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SECOND PART

On the study of the uniform convergence to a limit point
and of the periodic convergence to a periodic cycle.

27. The theorem, enunciated at the end of the previous chapter, has shown us that if one knows the behaviour of the sequence

$$\varphi(z)\ ,\ \varphi_2(z)\ ,\ \ldots,\ \varphi_n(z)\ ,\ \ldots$$

in an arbitrarily small area, consisting of one piece and inside a region R, wherein the interior points do not belong to E′ but only all the boundary points belong to E′, then one knows that the sequence has the same behaviour for all other points of the region R. We have remarked that the local study of the iteration gives some informations about the behaviour of the sequence of $\varphi_i(z)$ in the neighbourhood:

1) … about each *limit point with uniform convergence*: of each root of $z = \varphi(z)$ satisfying $|\ \varphi'(z)\ | < 1$;
2) … about each point belonging to a *periodic cycle*: of each point ζ which is a root of $z = \varphi_n(z)$ (1) satisfying $|\ \varphi_n'(z)\ | < 1$, and about the other n - 1 points

$$\zeta_1 = \varphi(\zeta)\ ,\ \ \zeta_2 = \varphi(\zeta_1),\ \ldots,\ \zeta_{n-1} = \varphi(\zeta_{n-2}), \qquad\qquad [\zeta = \varphi(\zeta_{n-1})]$$

generating a periodic cycle with ζ.

Now we want to examine carefully the study of the characters of a region R including in its interior either a limit point with uniform convergence, either a periodic cycle.

28. *The immediate domain of convergence to a limit point with uniform convergence*. - Let us point our attention, just for a moment, to the case of the limit point with uniform convergence $\zeta = \varphi(\zeta)$ satisfying $|\ \varphi'(\zeta)\ | < 1$ and we will see further that our conclusions are still valid for periodic cycles.
Thanks to a preliminary homographic mapping, one may always suppose that the considered limit point is the origin.
One may suppose also that the point at infinity (2) is a point which does not admit zero as the limit point of its images.
For example, a point $z = \varphi(z)$, different from the origin, may be sent to infinity

$$z_1 = \varphi(z) = a_1 z + a_2 z_2 + \ldots\ ,\ |\ a_1\ | < 1$$

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(1) One suppose, according with Kœnigs, that ζ is not a root of the equation $z = \varphi_p(z)$, where $p < n$. One says, whatever it is, that ζ belongs to the index n, where there is a *primitive* root of $z = \varphi_n(z)$, by analogy with binomial equations.
(2) This allows us to conclude that, in a small region surrounding zero, all functions $\varphi_n(z)$ $(n = 1,2,\ldots,\infty)$ have a finite branch of their inverse functions $\psi(z)$.

If $a_1 \neq 0$, then the origin has *one and only one preimage confusing with the point itself.*
If $a_1 = 0$, then the origin has at least two preimages confusing with the point itself.
Let ρ be a small enough positive number; if the point z describes the circle C, centered at O and with a radius ρ, then:

1. If $a_1 \neq 0$, the only preimage z_{-1} of z, as it becomes = 0 for $z = 0$, describes around the origin an analytic curve C_{-1}, enclosing C in its interior. As z describes the bounded area by C, including O, z_{-1}, since it is an analytic function of z in C, describes the area, bounded by C_{-1} and including O.

2. If $a_1 = 0$, $a_2 = 0$, …, $a_{p-1} = 0$, ($a_p \neq 0$), then the origin has p and only p preimages confusing with themselves. As z approaches to 0, then the p preimages approach to 0 and they generate one circular system; while in p times z describes C continuously in the positive direction, each one of the p previous preimage describes once an analytical closed curve C_{-1}, surrounding C; when z describes the simply connected area S, consisting of p superimposed *layers* (each one is the same as the surface of the circle C) which branch out each other in the point O, the area S is bounded by the circle C and it is described p times and it includes O in its interior; then each one of the p preimages of z (we have previously spoken about them) describes only once the interior of the previous curve C_{-1}; each interior point z of the circle C has only these p interior preimages of C_{-1}; each interior point of C_{-1} has all its images in C.
Moreover, we know that, if z is any interior point of C, satisfying $| z | < \rho$, then one gets

$$| z_1 | < H\rho \qquad (H < 1)$$

and following

$$| z_i | < H^i\rho$$

The sequence $z_1, z_2, \ldots, z_n$ has only O as its limit point, whatever z is in C.

We will study the properties of the *immediate domain of convergence of the point* O, which is the set of the points in the plane whose images admit O as the only limit point: these points can be linked to O by a simple path, whose points enjoy the previous property (to admit 0 for its only limit point of their images). This domain is the region R of the plane, consisting of one piece, including O and bounded by the perfect set E′: every interior point of this region admits O as the only limit point of its images; it can be linked to O by a simple path whose points are inside R so that they enjoy the previous property. A boundary point of the region R belongs to E′; its images are in E′: they do not admit O as their limit point since O does not belong to E′ and, as we saw before, it is surrounded by a circle C where no points of E′ can lie inside. *On the boundary of* R *the points of* E *are everywhere dense*, that's to say, in the entire area of the plane, including some boundary points of R, there are some roots points of $z = \varphi_n(z)$, for some proper values of n and satisfying $| \varphi_n'(z) | > 1$. Every interior point z of R has

O as limit point of its images; it follows that a certain z_i is inside C, such as all images of z. Therefore R can be defined as *the set of points consisting of one piece with* O, *where a image point fall into* C; *we will generate* R *by a process we will use later*.

While z describes the interior area of C, let us look for the area described by one or by some preimages vanishing with z: this is the area bounded by C_{-1}; for shortening, it is called the *preimage area of* C. One has seen that C_{-1} includes C. When z_{-1} describes the area C_{-1}, one or more preimages, vanishing with z_{-1}, describe an area C_{-2}, including C_{-1} in its interior, since C_{-1} includes C and since C_{-1} is the preimage of C.
C_{-2} will be called the *preimage* of rank 2 of C.
It is important to remark it: if $a_1 = 0$, then the point O is a critical point for the algebraic function $\psi(z)$, inverse of $\varphi(z)$; so there is a point vanishing $\varphi'(z)$; then the immediate domain R of convergence to O includes *in its interior* a critical point of the inverse function of $\varphi(z)$, the image of a point where $\varphi'(z)$ vanishes.

