(z) = 1$.
It is impossible when the second coefficient of the Taylor development of φ_{pi} (being $p_i a_n$) becomes infinite with p_i.

Therefore O is a point of E′ and the functions φ_p are not *normal*.

105. Leau (Thesis, 1897) has shown that a point $\zeta = \varphi(\zeta)$, satisfying $\varphi(\zeta) = 1$, could be surrounded by a small circle including some points where the images converge to ζ and by some points whose the *preimages*, defined by the branches of the inverse of φ (so that ζ is a fixed point) converge to ζ''.
Moreover I will show it, with the help of the preposition (established in the *Preliminaries*, §5) generalizing the Schwarz′s Lemma.

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Let us start with the simplest case where $\varphi(z)$ can be developed around $\zeta = 0$ by

$$z_1 = \varphi(z) = z + a_2 z^2 + \ldots \qquad (a_2 \neq 0).$$

Consider the circle γ, centered at α and *intersecting the origin*; then its equation is

$$zz' - \alpha z - \alpha z' = 0 \text{ (since both } z \text{ and } z' \text{ conjugates)}$$

where

$$z_1 z'_1 - \alpha' z_1 - \alpha z'_1 = [zz' - \alpha' z - \alpha z'] + a_2 z^2 (z' - \alpha') + \alpha'_2 z'^2 (z - \alpha) + \ldots,$$

the terms with order ≥ 4 are not written, if α is supposed to be infinitely small and with order 1 and if z is so chosen to lie in the interior of γ or on γ.

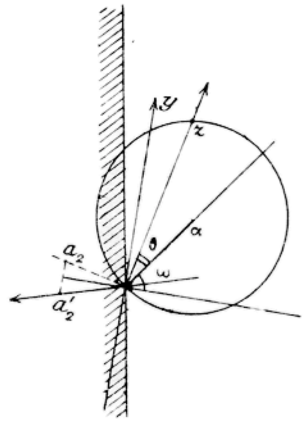

Fig. 26

If the point z lies on γ, one will get, by fixing $\alpha = re^{i\omega}$,

$$z - \alpha = re^{i(\omega + 2\theta)} \qquad z - \alpha = 2r\cos\theta e^{i(\omega+\theta)} \qquad z' - \alpha' = re^{-i(\omega+2\theta)}$$

at then

$$z_1 z'_1 - \alpha' z_1 - \alpha z'_1 = 4r^3 \cos 2\theta (a_2 e^{i\omega} + a'_2 e^{-i\omega}) + \ldots,$$

since the unwritten terms are of order ≥ 4 in r.
The main part of $z_1 z'_1 - \alpha' z_1 - \alpha z'_1$, whose sign fixes the position of z_1 in respect of the circle γ as soon as r is chosen to be small enough, is

$$4r^3 \cos^2\theta \, \mathrm{R}(a_2 e^{i\omega}) = 4r^3 \cos^2\theta \times [\text{real part of}(a_2 e^{i\omega})] .$$

with a constant sign if θ ranges from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, while z describes γ and on the condition that $\mathrm{Re}(a_2 e^{i\omega}) \neq 0$.

Put the sign – if α is chosen so that

$$\frac{\pi}{2} < \arg\alpha + \arg\alpha_2 < \frac{3\pi}{2},$$

since

$$\frac{\pi}{2} < \arg(a_2 e^{i\omega}) < \frac{3\pi}{2} \qquad \text{and} \qquad \mathrm{Re}(a_2 e^{i\omega}) < 0 .$$

Let us suppose that the argument ω of α is chosen so that $\mathrm{Re}(a2e^{i\omega}) < 0$: for this value of ω, one may choose r to be small enough ($r < r_0$) to obtain that the sign of $z_1 z'_1 - \alpha' z_1 - \alpha z'_1$ is constantly the sign - (minus), when z describes the circle γ, centered at α and with radius $r(\alpha = re^{i\omega})$.

The point z_1 is in the interior of the circle γ and it lies on γ when z would lie on O.

Then our lemma: O *is the only limit point of the images of every interior point of the circle* γ.

Let us make ω range within its extreme limits, from $\frac{\pi}{2} - \arg a_2$ to $\frac{3\pi}{2} - \arg a_2$, so that, for every value of ω, r is > 0 : $z_1 z'_1 - \alpha' z_1 - \alpha z'_1$ ($\alpha = re^{i\omega}$) is constantly < 0, while z describes the circle γ.
Then, one generates a domain Δ [its shape recalls a cardiod with an angle vanishing at O, where the tangent line (in the exterior of Δ) is the direction (-arg a_2), that's to say the direction going from O to a'_2, conjugate with a] so that O is the only limit point of the images of any interior point.

If one goes to the branch of the inverse function, vanishing at the origin, then one gets

$$z = \psi(z_1) = z_1 - a_2 z_1^2 + \dots ,$$

so that the same considerations as before retrieve a domain Δ', sensibly symmetric to Δ in respect of O, where any arbitrary point of Δ' has its preimages, defined by $\psi(z)$ converging to O and only to O.
Δ and Δ' share a common part, owing to their cardiod shape.

In each *interior* region of Δ, the sequence of $\varphi_i(z)$ converges uniformly to zero so that it generates a *normal* sequence.

Therefore Δ is inside the region R of the plane z, consisting of one piece and bounded by the set E′: O is the only limit point of the images of every interior point of R.

So R is the immediate domain of convergence of the point O. (We will see that R cannot be the total domain of O.)

It is clear that if:

1. we start from a small circle c, surrounding O and inside the set of Δ and Δ' ...
2. ... after we take the preimages of an area (c) for the branch of $\psi(z)$, vanishing at the origin;

then the portion of (c), inside Δ', has *some preimages converging to* O *and the portion of* (c) *in* Δ *has some preimages converging to the domain* R.

Therefore, R is the limit of the preimage areas (C_{-i}), defined by the branch of $\psi(z)$ vanishing in O ([1]).

106. Evidently, the treatment before supposes $a_2 \neq 0$, but it fails for $a_2 = 0$; but some more general considerations could be expressed and inspired by the previous cases that are easier to follow when the case where $a_2 \neq 0$ has been understood.

Generally, let

$$z_1 = \varphi(z) = z + a_p z^p + \ldots \qquad (a_p \neq 0, p \geq 2)$$

be the Taylor development around O.