Let us suppose on the contrary that $a_1 \neq 0$; starting from the previously examined small circle C, let us define ly the area C_{-1}, preimage of the area C and described by z_{-1} (the only preimage point of z) which is = 0 for $z = 0$ as z describes the area C.
In the same way C_{-2} is the preimage of C_{-1} and so on.
This process keeps on going without any difficulties so that the areas C_{-1}, C_{-2}, …, C_{-i} do not include the critical point of the branch of the algebraic function $\psi(z)$, inverse of $\varphi(z)$ which is equal to 0 as $z = 0$; all the determined areas will have one only layer; they will be simply connected so that each one of them is inside its preimage.

If this process goes on indefinitely so that no critical point of $\psi(z)$ is inside one of the areas C_{-i}, then R will be the limit area for C_{-i}, as i grows indefinitely ([1]).
I will show that this hypothesis is impossible.

29. In fact R will be a simple connected area ([2]) which is easy to map conformally ([3]) in the interior area of a circle $|Z| < 1$ (since *all the points* of E′ are outside R or they are boundary points of R; this conformal mapping $z = f(Z)$ maps any interior point z of R to a point Z with module $|Z| < 1$: this mapping is biunivocal and analytic for all interior points).

([1]) It is easy, in fact, to see that if the area C_{-i} does not include any critical point of $\psi(z)$, each one of the K preimages of z is an analytic function of z in C_{-i} ; each one describes an area with one only layer and bounded by an simple analytic boundary, if z describe the area C_{-i} and the *k areas*, *preimage to* C_{-i} do not share (two by two) any interior points, neither boundary points. Whenever it occurs, we will be interested only in the preimage area of C_{-i}, including the origin.
([2]) The relation $z_1 = \varphi(z)$ maps every point interior point z of R to a point z_1, still inside R, and, inversely, every interior point z_1 of R is mapped to one and only point z inside R.
([3]) For example, one may affirm that, calling $f_n(Z)$ the function applying on $|Z| < 1$ the area C_{-n} [$f_n(0) = 0$, $f'_n(0)$ real], $f(Z)$ is the limit of the sequence $f_1(Z)$, …, $f_n(Z)$, …, all uniformly defined in all the interior area of $|Z| < 1$.

We will make a correspondence between the origins of both planes z and Z [$f(0) = 0$] like the positive directions of the real axes of those planes [$f'(0)$ real].
Two points z and z_1, related by $z_1 = \varphi(z)$, are mapped to two other points Z and Z_1 (related within themselves) by the conformal map.
Whatever Z is in $| Z | < 1$, it is mapped to a point z for $z = f(Z)$; after z_1 for $z_1 = \varphi(z)$ is mapped to Z_1 for $z_1 = \varphi(Z_1)$.
So, each point Z in $| Z | < 1$ is mapped to a point Z_1, in $| Z_1 | < 1$ depending analytically on Z in $| Z | < 1$.
Inversely, a point Z_1 in $| Z_1 | < 1$ is mapped to one and only one point Z in $| Z | < 1$ depending analytically from Z_1. The holomorphic relation $Z_1 = \Phi(Z)$ in $| Z | < 1$ [such as its inverse $Z = \psi(Z_1)$ in $| Z | < 1$] is the transformed function of $z_1 = \varphi(z)$ with the substitution $z = f(Z)$.
The relation $Z_1 = \Phi(Z)$ maps the interior of the circle $| Z | < 1$ into itself and *it preserves the origin* (if $Z = 0$, $Z_1 = 0$); it is the same as the inverse function $Z = \psi(Z_1)$.
The mapping is biunivocal and analytic.
There is a well known result asserting that this mapping is just a rotation around the origin ([1]) so that one necessarily gets

$$Z_1 = Ze^{i\theta};$$

That's to say, in O one gets

$$\left|\frac{dZ_1}{dZ}\right|_0 = 1 \ .$$

But there is a result in contradiction with our hypothesis.
In fact, one gets

$$\left(\frac{dZ_1}{dZ}\right)_0 = \left(\frac{dZ_1}{dz_1}\right)_0 \left(\frac{dz_1}{dz}\right)_0 \left(\frac{dz}{dZ}\right)_0$$

$$\left(\frac{dz_1}{dZ_1}\right)_0 = f'(0) = \left(\frac{dz}{dZ}\right)_0 \quad \text{and} \quad \left(\frac{dz_1}{dz}\right)_0 = \varphi'(0) = a_1$$

as the hypothesis.
Therefore

$$\left(\frac{dZ_1}{dZ}\right)_0 = a_1$$

and it is supposed that $| a_1 | < 1$.

Therefore it is impossible to suppose, without any contradiction, that the region R does not include any critical point of $\psi(z)$ in its interior or any preimage of the critical point where $\varphi'(z)$ vanishes.

([1]) Read POINCARÉ, *Sur le groupes des equations lineares*, § 7. Fundamental Lemma.

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While making, as enunciated before, the preimages C_{-i} of the area C, there is always an area C_{-n} including a critical point of the inverse function of $\varphi(z)$ ([1]).

30. After considering that the first derivative vanishes in the point $\zeta = \varphi(\zeta)$ or it has its module < 1, we state the following fundamental theorem:

THEOREM. – *The immediate domain of convergence* R *of a limit point with uniform convergence* ζ ([2]) *always includes in its interior at least one critical point for the inverse algebraic function of* $\varphi(z)$.
This point is critical for the branch of that function becoming equal to ζ *as* $z = \zeta$; *it is the image of an interior point of* R, *satisfying* $\varphi'(z) = 0$. ([3])

The demonstration of that theorem, resulting from the previous analysis and based onto the conformal map, is all obvious when we try to define R, as we already did it before with the limit of preimages of the area C.

31. Here is a second demonstration which is the same as we did at the beginning but it does not use conformal mappings.
Let us take again

$$z_1 = \varphi(z) = a_1 z + \dots ,$$

whereby the branch vanishing at the origin, one gets

$$z_{-1} = \frac{1}{a_1} z + \dots .$$

If the process above, for generating the preimages C_{-1}, C_{-2}, … of the area C, indefinitely goes on without intersecting the interior of an area C_{-i} and keeping a critical point of the considered branch of $\psi(z)$, then it means that the function $\psi_n(z)$, inverse of $\varphi_n(z)$ has, whatever n is, a branch vanishing at the origin and analytic in C ([4]).