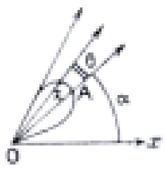

Fig. 27

Let us try to replace the previous circle γ with an analogous curve Γ intersecting the origin: we take a petal of the flower $r^n = a^n \cos n\theta$ (whose axis OA makes the angle α with OX) we try to determine n, α, a (when a is small enough) so that, $z = re^{i(\theta+\alpha)}$ describe that petal Γ.
z_1 remains in the interior of the petal and it would come over the petal unless z would come to O, so that it carries z_1 in O.

The petal Γ can be defined by

$$z^n = r^n e^{ni(\theta+\alpha)} = a^n e^{ni(\theta+\alpha)} \frac{e^{ni\theta} + e^{-ni\theta}}{2},$$

$$z^n - \frac{a^n e^{ni\alpha}}{2} = \frac{a^n}{2} e^{ni(2\theta+\alpha)},$$

$$z'^n - \frac{a^n e^{-ni\alpha}}{2} = \frac{a^n}{2} e^{-ni(2\theta+\alpha)},$$

since θ ranges from $-\frac{\pi}{2n}$ to $\frac{\pi}{2n}$. Therefore

$$(\Gamma) \qquad z^n z'^n - \frac{a^n e^{-ni\alpha}}{2} z^n - \frac{a^n e^{ni\alpha}}{2} z'^n = 0 .$$

Let us consider, since $a > 0$ and infinitely small of the first order, the following value:

$$\delta = z_1^n z_1'^n - \frac{a^n e^{-ni\alpha}}{2} z_1^n - \frac{a^n e^{ni\alpha}}{2} z_1'^n ;$$

([1]) If an area (C_{-i}) includes a critical point of the considered branch, then the process does not stop. I will show that the domain R always includes a critical point of $\psi(z)$.

one gets, by developing δ and recalling that z is on Γ

$$\delta = nz^{n+p-1}\, a_p\left[z'^n - \frac{a^n}{2} e^{-ni\alpha} \right] + nz'^{n+p-1} a'_p \left[z^n - \frac{a^n}{2} e^{ni\alpha}\right]$$

where the order of unwritten terms is > $2n + p - 1$, in respect of a, the most important and infinitesimally small.

If a is small enough, then the sign of δ is therefore, whatever z in on Γ, the sign of

$$\delta_1 = nz^{n+p-1} a_p\left[z'^n - \frac{a^n}{2} e^{-ni\alpha} \right] + nz'^{n+p-1} a'_p \left[z^n - \frac{a^n}{2} e^{ni\alpha}\right]$$

which is reduced, *assuming* $n = p - 1$, to

$$\delta_1 = nR\left[a_p \frac{a^n}{2} r^{2n} e^{ni\alpha}\right] = n\frac{a^n}{2} r^{2n} R\left[a_p\, e^{ni\alpha}\right].$$

Therefore, if the value assumed by α in respect of

$$\frac{\pi}{2} < n\alpha + \arg a_p < \frac{3\pi}{2} \qquad (\text{mod } 2\pi)$$

then α could be determined such small as, when z *describes the petal* Γ of the axis OA (α grows up from the intersection of OA with OX) in respect of OA, then *the sign of* δ *is constantly* the same of $\mathrm{Re}(a_p e^{ni\alpha}) < 0$.

Then, *since* δ is *always* < 0, z_1 is inside the considered petal Γ: it comes back to O only if z comes back to O (evidently, z_1 is in the interior of the considered petal, not in another petal of the same flower, because z_1 differs from z for an infinitely small quantity with order $p \geq 2$).

This reasoning is the same as the one we used for $p = 2$ and it immediately extends the lemma we needed at the case where Γ replaces the circle γ ([1]).

It proves that O *is the only limit point of the images of any interior point z of* Γ.

([1]) This extension may be done by conformally mapping Γ on a circle γ using the relation $Z = z^n$; then Z_1 becomes a regular function of Z in the circle γ, enjoying those properties, expected by our lemma. In O, it is verified (in the neighbourhoods of O, in Γ, $Z_1 = Z + (p{=}1)a_pZ_2 + Z_2\varepsilon(Z)$, for ε(Z) converging to zero with Z, being algebraic in Z), Z_1 is never analytic in Z, it is algebraic in Z, around the O; but, when Z converges to O from the interior of Γ, one see that Z_1 converges to zero, $\frac{aZ_1}{dZ}$ converges to 1, $\frac{d^2Z_1}{dZ^2}$ converges to a well determined finite limit, so that the lemma keeps on being applied.

The inequalities

$$\frac{\pi}{2} < n\alpha + \arg a_p < \frac{3\pi}{2} \qquad (\text{mod } 2\pi) \qquad , \qquad n = p - 1$$

prove that the direction, determined by the angle α, could be arbitrarily assumed in a number of $p - 1 = n$ angles, so that each one of them is equal to $\frac{\pi}{n}$ (the dashed angles), regularly placed around O.

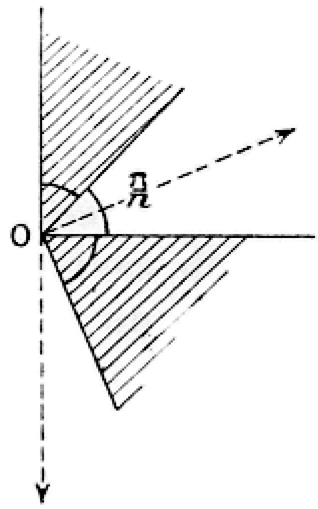

Fig. 28

Thus, every determined α is mapped to a small enough *a* enjoying the previous property (it may assume the biggest number *a* while enjoying that property).

After, as α assumes all proper values, Γ sweeps a domain Δ including $n = p - 1$ points coming back to O, whose tangent lines are the bisectors of the angles splitting those other angles, swept by OA.

O is the only limit point of the images of every interior point of Δ.

If one considers the branch ψ(*z*), inverse of φ and vanishing in O, then, for $a_2 \neq 0$, the result is,

$$z = \psi(z_1) = z_1 - a_p z_1^p + \dots ,$$

now one may fall on the domain Δ', the analogous of Δ, defining now its points as the bisectors of the dashed angles (Δ' is sensibly symmetric of Δ, in respect of any straight line determining the limit angles of α in the previous inequalities).

Both Δ and Δ′ share some parts (in general 2*n* parts, opposite to each other in respect of O), between the 2*n* limit half-lines of the dashed angles.

Every interior point of Δ' has some preimages, which are defined by $\psi(z)$ and converging to O.

Therefore, if:

1. one examines a small circle c, surrounding the origin and enough small to be in the interior of all parts of both Δ and Δ' ;
2. one considers the preimages of the area (c) including O *with the help of the branch* $\psi(z)$ *vanishing in* O;

then the preimages (c_{-i}) converge for $i \to \infty$ to an area (R), which is bounded by the set E′ and attaining to O; O is the boundary point of that area for p - 1 points whose tangent lines are still tangent in the points of Δ.

This area (R) may consist of more disconnected distinct regions attaining to O.
(R) *will be the immediate domain of convergence to* O.

The whole truth consists of asserting that the parts of (c), in the interior of Δ', converge to zero: they are in the interior of Δ and converging to R. In every interior area of R, the sequence of φ_i is normal and it converges uniformly to zero. Then, R includes Δ.

We see further that R cannot be the total domain of convergence to O.

107. In particular, if $p = 3$, then a simpler result may come out.
Let us consider

$$z_1 = z + a_3 z^3 + \ldots$$
$$z'_1 = z + a'_3 z^{13} + \ldots$$

and let us search again for a circle Γ intersecting O and including its image.

It is necessary to determine the complex number $\alpha = re^{i\omega}$ so that if $zz' - \alpha' z - \alpha z' = 0$ with

$$z - \alpha = re^{i(\omega + 2\theta)}, \qquad z = 2r\cos\theta\, e^{i(\omega+\theta)}$$

where θ ranges from $-\frac{\pi}{2n}$ to $\frac{\pi}{2n}$.

The result is $\delta_1 = z_1 z'_1 - \alpha' z_1 - \alpha z'_1 < 0$

Now

$$\delta_1 = R\left[a'_3 z'^3 (z - \alpha)\right] + \ldots$$

where unwritten terms are of degree > 4 in r, which is assumed to be infinitely small.

δ_1 has the sign of $\mathrm{Re}\left[a'_3 z'^3 (z - \alpha)\right]$, if r is small enough; this last quantity is anything else but

$$\mathrm{Re}\left[a'_3 8r^4 \cos^3\theta e^{i(2\omega + \theta)}\right] \;;$$

where θ ranges from $-\frac{\pi}{2n}$ to $\frac{\pi}{2n}$: this last quantity has not a constant sign: it is < 0, if θ is assumed to be

$$2\omega + \arg a'_3 \equiv \pi \qquad (\text{mod } 2\pi)$$

It determines these two opposite directions that α should take.
r may assume a small enough value on both directions so that δ_1 is < 0 while *z* describes Γ and on the consequence z_1 *is in the interior of* Γ.
Therefore Δ reaches to O from two opposite points; it is easier than assuming Δ as the set of two biggest circles, which are tangent outside O so that they satisfy the last enunciated condition; the limit of the domain of convergence to O is defined as soon as the preimages of the interior areas of the two circles are considered.

108. The notions before are applied to examples about the singular fractions in the fundamental circle, notified by Fatou (*Comptes Rendus*, May 21th 1917).

Let us consider

$$z_1 = z + \sum \frac{b_i}{a_i - z} \qquad (b_i > 0,\ a_i \text{ real})$$

(The real axis is the fundamental circle.)
The point at infinity is a singular limit point. In fact, around the point $z = \infty$

$$z_1 = z - \frac{\theta_1}{z} + \frac{\theta_2}{z^2} + \ldots \qquad \theta_1 = \sum b_i, \qquad \theta_2 = -\sum \frac{b_i}{a_i}, \qquad \ldots,$$

the sequence of the second member converges for a big enough *z*.
Fix $z = \frac{1}{Z}, z_1 = \frac{1}{Z_1}$, then assume that

$$z = \frac{1}{Z_1} = \frac{1}{Z} - \theta_1 Z + \theta_2 Z_2 + \ldots,$$

where $Z_1 = \frac{Z}{1 - \theta_1 Z^2 + \ldots}$,
since the denominator converges for a small enough Z, where

$$Z_1 = Z + \theta_1 Z^3 + \ldots \ .$$

in the case when $a_2 = 0$ and $a_3 \neq 0$. Z = 0 is the limit point of the images of every point in the interior of the fundamental circle.
In these case, E′ is the entire fundamental circle (*read* FATOU, p.807).
In this example, the two opposite directions (indicated before) are taken from the limit singular point to the centre of the fundamental circle ([1]) by a straight line.

([1]) Here, the perpendicular line in Z = 0 in respect of the real axis.

Γ is any tangent circle to the fundamental circle in Z = 0, that's tangent to say the real axis.

109. *Other examples* :

1. $z_1 = z + z^2$. The origin is the singular limit point where the images of the point $z = -\frac{1}{2}$ converge to; in this point, $\varphi'(z)$ vanishes. Infinity is the other limit point.

The circle Γ with diameter (0,-1) belongs to the domain of O.
In fact, the condition for

$$\mathrm{Re}\left(\frac{1}{z_1}\right) \leq \mathrm{Re}\left(\frac{1}{z}\right) \qquad \text{where} \qquad \mathrm{Re}\left(\frac{1}{z_1} - \frac{1}{z}\right) = -\ \mathrm{Re}\left(\frac{1}{1+z}\right) \leq 0$$

is

$$\mathrm{Re}(1+z) \geq 0 \qquad \text{where} \qquad \mathrm{Re}(z) \geq -1$$

Then, for any point of the circle Γ with diameter (0,-1), except for –1 that belongs to E′ since it is an preimage of O, one gets

$$\mathrm{Re}\left(\frac{1}{z_1}\right) < \mathrm{Re}\left(\frac{1}{z}\right),$$

and, following, the image is *in the interior of* Γ, since the equation of Γ is

$$\mathrm{Re}\left(\frac{1}{z}\right) = -1 \ .$$

Here $a_2 = 1$. The direction of the tangent line in the point of the domain of O, since it is $(-\arg a_2)$, is the direction of OX since $\arg a_2 = 0$.