([1]) For defining the preimages of the area C_{-n}, including a critical point, we may imagine that z describes the area C_{-n}; then, in general, there are two of its preimages, becoming equal if z assumes a critical value; since the considered point is critical for the branch $\psi(z)$, vanishing at the origin, then one of those preimages is given by the branch $\psi(z)$; when z describes the area C_{-n} once and only once, then the set of the preimages (all of them tend to assume the same value), since z becomes a critical point, describes an area $C_{-(n+1)}$, including C in its interior, with one layer and bounded by only one analytic curve.
The area $C_{-(n+1)}$ is the preimage C_{-n}, which we needed to consider into R.
([2]) There is nothing special when I say that for the case where ζ, limit point with uniform convergence, is the point at infinity of the plane; by an homographic mapping, one may take the point back to a finite distance: but it is only a convenience of language. It occurs whenever $\varphi(z)$ is a polynomial; then the infinity will be a point with uniform convergence and so a critical point of the inverse function $\psi(z)$.
([3]) One realizes that the previous theorem is still verified for the iteration of an *transcendent entire* function in the neighbourhood of a roots of $z = \varphi(z)$, retrieving $|\varphi'(z)| < 1$.
([4]) Since we took the precaution of sending to infinity a point which does not admit zero as the limit point of its image points, it hasn't in C any pole of ψ_n(z) (whatever is n) as we choose an enough small C.

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THEOREM. – *If a function* ω = F(*z*), *holomorphic in a circle* C, *centered at* O *and with radius* r, *satisfying* F(0) = 0 *and* F′(0) = 1, *never assumes the same value in two distinct points*, *then*, as *z describes the circle* C *with radius* ρ, ω *describes a line* L *whose the minimum distance from the origin is*

$$d > \frac{-1+\sqrt{2}}{4}\rho \ , \ (^1)$$

The branch of $\psi_n(z)$, inverse of $\varphi_n(z)$ and vanishing at the origin, is analytical in C and on C (if C is small enough); its Taylor development in C is

$$z_{-n} = \varphi_n(z) = \left(\frac{1}{a_1}\right)^n z + \dots \quad .$$

Since this branch, which is the inverse of a rational fraction, cannot assume the same value in two different points *z* and *z′*, then the previous theorem cannot be applied; moreover, $\frac{z_{-n}}{\left(\frac{1}{a_1}\right)^n}$ vanishes at the origin and it has its derived is equal to 1 at the origin;

as *z* describes C, z_{-n} describes C_{-n}, which is the preimage of C of *n* rank. Therefore, it follows that the shortest distance, from C_{-n} to the origin, will be estimated as

$$d_n > \frac{\sqrt{2}-1}{4}\rho\left|\frac{1}{a_1}\right|^n ,$$

since ρ is the radius of C.

This means that, for *n* big enough, the area C_{-n} includes a circle, centered at O and with a radius as big as one wants (since $\left|\frac{1}{a_1}\right|^n$ is as big as one wants, for an index *n* enough big, due to $|a_1| < 1$).

But one shall choose a radius, *equal to the distance of a critical point to the origin*, *for the branch of* ψ(*z*), *inverse of* φ(z) *and vanishing at the origin*; it may happen that the area C_{-n} includes a critical point of the considered branch in its interior, in opposite to the hypothesis in the beginning.

32. *Remark.* – After proving the previous theorem, it is necessary here to find R, starting from C.

Let us imagine to unfold on the plane *z* the Riemann surface R with K layers of the functions ψ(*z*), algebraic and inverse of φ(*z*).

Let us start from C; if it is small enough and if φ′(0) ≠ 0, then it does not include any branch point of R .

C intersects *k* superimposed areas in R: as *z* describes each one of those areas, z_{-1} describes in the analytic plane one simple area, bounded by a simple analytic curve, and there is only one area C_{-1} of those areas including C in its interior.

([1]) Read KLEIN and FRICKE, *Fonctions automorphes*, p. 500.

The other (k - 1) areas do not share any common points with C_{-1}.
After, if the area C_{-1}, like C_1, does not include branch point of R, then one can work with it like working with C: the process goes on until it gets to an area C_{-i}, bounded by a simple analytic curve C_{-i} including a branch point of R in its interior.

If one supposes that the origin belongs to an upper layer of R, then C_{-1} will be the preimage of the area intersected by C in the upper layer and the (k - 1) other areas, described by z_{-1}, will be the preimage of the k - 1 areas intersected by C in the 2^{nd}, 3^{rd}, … k^{th} layer of R.

It will be the same so that the curve C_{-i} won't enclose in its interior ([1]) any branch point of R and precisely any point branching the *upper layer* of R with a lower layer.

If the curve C_{-i} includes a point branching between two layers of R, distinct from the upper layer, then this point *won't be critical for the branch of the function* $\psi(z)$ *vanishing at the origin*: this branch will be holomorphic in the area C_{-i}; then C_{-i} intersects in the upper layer a simple area whose preimage $C_{-(i+1)}$ will be a simple area including C_{-i}.

In that case one should examine the interior layers (of the curve C_{-I})branching in a point P ; in general there are two layers.
The curve C_{-i} intersects a simple connected area Σ (mapped to itself) in those two layers; as z is describes the interior of that area Σ, there are two preimages (which become the same when z approaches to P) describing the same simple area Σ_{-1}, with one only layer and bounded by only one analytic boundary: if z describes the whole area Σ, each one of these preimages describes the whole area Σ_{-1}.

Therefore one may say that Σ_{-1} is described twice, once for each one of the two preimages of the considered z; as z describes once the area Σ, in such a way that each point, since it is inside C_{-i} is mapped to two preimages inside Σ_{-1}. But the area Σ_{-1} won't share any common points with the area inside $C_{-(i+1)}$ and Σ_{-1} cannot be considered in the generation of the immediate domain R of convergence of O.

If that area is always outside the curves C_{-i}, then Σ_{-1} is inside a certain preimage of the simple area R; it belongs, as we'll see further, to the *total* domain of convergence to O, but it won't share any common point with the *immediate* domain.

Then now one realizes the process of making R, so that each of those C_{-i} intersects the upper layer in an area, not including any point where this layer branches out with a lower layer of a critical point for the branch of $\psi(z)$ vanishing at the origin.

([1]) The interior of C_{-i} is always intended as the region of the plane, bounded by C_{-i} and including the origin.