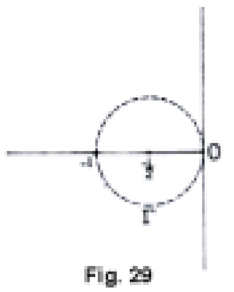

Fig. 29

Therefore, if one takes the preimages of the area (Γ), then the simply connected areas (Γ_{-i}) converge to the domain R_0 ([1]) of the origin.

The curves Γ_{-i} (the boundaries of those areas) converge to a continuous line being E′ and splitting R_0 from R_∞ (R_∞ is the domain of the point at infinity).

([1]) R_0 like R_∞, both simply connected and consisting of one piece, are simultaneously the total domain and the immediate domain of convergence to the limit point.

Outside Γ, one gets

$$|\varphi'(z)| > |2z - 1| > 1 .$$

Some simple considerations, already explained in the third part of this Memoir, prove that the curves Γ_{-i} converge *uniformly* to their limit E′, which is a *simple closed Jordan curve* intersecting either O and –1 while it is tangent in O with the direction OX_1.

110.

2. Let us take $z_1 = z + z^3$. The zeros of $\varphi'(z)$ are $z = \pm\frac{i}{\sqrt{3}}$; it is assured that their images converge uniformly to O, since they lie on the imaginary axis.

Therefore no other limit point (or periodic cycle) but O.

The immediate domain of the point O consists of two simply connected areas, symmetric in respect of O and tangent to OX in O; these two areas are respectively obtained by starting from two circles Γ and Γ′: symmetric in respect of O and tangent to OX in O, with any radius $\le \frac{1}{2}$.

It may be supposed that the radius is $= \frac{1}{2}$, since Γ and Γ′ intersect the two imaginary points *i* and *-i* , preimages of O.
In fact, it is verified that

$$\mathrm{Im}\left(\frac{1}{z_1}\right) - \mathrm{Im}\left(\frac{1}{z}\right) = -\mathrm{Im}\left(\frac{z}{1+z^2}\right),$$

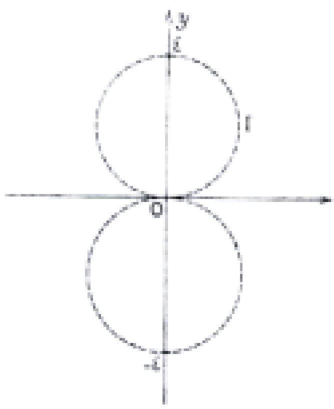

Fig. 30

whose sign is the same as $\mathrm{Im}\left(z + \frac{1}{z}\right)$, two inverse complex numbers whose imaginary parts have opposite signs.

Therefore if $|z| < 1$ and if $\mathrm{Im}(z) > 0$, then we get

$$\mathrm{Im}\left(\frac{1}{z_1}\right) < \mathrm{Im}\left(\frac{1}{z}\right).$$

Therefore, if z is in the interior of Γ or on Γ (so that $z = i$), then one gets an interior point z_1 of Γ, since the equation of Γ is $\mathrm{Im}\left(\frac{1}{z}\right) = -1$; any interior point satisfies the following inequality $\mathrm{Im}\left(\frac{1}{z}\right) < -1$. Γ is interior in R_0.

Then one sees that $|\ \varphi'(z)\ | = 1$ is a lemniscate with two focuses in $+\frac{i}{\sqrt{3}}$ and $-\frac{i}{\sqrt{3}}$ respectively; it intersects O and it is in the interior of Γ and Γ′.

Therefore, on Γ and Γ′, outside Γ and Γ′, the following inequality $|\ \varphi'(z)\ | > 1$ holds.

In conclusion, the boundary of R_0 (the immediate domain of O) consists of two simple closed Jordan curves which are both tangent to OX in O and symmetric with each other in respect of OX.

R_0 consists of two simply connected symmetric half-parts ([1]); each one of the two halves of R_0 includes a critical point of $\psi(z)$.

If one generates the total domain of convergence to O by taking the preimages of each one of the two areas generating R_0, then one gets an infinity of areas (each one is bounded by a simple closed Jordan curve) whose the linear dimensions converge to zero so that these areas are grouped like the areas appearing during the study of the example

$$z_1 = \frac{-z^3 + 3z}{2}$$

the set of their boundaries generates only a continuous line which is the boundary of the simply connected domain R_∞, consisting of one piece. R_∞ is at the same time both the immediate domain and the total domain of convergence to infinity. The continuous line, bordering R_∞, is a continuous closed curve, represented by the following equations

$$v = f(t), \qquad y = g(t) \text{(since } f \text{ and } g \text{ are continuous)}$$
$$f(a) = f(b), \qquad g(a) = g(a) \qquad (a \leq t \leq b)$$

but it is not a *simple* Jordan curve. The curve has some double points everywhere dense on itself: they are preimages of the origin.

All these points are accessible from the interior of R_∞ as the origin is accessible from the two sides (both positive and negative) of the real axis ([2]).

([1]) Each one of the two parts turns into itself by $z_1 = z + z^3$ and by the two branches of the inverse function $\psi(z)$, permuting around a interior critical point in the considered part.

([2]) For $z_1 = z - z^3$, the critical points would be real, the OY axis would be replaced by the axis OX; and the two parts of R_0 would lie, like the domains R_1 and R'_1 in the example $z_1 = \frac{-z^3 + 3z}{2}$.

111. *The interior critical points of* $\psi(z)$ *in the immediate domain of a point* $\zeta = \varphi(\zeta)$, satisfying $\varphi'(\zeta) = 1$ ([1]).

Firstly, let us send to infinity a point $z = \varphi(z)$, satisfying $|\varphi'(z)| > 1$ (a point of E); it may happen that the point at infinity does not admit the point ζ (it may be supposed to be the origin) as the limit point of its images.
Let us trace, around $\zeta = 0$, a small circle bounding the area (C); we will consider the preimages (C_{-i}) of (C) for by the branch (of $\varphi(z)$) vanishing at the origin.
If, whatever is i, none of (C_{-i}) includes the critical point of the examined branch, then the following functions

$$z_{-1} = \psi(z) = z + \lambda_2 z^2 + \ldots$$
$$z_{-2} = \psi_2(z) = z + \ldots$$

are obtained by the iteration of the branch $\psi(z)$ and they are respectively the branches of the inverse functions of $\varphi(z)$, $\varphi_2(z)$, … vanishing at the origin and holomorphic in (C) [they have not poles or critical points, because (C) does not include any image of the point at infinity].