But we know that a certain contour C_{-i} (1) includes a such point P inside. In this point the upper layer branches out in a certain number of lower layers: in general only with the immediately lower layer (2).
For simplifying, let us suppose that only the branch point P of R figures in C_{-i}; then the curve C_{-i} intersects in R a certain number of simply connected superimposed areas, whose the highest one: 1. includes the origin; 2. it is bounded by the curve C_{-i}, 3. it is crossed twice; 4. it includes two superimposed layers branching in P.

In a general manner, it is considered the area S_{-i}, consisting of one piece and intersected in R by C_{-i} and *including the origin in the highest layer*: this area will be mapped to itself many times, moreover it will include the part of the highest layer, inside C_{-i} and the interior parts of C_{-i} of all the layers consisting of one piece with the highest layer in the interior of C_{-I}: that's to say of all those layers (inside the curve C_{-i}) whose any point can be joint to the point O of the highest layer by an interior continuous path of the curve C_{-i} and traced onto R.
The area S_{-i}, determined by C_{-i}, has many layers and it would be simply connected, if it admits that C_{-i} is described many times by the contour only: this occurs for example if it includes only a branch point for the algebraic function $\psi(z)$.
But it may happen that this area is multiply connected: a very simple case arises when the curve C_{-i} encloses in its interior two branch points that join the first and second layer only inside the curve.
Then, evidently, the considered area S_{-i} will be bounded by two superimposed contours in C_{-I} : the first contour is entirely traced onto the upper layer, the second is entirely traced onto the lower layer. Therefore, whatever it is, one will consider the area S_{-i}, consisting of one piece and intersected in the Riemann surface by C_{-i}, including the origin in the upper layer ;
As z_{-i} describes this surface, then one of preimages $z_{-(i+1)}$, vanishing at the origin, describes a plane area $C_{-(i+1)}$ with only one layer and including the curve C_{-i} in its interior and the plane area bounded by C_{-i}; the plane area $C_{-(i+1)}$ is bounded by so many contours as the contours of the area consisting of one piece and intersected in R by C_{-i}.
The plane area $C_{-(i+1)}$ is the *locus* of the points that can be joint to O by a path whose points have their images in the interior of the curve C_{-i}. One may say that this area includes O and it is the preimage of the area bounded by C_{-i} and including O; as z describes the area C_{-i}, the preimage, vanishing at the origin, and all preimages, branching with it in the interior of C_{-i}, describe the new examined area.

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(1) It may be the first boundary C, if $\varphi'(0) = 0$ with $\varphi(0) = 0$.
(2) I recall that R, since it belongs to the *genus zero* and since $\varphi(z)$ is general, the upper layer n°1 is uniform with the second layer for two branch points; each one of the layers n°2,3, … , k-1 is uniform for two branch points with both the previous and the following layer; finally the lower layer n° k is uniform by two points to the layer (k - 1). (Read PICARD, *Traité d' Analyse*, t.2 , p. 416, § 16).

It may happen that this last new plane area (the preimage of the area C_{-i}) is not simply connected (I called this area $C_{-(i+1)}$). The way to apply $C_{-(i+1)}$ is the same as C_{-i}: so, one searches for the preimage $C_{-(i+2)}$ of $C_{-(i+1)}$, including O; there is the place of the points in the plane which can be joint to O by a simple path whose points have their images in the interior of the area $C_{-(i+1)}$; the determination of the area $C_{-(i+2)}$ is the result from the test of a set (in R) consisting of one piece, with the origin of the upper layer and generated by portions of the layers (of R) projecting themselves in the interior of the plane area $C_{-(i+1)}$.

This set is an area $S_{-(i+1)}$, consisting of one piece and including the origin of the upper layer, mapped to itself for many times; all points of $S_{-(i+1)}$ are projected in the interior of $C_{-(i+1)}$ and all of them can be joint to origin (on the highest layer) by a continuous path, traced onto R and projecting in the interior of the plane area $C_{-(i+1)}$.
As $z_{-(i+1)}$ describes the area $S_{-(i+1)}$, the vanishing preimage, when $z_{-(i+1)}$ coincides with the origin (lying on the upper layer), describes a plane area $C_{-(i+2)}$, consisting of one piece, including the origin O and the plane area $C_{-(i+1)}$, whose order of connection is the same as the area $S_{-(i+1)}$.
This process, valid in any case, may continue indefinitely. Each one of the plane areas C, C_{-1}, C_{-2}, …, C_{-i}, … is included in the interior of its image area.
R, *the immediate domain of convergence to* O *is the limit of the area* C_{-i} *while the index i increases indefinitely*.

Each interior point of R is also inside all C_{-j}, whose index j exceeds a given number.
In fact, each interior point of R can be joint to O by a simple interior path of R whose images admit O as the only limit point so that, starting from a certain rank, they are inside the area C; this property has ruled the generation of all regions C_{-i} since each region C_{-i} is the set of the points which can be joint to O by a path where the i^{th} image is the interior of the area C wherein all images converge only to O.
The boundary of R, as we saw before, consists univocally in points of E′, and on that boundary the points of E [$z = \varphi(z)$, $|\varphi'_n(z)| > 1$] are everywhere dense.
More often, in the reported examples, all C_{-i} will be simply connected and R too.
The set E′ will be a simple continuous line. Consider the previous example

$$\left(z_1 = \frac{z^k}{z^k + 2} \right)$$

where R has an infinite order of connection. By the transformation $z_1 = \varphi(z)$, every interior point z of R is mapped to an interior point z_1 of R.
Inversely an interior point z_1 of R is mapped at least to two interior preimages of R, according to the previous fundamental theorem.

33. *The case of the periodic cycle.* – We will work now with a group of p distinct points $\zeta, \zeta_1, \zeta_2, \ldots \zeta_{p-1}$ roots of $z = \varphi_p(z)$, satisfying $|\varphi'_p(z)| < 1$. One gets

$$\zeta_1 = \varphi(\zeta)\ ,\ \zeta_2 = \varphi(\zeta_1)\ ,\ \ldots,\ \zeta_{p-1} = \varphi(\zeta_{p-2}),\ \zeta = \varphi(\zeta_{p-1})$$
$$\varphi'_p = \varphi'(\zeta)\varphi'(\zeta_1)\ \varphi'(\zeta_2) \ldots \varphi'(\zeta_{p-1})\ .$$

We will examine the substitution $z_p = \varphi_p(z)$ in the neighbourhood of each previous point: for example in the neighbourhood of ζ. In the same way as we did before with this substitution $z_p = \varphi_p(z)$, we may define the immediate domain of convergence R to ζ by starting from a small circle surrounding ζ.
ζ will be the limit of the preimages (ζ included) of the area C by the branch of $\psi(z)$, inverse of $\varphi_p(z)$, which is equal to ζ for $z = \zeta$.
R encloses a critical point for the considered branch of $\psi_p(z)$ in its interior.