The first derivative of these functions is = 1 at the origin; each one of them assumes only once a values in (C): if z_1 and z_2 are distinct in (C), then $\psi_i(z_1) \neq \psi_i(z_2)$, whatever is the value of i. Then one may apply the Kœbe's theorem and recall § 31. $\psi_i(z)$ satisfies all conditions of validity for that theorem; as z describes C, then the point z_{-i} describes a closed line C_{-i} so that its shortest distance to the origin is $d > \frac{\sqrt{2}-1}{4}\rho$, where ρ is the radius of C.

All the curves C_{-i} lie at a finite distance from O: this is impossible since it has been shown that the parts of an interior circle C of the domain, that we defined Δ', have some preimages converging to O.
Therefore, as it is been established that the shortest distance from C_{-i} to O converges to zero for $i \to \infty$, then (C_{-i}) tends, while i increases, to include a critical point of the branch $\psi(z)$ vanishing in O, whatever small the starting area (C) is. It is been shown previously that only the parts of (C_{-i}), belonging to Δ but not to Δ', converge to zero; these parts, belonging to Δ, are in the interior of the immediate domain R_0.

In conclusion *the immediate domain of the origin* includes always a critical point for the branch of $\psi(z)$ vanishing at the origin; moreover, but with less rigour, there is a critical point of $\psi(z)$ (an image of a point satisfying $\varphi'(z) = 0$) whose images admit O as the only limit point.

([1]) All we say is to be applied to

$\zeta = \varphi(\zeta)$, $\qquad \varphi'(\zeta) = e^{2\pi i \frac{p}{q}}$ $\qquad$ (*p* and *q* are integer numbers)

112. Thus, our method proves that *if any critical point of* $\psi(z)$ *does not admit the origin as the limit point of its images*, then *the origin cannot belong to* E′.

In fact, since (C) has been chosen, in respect of this hypothesis, to be as small as it does not include the image of any point of (C), then all ψ_i are holomorphic in (C); moreover there are both *upper* and *lower bound*s for $|\ \psi_i(z)\ |$, while z describes any determined circle inside (C); these limitations are *independent from the index i* [*read* KLEIN et FRICKE, 2nd volume, p. 500 and 506, for the upper bound of $|\ \psi_i(z)\ |$].

The deduction is that O is in the interior of a simply connected region A, consisting of one piece and inside all (C_{-i}); A is biunivocally mapped to itself by $z_1 = \varphi(z)$ and by the branch of $z_{-1} = \varphi(z)$ vanishing at the origin.
A could be conformally mapped by $z = f(Z)$ in the interior of a circle, centered at O.

So $Z_1 = \Phi(Z)$, the transformed mapping between Z_1 and Z, is just a rotation $Z_1 = Ze^{i\omega}$ (1), that preserves any circle centered at O.
Then $z_1 = \varphi(z)$ preserves an infinity of small analytic curves crossing A and surrounding O (like the *centres* of the differential equations of the first order).

In the interior of one of that curves, all $\varphi_i(z)$ are bounded since all $z_i = \varphi_i(z)$ map biunivocally the area inside the curve to itself.

The $\varphi_i(z)$ generates a *normal* sequence in the interior of the area A and O does not belong to E′.
It contradicts the already discovered result stating that *any infinite sequence*, *extracted from the sequence of* $\varphi_i(z)$, *is not normal in* O.

Therefore if O belongs to E′, then O shall be the limit point of the images of a critical point of $\psi(z)$.

113. *Some considerations about the points* (2)

$$\zeta = \varphi(\zeta), \qquad \varphi'(\zeta) = e^{i\theta} .$$

when θ is *incommensurable with* 2π (it is always supposed $\zeta = 0$).

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(1) $e^{i\omega}$ is the value of $\varphi'(\zeta)$ at the point $\zeta = \varphi(\zeta)$, and, in our hypotheses, $\omega = 2\pi\frac{p}{q}$, (p, q integers). If one gets $\varphi'(\zeta) = 1$, then simply $e^{i\omega} = 1$ is retrieved, where $Z_1 = Z$: it is impossible, due to $z_1 \neq z$ around O. For $\omega = 2\pi\frac{p}{q}$, a conformal map is impossible with a rational $\varphi(z)$, since $z_q = \varphi_q(z)$ would become, for this map, $Z_q = Z(e^{i\omega q} = 1)$; this implies $\varphi_q(z) \equiv z$ around O. Evidently, it is impossible.
(2) What we are going to deal with is applied to periodic cycles, satisfying $\zeta = \varphi_p(\zeta)$, $\varphi'_p(\zeta) = e^{i\theta}$, when θ is incommensurable with 2π.

The simultaneous application of the theorem of normal sequence with the Kœbe's theorem, previously mentioned, allows us to reduce the number of possibilities at only two cases.
Let us surround, as we did before, the point $\zeta = 0$ with a small curve C (a circle), wherein we take the preimages coming from the branch of $\psi(z)$, the inverse function of $\varphi(z)$ and vanishing at the origin:

$$\varphi(z) = ze^{i\theta} + \dots$$
$$\psi(z) = ze^{-i\theta} + \dots$$

It is been shown that, if none of the areas (C_{-i}), the preimages of (C), includes a critical point of the considered branch of $\psi(z)$, then O is the centre of a region A, crossed by analytical curves, generated by $z_1 = \varphi(z)$ like it happens by $z_{-1} = \psi(z)$.

Around O, an holomorphic function

$$Z = f(z)$$

may be found so that, among Z and Z_1, the same function becomes

$$Z_1 = Ze^{i\theta} ;$$

$f(z)$ satisfies the functional equation

$$f[\varphi(z)] = e^{i\theta}f(z) ,$$

which is called the Schrœder's equation.

This equation *has an holomorphic solution $f(z)$* in O, if O is not a limit point of the images of the critical points of the branch $\psi(z)$ vanishing at the origin.