The immediate domains of convergence to $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$, of the substitution $z_p = \varphi_p(z)$, simply come up by taking the iterations of rank 1, 2, …, p - 1 of the domain R.
(The iteration of rank p of R is the same as R.)

Then the domains $R_1, R_2, \ldots, R_{p-1}$, surrounding respectively $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$ come up; each one of them consists of one piece; an arbitrary pair of them won't share any common points; $R_1, R_2, \ldots, R_{p-1}$ can be arranged circularly so that, whatever is the path inside one of them (R for example) joining an interior point of R to the point ζ, then its images are respectively inside $R, R_1, R_2, \ldots, R_{p-1}, R, R_1, \ldots$ periodically repeating.
The only limit points of this circular group of regions is the system of p points $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$ which are the only distinct images of ζ.
In the circular way, each one of $R, R_1, R_2, \ldots, R_{p-1}$ is the iteration of the previous one by the substitution

$$z_1 = \varphi(z)\ [R_1 = \varphi(R),\ R_2 = \varphi(R_1),\ \ldots,\ R = \varphi(R_{p-1})]\ .$$

This is easy to be shown.
Each one of the R_i includes at least a critical point for the branch of the inverse function of $\varphi_p(z)$ which becomes equal to ζ_i for $z = \zeta_i$. Again, the critical points of $\psi_p(z)$, inverse of $\varphi_p(z)$, are the critical points of $\psi(z)$, inverse of $\varphi(z)$ and of their images of rank 1, 2, …, (p - 1). In general $\psi(z)$ has $2(k - 1)$ critical points, which are the images of rank 1 of $2(k - 1)$ points satisfying $\varphi'(z) = 0$ [$\varphi(z)$ is of rank k], therefore $\psi_p(z)$ has $2p(k - 1)$ critical points in all.
Necessarily, it may happen that at least one critical point of $\psi(z)$ is inside a region R_i, and its p - 1 images are respectively inside R_j which follow R_i in a circular way.
So *there is at least one point where the function* $\varphi'(z)$ *vanishes and it falls in one of the p domains* $R, R_1, R_2, \ldots, R_{p-1}$, *that's to say whose images admit* $\zeta_1, \zeta_2, \ldots, \zeta_{p-1}$ *generate periodic cycle*.

In the case of the periodic cycle, E′ encloses a perfect continuous linear set, splitting the plane in many regions, such that it is impossible to move from one region to another by a simple path without crossing a point of E′.
So the regions R, R_1, R_2, …, R_{p-1} are separated by E′.
Their boundaries belong to E′ and enclose continuous lines separating each interior point of R_i from any exterior point (*read* n° 26, 3°).

34. IMPORTANT CONCLUSION. – *For any rational fraction* φ(*z*), *the number of limit points with uniform convergence or of the periodic cycle is finite.*

In fact, it is clear that, since every limit point with uniform convergence shall be the only limit point of the images of a point satisfying at least φ′(*z*) = 0 and since the *p* points of a periodic cycle shall be also the limit points of the images of at least a point satisfying φ′(*z*) = 0, the number of the limit points with uniform convergence, increased by the number of periodic cycles, cannot be bigger than the number of the points where φ′(*z*) = 0: generally this number is 2(*k* - 1) if φ(*z*) is of degree *k*; however it is not bigger than 2(*k* - 1).

I recall that the fundamental theorem which obliges at least one point, that satisfies φ′(*z*) = 0, to have some images converging uniformly to a limit point or converging periodically to the points of a periodic cycle, is still verified if φ(*z*) is a transcendental entire function.

But we are going to get out some conclusions that cannot be extended to all transcendental entire functions φ(*z*), since in general it may happen that a transcendental function admits an infinity of points where φ′(*z*) = 0.
So the doubt about periodic cycles is well solved.

Koenigs writes, in the section 34 of his Memoir, *Sur les integrals de certaines équationes fonctionelle* (*On integrals of certain functional equations*), published by *Annales de l'Ecole normale supérieure* in 1884 (page 401 of the supplement):

> « *The importance of the division of the plane in regions, according to the limit point where its preimages converge to, is once more in evidence. But one knows what's the difficulty related to the problem arising from the existence of an infinity of periodic cycles, whatever big is the index of a group itself. Cayley posed the problem in the case of the Newton's method; but, in the same case of a simple entire polynomial, the number of these groups may be infinite ...* » .

This last assertion holds no more now because, for every entire polynomial, the number of periodic cycles is FINITE.
We will examine it further while approach to the Newton's method, so that, we will see some new and interesting circumstances come up when the equation degree is over 2.

35. *Applications*. – Immediately, one may use the general theorem to study the problem of the iteration in a more general way than we did up to now.

At present, the only known examples of the study of an iteration in the entire plane have been published by Fatou in its two Notes appearing in *Comptes Rendus* of 1906 and 1917 (October 15th 1906, May 21th 1917).
One will see further that very simple cases may occur in the iteration and what are those circumstances, related to this simplicity.

1. The Note of 1906 studies the fraction $z_1 = \frac{z^k}{z^k + 2}$ (coming out, by the auxiliary homographic mapping $z = \frac{1}{Z}$, from the original polynomial $Z_1 = 2Z^k + 1$).
For that fractions, the origin is a limit point with uniform convergence. The circle C_0, centered at O and examined by Koenigs during the local studies around the origin, keeps inside the only two critical points of the inverse function: they are $z = 0$ and $z = 1$. Then it is assured that points admit the origin as the only limit point of their images.

Therefore no other limit points with uniform convergence exist, neither periodic cycles $z = \varphi_n(z)$ satisfying $|\varphi'_n(z)| < 1$.

R, the immediate domain of convergence to O, consists of the entire plane except an everywhere discontinuous perfect set which is E′.
So R is the total domain of convergence to O.
The simplicity of the answer lies on the fact that the inverse of $\varphi(z)$ has only two critical points and both of them belong to the domain R of the origin.

2. Furthermore, Fatou examines the particular example $z_1 = \frac{z + z^2}{2}$, where both the origin and the point at infinity are limit points with uniform convergence.

The critical points of the inverse function are $z = -\frac{1}{8}$ and $z = \infty$.

It is really easy to prove that $z = -\frac{1}{8}$ belongs to the immediate domain of convergence R_0 to the origin and that $z = \infty$ belongs to the domain R_∞ of the point at infinity.
As one surrounds the origin with a enough small circle C (for example $|z| = \frac{1}{2}$) and as one takes the preimages of the area C, one generates some areas C_{-1}, C_{-2}, ... enclosing the origin and lying at a finite distance.

All of them are inside a certain circle Γ with so big enough radius to include both the origin and the point at infinity: the area, bounded by Γ and including ∞, is mapped by

φ(z) into a part of itself: this new region surrounds ∞, that's to say the image of the curve Γ is a curve Γ_1, completely surrounding Γ ([1]).
(For example, one may take the circle | z | = 4 as Γ).

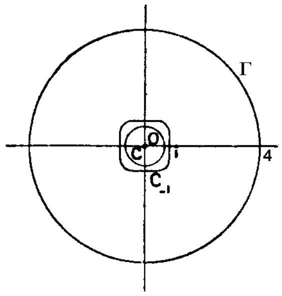

Fig. 16

All the areas C_{-i} are simply connected; they have as limit an simply connected area R_0, inside Γ, which will be the domain of the origin. In the same way one takes the preimages of the area Γ, bounded by the curve Γ and including the point at infinity.

The preimages are all *outside* R_0: their limit is a simply connected area R_∞ including the point at infinity; this area is the domain of the point at infinity.
The areas R_0 and R_∞ doesn't share any common point.

In this case E′ is a set splitting the plane at least in two regions R_0 and R_∞; E′ includes a continuous line.
The boundaries of both R_0 and R_∞ are continuous lines belonging to E′.

But each point of E′ is a limit for the preimages of any point of the plane (except the point at infinity); if one chooses any point in R_0, then all its preimages lie in R_0; if one chooses any point in R_∞ , then all its preimages lies in R_∞ ([2]).

This proves that any point of E′ acts like a boundary for both R_0 and R_∞, and inversely.

Therefore both R_0 *and* R_∞ *share as the common boundary the continuous line* E′ *where the preimages of the curve* C *and those of* Γ *converge to*. (we will see later that E′ is a *Jordan curve* in this case).

3.What has been said about $z_1 = \frac{z+z^2}{2}$ can be evidently be extended to *any fraction of 2nd degree, holding two limit points with uniform convergence*, that's to say two points ζ_1 and ζ_2 satisfying $z = \varphi(z)$ and $|\varphi(z)| < 1$.

([1]) So the area, bounded by Γ and including ∞, becomes the bounded area by Γ_1 and including ∞, which is inside the previous one.
([2]) Because for each one of these domains, R_0 for example, one and only one a critical point is enclosed for ψ(z); that point is critical and it is equal to zero at the origin. An interior point of R_0 is mapped to two preimages inside R_0 and to one image inside R_0.

Such a fraction has some complex coefficients ranging over certain limits (for example, every coefficient can be represented by a point of the complex plane lying in *certain areas* of its plane); but these limits are less restrictive than those forcing $z_1 = \varphi(z)$ to admit a fundamental circle.

Fatou showed, in its Note of May 21th 1917, that a fraction $z_1 = \varphi(z)$, preserving the interior of the fundamental circle, may be reduced by an homographic mapping to another fraction whose *all coefficients are real* ; it is clear that the condition of reality (i.e.: of being real number) coincides with a relation of equality [that's to say: the imaginary part is null ([1])]; then the restrictions, imposed to coefficients, are not anything else but *inequalities*; the points, representing coefficients, can describe *two dimensions areas, each one in its plane*.

For understanding, it suffices to start from a fraction of 2nd degree with two limit points with uniform convergence; the roots of the equation $z = \varphi(z)$ depend always on the parameters of $\varphi(z)$ such as the value assumed by $\varphi'(z)$ at one of those two roots; while considering the two roots of $z = \varphi(z)$, satisfying $| \varphi'(z) | < 1$, in the particular case where if the coefficients of $\varphi(z)$ describe (each one on its plane) small areas surrounding the starting values, then their roots continuously range and retrieve always $| \varphi'(z) | < 1$.

Therefore, since ζ_1 and ζ_2 are the two limit points with uniform convergence, each one of them is the limit point for the images of a critical point of the inverse function of $\varphi(z)$. There are two critical points and they can be obtained easily: if $z_1 = \frac{az^2 + bz + c}{a'z^2 + b'z + c'}$ is the examined fraction $\varphi(z)$, then the two critical points can be obtained by setting to zero the discriminant of the following equation

$$(a'z_1 - a)z^2 + (b'z_1 - b)z + c'z_1 - c = 0 \qquad \text{so} \qquad (b'z_1 - b)^2 - 4(a'z_1 - a)(c'z_1 - c) = 0$$

Except for ζ_1 and ζ_2, there are no any other limit points with uniform convergence, neither a periodic cycle. Let us start from two small enough circles, C and Γ, surrounding respectively ζ_1 and ζ_2; then let us search for the preimages of the areas bounded by the two circles: one will see that all preimages of C are simply connected areas C_{-1}, C_{-2}, …, leaving Γ outside; the preimages of Γ, Γ_{-1} and Γ_{-2}, are simply connected too and they leave C outside. The areas C_{-i} have a simply connected area R_1 as limit: this area is the immediate domain of convergence to ζ_1; the area Γ_{-i} has an area R_2 as domain of ζ_2; and one will see, as before, that E′ *is the common boundary*, *still a continuous line*, *splitting the extended plane in two complementary regions* R_1 *and* R_2.

This continuous line is the common limit of preimages C_{-i} and Γ_{-i} of the curves C and Γ.

([1]) The point, related to that coefficient, lies on the real axis or over a circumference which is the transformed of that axis by the homographic substitution, applied onto the variable.

Every point of the plane can belong to R_1 or to R_2 or to their common boundary E′.
The nature of the continuous line E′ (we already know that it is not an area) will be examined later.