Inversely, if the equation has an holomorphic solution in O, then every point z, near to O, has its images on the analytic curve coming out from $| Z | =$ constant by the conformal map $Z = f(z)$, $[f(0) = 0]$; on that curve, its images are as everywhere dense as the image points of Z (generated by $Z_1 = Ze^{i\theta}$) are dense on $| Z | =$ constant.

Therefore $z = 0$ *cannot be the limit point for the images of any point in the plane*.

114. Therefore the following assertions are equivalent:

1. O is a center;
2. O is not the limit point of the images of any critical point of $\psi(z)$.

It is possible to have some holomorphic functions $f(z)$ in O, so that O is a center for these functions; it is easy to see, if we start from

$$Z_1 = Ze^{i\theta} ,$$

so that $f(z)$ is both holomorphic and vanishing in O [$f(0) = 0$], then z_1 is retrieved from z by the following equation

$$f(z_1) = f(z)e^{i\theta} ;$$

this last equation admits an holomorphic solution $z_1 = \varphi(z)$, which vanish in O and it can be developed by

$$z_1 = \varphi(z) = ze^{i\theta} + \dots .$$

Here, the curves come out for $| f(z) | =$ constant.
In the example of the equation $Z = 2z - z^2$, it is retrieved:

$$2z_1 - z_1^2 = e^{i\theta}(2z - z^2) \qquad \text{and} \qquad z_1 = 1 - \sqrt{1-(2z-z^2)\, e^{i\theta}}$$

$$z_1 = \varphi(z) = ze^{i\theta} - \frac{z^2}{2}(e^{i\theta} - e^{i2\theta}) + \dots \quad (^1)$$

For this transformation, the origin is the centre and the preserved curves are the Cassini's ovals with the two focuses, located in 0 and 2

$$| 2z - z^2 | = K ,$$

surrounding the origin ($K < 1$); all that ovals are inside the petal of the lemniscate L

$$|2z - z^2| = 1,$$

surrounding the origin.

115. Therefore, given a rational fraction $\varphi(z)$, the problem is to understand if $\zeta = 0$ is a centre point, that's to say:

1. If the equation $f[\varphi(z)] = e^{i\theta}f(z)$ has an holomorphic solution (vanishing in O) around the origin.

2. If the same equation above has not an holomorphic solution, then one is assured that $\zeta = 0$ is the limit point for the images of a critical point of the branch $\psi(z)$ vanishing in O.

All cases may be reduced to these two hypotheses.

We know that, since $\varphi(z)$ is rational, the second hypothesis may happen only if θ is incommensurable to 2π, as we discovered in the previous section; we saw the impossibility of the first hypothesis stating that $\zeta = 0$ *is*, when θ is commensurable to 2π, *a point of* E′. It comes out since $\zeta = 0$ is the limit of the images of a critical point of the branch $\psi(z)$ vanishing in O.

(1) This sequence is entire in z and it converges in the circle, centered at O and tangent internally in the petal of the lemniscata L, $|2z - z^2| = 1$, surrounding the origin.

We may say that, whenever the first hypotheses is verified, the interior area (Γ) of a curve $| f(z) |$ = constant (when the constant is small enough), is mapped biunivocally to itself by $z_1 = \varphi(z)$ and by its inverse $z_{-1} = \psi(z)$; it is assured that the sequence of $\varphi_i(z)$ is bounded in (Γ) since all $z_i = \varphi_i(z)$ map (Γ) to itself: therefore the sequence of φ_i is *normal* in Γ.
Then O does not belong to E′.
So, the first hypothesis can be proven only if $\zeta = 0$ does not belong to E′.

In all the cases where $\zeta = 0$ belongs to E′, it may be affirmed that only the second hypothesis is proven and so the Schrœder's equation has not any holomorphic solution vanishing in O.

116. Let us show effectively that if $\zeta = 0$ does not belong to E′, that's to say if the $\varphi_i(z)$ generate a *normal* sequence ([1]) in the area R (consisting of one piece and bounded by E′) including the considered point $\zeta = 0$, then *the first hypothesis is always verified*.

In fact, let us surround $\zeta = 0$ with a small interior area D of R, wherein the sequence $\varphi_i(z)$ is normal. Then both the sequence of $\varphi'_i(z)$ and of $\frac{1}{\varphi'_i(z)}$ are normal too.

In O it is retrieved $| \varphi'_i(z) | = 1$, whatever i is. These $| \varphi'_i(z) |$, bounded in O, are bounded in D too like $\left|\frac{1}{\varphi'_i(z)}\right|$. Therefore, if D is small enough D, then it is retrieved

$$\frac{1}{K} < |\varphi'_i(z)| < K \qquad (K > 1)$$

whatever i is. Now the upper bound of $| \varphi_i(z) |$ may be fixed in D.
The $\varphi_i(z)$ (vanishing in O) have all their derivatives with module = 1 in O; any extracted sequence from $\varphi_i(z)$ cannot converge uniformly to a constant value in D, since it is necessary that the values in D of the derivatives of the functions (in this sequence) converge to zero.

Therefore a number N can be found so that, whatever i is and whatever the complex number a is, none of φ_i do not assume the value a in D more than N times.

The sequence of $\frac{\varphi_i}{z}$ is *normal* in D like the sequence of φ_i, and it is bounded too. If

$$z_1 = \varphi(z) = ze^{i\theta} + \ldots$$

then

$$z_k = \varphi_k(z) = ze^{k i \theta} + \ldots \qquad :$$

([1]) It may be always supposed that the infinity is a point of E, so that in R there is no pole for the $\varphi_i(z)$ and for their inverse functions, since those poles are preimages and all the images of the infinity lie only in E′: that's to say they are not in the interior of R.

therefore, for z = 0, all the $\frac{\varphi_k}{z}$ are holomorphic and the point 1 is a limit point for the values of these functions in O.
Then let us choose a sequence with index n_1, n_2, … so that the values $e^{n_1 i\theta}, e^{n_2 i\theta}$, … converge to 1.

In O, the sequence of $\Phi_k(z) = \frac{\varphi_k}{z}$ converges to 1.

Therefore, from the sequence of n_i, a sequence of N_1, N_2, … may be extracted so that the sequence of $\Phi_{N1}(z)$, ..., $\Phi_{Ni}(z)$ converges uniformly in D to both the constant 1 and to a function $\Phi(z)$, holomorphic in O, assuming the value 1 in O.
In the second hypothesis, since $\Phi(O) = 1$, all functions $\Phi_{Ni}(z)$ start, from a certain rank, to assume the value 1 in a small circle, centered at O.

Therefore, all equations $\Phi_{Ni}(z) = 1$ and $\varphi_{Ni}(z) = z$ have, beginning from a certain rank, solutions in the whole circle which is centered at O.

Then O would be a point of E′: *but it is absurd*.

Therefore, the only hypothesis is that the sequence of $\Phi_{Ni}(z)$ converges to the constant 1 in D; that's to say, the sequence of $\varphi_{Ni}(z)$ converges *uniformly to z* in D. (One sees here the centre in z = 0.)
In this case, a circle may be traced around O, wherein no function $\varphi_i(z)$ assumes the same value in two distinct points.

In fact, an opposite behaviour would mean to admit:

1. the existence of an infinite sequence, of radius ρ_i converging to zero;
2. the existence of pairs of distinct points z_i, z'_i, respectively interior in the circles, centered at O and with radius ρ_i ;

as the indexes n_i increase indefinitely, then

$$\varphi_{ni}(z_i) = \varphi_{ni}(z'_i)$$

From the sequence of φ_{ni}, a sub-sequence φ_{Ni} may be extracted, so that φ_{Ni} converges to a non constant function $f(z)$, since all $\varphi'_{ni}(z)$ are =1 in O; finally, $f(z)$ vanishes in O and its first order derived has module = 1 at the origin.

Therefore $f(z)$ maps conformally a small enough circle, centered at O, *on a nowhere-overlapping-area*, that's to say $f(z)$ assumes two distinct values in two arbitrary distinct points in a small enough circle, centered at O (i.e.: a *one-to-one mapping*); if i is big enough, then φ_{Ni} differs as little as desired from $f(z)$; two distinct points, near to O, cannot retrieve the same value, as it is supposed in the hypothesis.

The contradiction shows that one may find a small circle γ, surrounding O, wherein none of $\varphi_i(z)$ assumes the same value in two distinct points.

Since $\varphi_i(z)$ are holomorphic in γ so that their derivatives are = 1 in module at O and since $\varphi_i(z)$ enjoys the just shown property, then the Kœbe's theorem can be applied.

The images of the area (γ) are simply connected areas, but not overlapping anywhere and whose contours are at a finite distance (bigger than a fixed limit) from the origin.

The existence of an area A is deduced: A is inside all (γ_i); it is simply connected and it surrounds O. A is biunivocally mapped to itself by $z_1 = \varphi(z)$ and by the branch of $\psi(z)$ that vanishes in O.
The conformal map of A on a circle, centered at O, preserves the origin and it proves again that the origin is a *centre*. O could be surrounded by a family of analytical curves, biunivocally preserved by $z_1 = \varphi(z)$.

The images of a point z (sufficiently near to O) are everywhere dense on the analytical curve intersecting the point z [in particular, there is a sequence with index n_1, n_2, … so that $\varphi_{ni}(z)$ converges uniformly to z in A]. The Schrœder's equation

$$F[\varphi(z)] = e^{i\theta}F(z)$$

vanishes at the origin and it is holomorphic in O.

According with the properties of the domains R, bounded by E′ (1), F(z) is holomorphic in every domain R, which is traversed by some analytic curves preserved by $z_1 = \varphi(z)$.
$\zeta = 0$ is not the limit for the images of any point of the plane.

117. So, when θ is incommensurable to 2π, we will have to choose one of these two following hypotheses :

1. $\zeta = 0$ and it does not belong to E′. Then ζ is a *centre*.

Every area R, consisting of one piece, surrounds the centre ζ ; R is bounded by E′ and it is traversed by analytical curves which are preserved by $z_1 = \varphi(z)$.

The domain R does not include any critical point for the branch $\psi(z)$, inverse of $\varphi(z)$ and vanishing at the origin (2).
It may be easily noticed that none of the preimages of the domain R can include a critical point of the algebraic function $\psi(z)$ while $\zeta = 0$ keeps on being a centre.

(1) Every sequence, extracted from $\varphi_{ni}(z)$ and converging uniformly in a part of R, converges uniformly in all the interior of R.
(2) Neither for the iterations $\psi_i(z)$ of $\psi(z)$, the inverse branches of $\varphi_i(z)$ vanishing in O.

The Schrœder's equation has an holomorphic solution F(z) in the entire R.
For that solution, the boundary points of R (points of E′) are singular essential points and the fundamental theorem, demonstrated in the 1st Part with regard to the points of E′ and to the only pair of exceptional values, is alike the Picard's theorem which is applied to the solution F(z) of the Schrœder's equation.

All of above follow both the theorem and the supposition that $\varphi_i(z)$ is *normal* in $\zeta = 0$.

2. $\zeta = 0$ and it belongs to E′.
Then, it is the limit of its images of a critical point of the branch $\psi(z)$ vanishing in $\zeta = 0$. The Schrœder's equation has no holomorphic solution at the origin. $\zeta = 0$ is not a centre.

I won't spend more words ([1]) except for the case when both hypotheses exclude mutually. In that case I would need more time and more knowledge for studying the question thoroughly, but I have not it now.

I intend to come back in future because it may happen that the application of the Schrœder's equation could help me to solve this question.

In fact if one searches for the Taylor's coefficients satisfying

$$F[\varphi(z)] = e^{i\theta}F(z)$$

where

$$\varphi(z) = ze^{i\theta} + \dots ,$$

then these coefficients may be determined progressively when θ is incommensurable to 2π (the formal calculus is generally impossible if θ is commensurable to 2π).

I tried to show the convergence of this development by the method of majorities but all majorities I obtained are divergent!

Anyway, I showed that, when $\varphi(z)$ is holomorphic around the origin, then the equation may have an holomorphic solution in some cases, so that the existence of centres is proven.
[Given the example where $F(z) = 2z - z^2$, since $\varphi(z)$ is an algebraic function in z.]

It still remains to know if a rational $\varphi(z)$ may have centres: it would be interesting to prove it or not by building related examples.