4. The general examples, shown in 3) , allow us to examine more general cases too.
In the previous example, both the areas C_{-i}, like Γ_{-i}, are simply connected and their limits share the same boundary; this is strictly important: for example, starting from a certain rank, C_{-i} includes a critical point for the determination of $\psi(z)$ which become equal to ζ_1 in ζ_2; then when z_{-i} describes the interior of the area C_{-i}, then *its two preimages* describe the interior of an simply connected area $C_{-(i+1)}$ including C_{-i} in its interior.
As $z_{-(i+1)}$ describes the contour $C_{-(i+1)}$ once in the positive direction, z_{-i} describes C_{-i} twice in the positive direction; one may say the same about C_{-j} and $C_{-(j+1)}$ for $j > i$; the same happens for Γ_{-i}, starting from a certain rank.

The fundamental matter is that ALL THE PREIMAGES of an area C_{-i} are inside $C_{-(i+1)}$; and, the same happen for the curves Γ_{-i}, whatever is the index i.

It would happen that R_1, limit of C_{-i}, has a part of E′ as the boundary which lies aside to R_2, the limit for Γ_{-i}, for all boundary points, because every point of E′ is the limit for the preimages of a point of C_{-i} like for the preimages of a point of Γ_{-i}.

Then we get to the same conclusion of 3).

a. Therefore, what we examined before can be now applied *to fractions of any degree*, *with two limit points with uniform convergence*, *so that its inverse function* $\psi(z)$ *has only two critical points* ; for example, any fraction of the type

$$z_1 = \varphi(z) = \frac{az^k + b}{cz^k + d}$$

where a, b, c, d are some constants, so that the equation $z = \varphi(z)$ admits two roots (ζ_1 and ζ_2) satisfying $| \varphi'(z) | < 1$.

Then, it is easy to reduce any fraction to the previous type ($k = 2$) by a linear auxiliary substitution in z, where the fraction is of 2^{nd} degree.

For such fraction, the plane splits in two simply connected regions R_1 and R_2, respectively including ζ_1 and ζ_2 and separated by a continuous line E′.
R_1 and R_2 are respectively the *locus* of points whose images converge to ζ_1 and ζ_2, respectively. Each point of E′ has all its images in E′.

On E′ the points of E [$z = \varphi_n(z)$, $| \varphi'_n(z) | > 1$] have a finite number of images and they are everywhere dense.

b. But one may still generalize this case by avoiding the restriction so that φ(*z*) has *only two distinct points satisfying* φ′(*z*) = 0 [corresponding to the two critical points of the inverse of φ (*z*)].

Let us consider a fraction of *k* degree so that there are two limit points with uniform convergence ζ_1 and ζ_2; when ζ_1 and ζ_2 are surrounded by two bounded small areas C and Γ, one considers the preimages of C and Γ with the help of the branches of ψ(*z*), becoming equal to ζ_1 and ζ_2 in ζ_1 and ζ_2 respectively, as we explained at sections 4 and 5 of this Memoir.

Now let us suppose to proceed until we find a simply connected area C_{-i} including ζ_1, so that when z_{-i} describes C_{-i}, *all k preimages* $z_{-(i+1)}$ describe a simply connected area $C_{-(i+1)}$, including C_{-i} in its interior ; in this way, as $z_{-(i+1)}$ describes once the simple contour C_{-i} in the positive direction, z_{-i} describes *k* times the simply boundary C_{-i} in the positive direction.

In general, C_{-i} shall include in its interior (*k* - 1) critical points of the function ψ(*z*), inverse of φ(*z*), so that, starting from ζ_1 with any determination of ψ(*z*), one may come back to ζ_1 with an interior path of C_{-i} by any desired determination of ψ(*z*).

One realizes it while considering the Riemann surface R with *k* layers, related to ψ(*z*): it happens that the boundary C_{-i} intersects in R a simply connected area with *k* superimposed layers including ζ_1, bounded by C_{-i} and intersected *k* times.
[In general, it needs that each layer of R is joined to the following layer by a critical point inside C_{-i}; therefore it needs the existence of (*k* - 1) critical points of ψ(*z*) in C_{-i}.]

In the same way, let us suppose to proceed on Γ so that one reaches to a preimage Γ_{-j} enjoying the same properties as C_{-i} in ζ, relatively to ζ_2; that's to say, when z_{-j} describes the interior of the simply connected area Γ_{-j}, all *k* preimages $z_{-(j+1)}$ describe the interior of a simply connected area $\Gamma_{-(j+1)}$ including Γ_{-j} in its interior.

Then all the preimages of the area C are simply connected like those of the area Γ.

The first preimages have a simply connected area R_1 as limit: this is the domain of convergence to ζ_1; the second preimages have a simply connected area R_2 as limit: it is the domain of convergence to ζ_2; both R_1 and R_2 share a continuous line E′ as common boundary; E′ splits these two areas and it is the common limit of the simple curves C_{-i} and Γ_{-j}.

Each point of the plane belongs to R_1, or R_2, or E′. One will see further the nature of E′ and that E′ is a *Jordan curve* for very general hypotheses.

Let us remark that, about what said before, *one may start just from* C_{-i} *and* Γ_{-j} *instead of* C *and* Γ, because properties of R_1, R_2 and their common limit E′ are better drawn when we start immediately from C_{-i} and Γ_{-i}.

If C_{-i} and Γ_{-j} are *a priori* two curves surrounding respectively ζ_1 and ζ_2 and enjoying, relatively to both ζ_1 and ζ_2, the previous properties so that their *k* preimages of rank 1 are, for ζ_1, confused with $C_{-(i+1)}$, for ζ_2, with $\Gamma_{-(j+1)}$, so that the area $C_{-(i+1)}$ includes C_{-i} and ζ_1, the area $\Gamma_{-(j+1)}$ includes Γ_{-j} and ζ_2 ... : then one may affirm that C_{-p} and Γ_{-p} converge to the same limit E′ which splits the domains R_1 and R_2 to ζ_1 and ζ_2, respectively.

Some of the functions in the fundamental circle, notified by Fatou (*Comptes Rendus*, May 21th 1917), belong to this case.

These functions admit two limit points ζ_1 and ζ_2 with uniform convergence: they are symmetric with the fundamental circle [*read* n° 26 and following, and n° 20 of this Memoir (2°)].