([1]) *Read* Note ([1]), § 103.

translated by Alessandro Rosa - *released* on May 15th, 2001

ADDITIONAL NOTE

118. I have realized, before printing this Memoir, that it is not properly right to affirm that the equation $z = \varphi(z) = \frac{P(z)}{Q(z)}$ has always a solution ζ satisfying | φ′(*z*) | > 1.

This is right when this equation *has no double root* ; but it may occur that some double roots satisfy both $\zeta = \varphi(\zeta)$ and $\varphi'(\zeta) = 1$ so that we would be obliged to conclude that:

1. There is at least one root of $\zeta = \varphi(\zeta)$, satisfying $|\varphi'(\zeta)| > 1$;
2. Or there is at least one root of $\zeta = \varphi(\zeta)$, satisfying $|\varphi'(\zeta)| = 1$.

In the first case, all we said in the 1st part of this Memoir (the sets E and E′, their properties ...) is absolutely correct. All properties *are verified in the second case* but they need to be introduced differently. As we read in the section § 104 of this Memoir: the functions $\varphi'_n(z)$ *do not generate any normal family* in the point ζ = φ(ζ) satisfying φ′(ζ) = 1. Therefore, in an arbitrarily small circle centered at ζ, every determined value (except two of them) is retrieved by a certain function $\varphi_n(z)$. These exceptional values, as we read in the 1st Part, come out only if $z_1 = \varphi(z)$ is reduced to $z_1 = z^{\pm m}$ or to a polynomial by an homographic transformation. But, surely, ζ *is not an exceptional value, neither it is one of the preimages*. (In none of its preimages the φ_n are *normal*.)

Therefore, ζ *is necessarily a limit point of its own preimages*. If ζ is surrounded by a sufficiently small circle Γ, then we may find a sequence $\zeta_{-n1}, \zeta_{-n2}, \ldots$ of preimages of ζ, so that they are both interior in Γ and converging to ζ.
There′s something more.
The *local study* of the neighbourhoods of ζ, made in §105 of the 4th Part, proves that, if ζ is surrounded by a sufficiently small circle Γ, then two domains Δ_1 and Δ'_1 (1) can be found, so that these two domains are interior in Γ and share ζ as a boundary point.
Δ_1 and Δ'_1 share one or more common areas.

(1) For example, if $\zeta = 0$ and if $z_1 = \varphi(z) = z + a_2 z^2 + \ldots$, $a_2 \neq 0$, one may take, on the condition for Γ to be sufficiently small, the area Δ_1, interior in two proper circles so that Δ_1 intersects ζ and is tangent to Γ; then one may take the area Δ'_1, the symmetric of Δ_1 in relation to ζ.

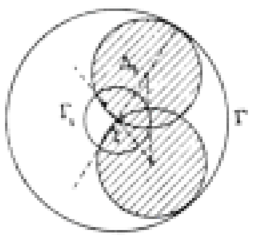

Fig. 31

Then it is visible that $\Delta_1 + \Delta'_1$ includes a sufficiently small circle Γ_1 with centre in ζ.

On one hand, each point of Δ_1 has *all its images in* Γ (*and in* Δ_1) *converging only to* ζ.
On the other hand, each point of Δ'_1 *has all its preimages* [by the branch of the function $\psi(z)$, inverse of $\varphi(z)$ where $\zeta = \psi(\zeta)$] *in the interior of* Γ (*and of* Δ'_1) *converging only to* ζ ([1]). Let us add that in the domain consisting of the union of Δ_1 with Δ'_1, one may find a sufficiently small circle Γ'_1, centered at ζ. Therefore, every point of Γ'_1 belongs to Δ_1 or to Δ'_1. Let us examine a sufficiently small circle Γ that leaves a certain preimage ζ_{-p} of ζ at its exterior so that the previous construction is possible.
In Γ an infinite sequence of preimages of ζ may be found:

$$\zeta_{-n1}, \zeta_{-n2}, \dots, \zeta_{-np}, \dots \qquad (p < n_1 < n_2 < \dots)$$

so that it converges to ζ.
By the previous construction, starting from a certain index n_i, the preimages $\zeta_{-n}, \zeta_{-n_{i+1}}, \dots$ are in the interior of a circle Γ_1. But ζ_{-n_i}, since it has a image ζ_{-p} (outside Γ), does not belong to Δ_1; therefore it belongs to Δ'_1, that's to say *all its preimages*, determined by the branch of $\psi(z)$ (so that ζ is a fixed point), *are in the interior of* Γ *and they converge to* ζ.
ζ_{-n} may be surrounded by a sufficiently small circle γ so that *all the preimages areas of the circle*, *by the considered branch of* $\psi(z)$, are simple connected plane areas converging to ζ.
Therefore, in every circle, centered at ζ and being as small as we want, one may find a plane area with one only contour γ_{-N}, preimage of the area of the previous circle γ.
It means, whatever small a plane area D (surrounding ζ) is, that the entire circle γ is in the interior of a layer of a certain iteration D_N of the area D; or it would mean that *on a certain layer of* D_N, *all points*, *projecting in the interior of* γ *or on* γ, *are interior points of* D_N. Now, a sufficiently small circle γ may be chosen so that the iteration γ_{n_i}, including ζ, is a small simple area surrounding ζ and chosen properly for the area D.
As it is been read in the 1st Part (3rd corollary of the fundamental theorem) and according to the section 5 of *Preliminaries*, γ always keeps at least one root inside of the equation $z = \varphi_{N+n_i}(z)$ satisfying $|\varphi'_{N+n_i}(z)| > 1$, since the iteration of rank N+n_i of γ is D_N: D is the same as the iteration of rank n_i of γ.
Therefore it is necessary to prove that *any point* ζ, *satisfying both* $\zeta = \varphi(\zeta)$ and $\varphi'(\zeta) = 1$, *is the limit of the roots-points of the equation* $z = \varphi_n(z)$ *satisfying* $|\varphi'_n(z)| > 1$.
The existence of E has been proved: it includes a countable infinity of points; moreover, it is correct to treat its derived set, E′.
This explanation has been pursued in this Memoir.

([1]) In Δ'_1, the branch of $\psi(z)$ is equal to ζ in ζ; and all iterations of that branch are holomorphic.