If ζ_1 is interior and ζ_2 is exterior, if R_1 consists of the interior and R_2 consists of the exterior of the circle, then E′ is the same of the splitting circumference.

One may *a priori choose*, *for the curves* C_{-i} *and* Γ_{-i}, *some circles*, *very near to the fundamental circle and having* ζ_1 *and* ζ_2 *as Poncelet points* ([1]).

But, starting from a rational fraction which, as I remarked before, has some coefficients related to conditions of *equality* (conditions of being real values), one may define more general properties than the property of preserving the fundamental circle.

If the coefficients of the original fraction range in a small interval, then each one of the coefficients keeps on lying in a small enough area surrounding the initial original value: one will get a much more general rational fraction (whose coefficients satisfy only inequalities) which won't preserve the fundamental circle anymore, but, like the original fraction, one the following fundamental properties which are related to:

1. the existence of two limit points ζ_1 and ζ_2 with uniform convergence;
2. the existence of two curves C_{-i} and Γ_{-j}, surrounding the two limit points.
(If the variation of the coefficients is small enough, then one may keep the same circles C_{-i} and Γ_{-j} of the original fraction.)

For this fraction, the splitting of the plane into two regions R_1 and R_2 is valid due to the process before.
The continuous set E′, splitting R_1 and R_2, is near to the fundamental circle keeping the original fraction, since it is included in the circles C_{-i} and Γ_{-j}, which have been chosen for the original fraction as near as possible to the fundamental circle.

([1]) Read the Schwarz's Lemma, in the *Preliminaries*.

In this case, one may say that *the continuous line* E′ *changes continuously as the parameters of the fraction* φ(*z*): this result that is far to be evident *a priori*.

The two limit points with uniform convergence change continuously like the parameters of φ: this is a consequence coming out from the continuity of φ(*z*) in relation to that parameters [in the same way, it is proved the continuity of φ′(*z*)].

36. *The total domain of convergence to a limit point with uniform convergence or to a periodic cycle*. – It is useful to think about it as the set of all points of the plane, whose images admit the considered point for the only limit point or the considered circular group only for a periodic cycle.

We relieve this explanation by dealing only with the case of the limit point with uniform convergence.

Up to now, we have been studying the *immediate domain of convergence* R to the considered limit point ζ [ζ = φ(ζ) | φ′(ζ) | < 1].

The boundary points of this domain, consisting of one piece and including ζ in its interior, belongs to E′ and its interior points do not belong to E′ and they cannot be joint by a path whose at least one point does not belong to E′.

We have recognized that R necessarily includes at least one point of φ′(*z*) = 0 in its interior so that its image is the critical point of the branch of ψ(*z*), the inverse function of φ(*z*), where ζ is a fixed point.

This shows that:

1. Every interior point *z* of R is mapped to an image which is still inside R (like all images of any rank);

2. Every interior point *z* of R has *at least two preimages* z_{-1} *of rank* 1 *inside* R; these two preimages coincide when φ′(*z*) vanishes, so that *z* is the same as an interior critical point in R for the branch ψ(*z*) we're interested in.

But we will see further that not *all preimages* of an interior point of R are inside R.

It is true for example if, starting from a certain rank *i*, all curves C_{-i} (each one is preimage of the other and whose limit for $i = \infty$ is the boundary set of R) consist of *only one contour* bounding a *simply connected area* of the plane, so that every C_{-i} (starting from a certain rank) intersects in the Riemann surface R (with *k* layers and related to ψ(*z*)) a simply connected area S_{-i} with *k* superimposed layers including ζ and bounded by the curve C_{-i}, traversed *k* times.

Then, as z describes the interior of C_{-i}, the set of its k preimages describes the interior of a simple curve $C_{-(i+1)}$, surrounding C_{-i} and including C_{-i} in its interior.

Since R is the limit for $i = \infty$ of the interior of C_{-i}, it is clear that, as z describes the interior of R, then the set of its K preimages describes the interior of R.

If one considers the simply connected portion S (with k layers) of R, projecting on R, then, while z describes S, *every one of its preimages is a uniform function of z on* S *and it describes the interior of* R *as z describes the interior of* S.
Then in general S includes k - 1 critical points for $\psi(z)$.

For anyone of these critical points, it happens that, *starting from* ζ *with any arbitrary determination of* $\psi(z)$, *one may come back to* ζ *with any other chosen determination of* $\psi(z)$ *by following an interior continuous path in* R ([1]).

In this case, *occurring at the* 2°, 3° *and* 4° *of the previous applications*, *the total domain* D *of convergence to* ζ *is the same as the immediate domain* R ([2]).

In fact, if an interior point z of a total domain admits ζ as the only limit point of its images, then all images z_i belong to the domain R including ζ.

Therefore D may be defined as *the set of those points whose an image* (*of an arbitrary high rank*) *belongs to* R.

Therefore D *consists of* R *and of all preimages of the area* R.

37. In the previous cases the area R coincided with its k preimages of rank 1; therefore it coincided with all its preimages.

The *total* domain of convergence D to ζ is the same as R (the *immediate* domain).

The necessary and sufficient condition for this circumstance is: let R be a domain, so that, as we start from ζ with an arbitrary determination of $\psi(z)$, then one may come back to ζ by an interior path of R and with any chosen determination of $\psi(z)$.

([1]) Definitely, this is the necessary and sufficient condition for R for being the same as D, since each point of z, inside R, is mapped to k preimages, inside R and so that one may go from one to another preimage without escaping from R: so z describes interior paths of R.
([2]) In the previous example like the first application ,

$$z_1 = \frac{z^k}{z^k + 2},$$

R is still the same as D, because the *necessary* and *sufficient* condition is still verified, since R includes all critical points of $\psi(z)$.

That's to say, the set of the points of R [the Riemann surface of ψ(*z*)], projecting in the interior of R, shall generate a surface S consisting of one piece and with *k* layers (1) (all layers of R) and including ζ.

It requires, in the case when all critical points are *simple*, that's to say when they do not link anything more than two layers of the surface R or than permuting only two determinations of ψ(*z*) so that if R is simply connected (2), it includes in its interior at least *k* - 1 critical points which allow to permute each one of the *k* determinations of ψ(*z*) with any of other (*k* - 1) ones by proper interior paths of R without escaping from R. If the critical points are *multiple*, then they shall rely at least for (*k* - 1) simple critical points (3).

38. It is not *a priori* difficult to find out that this condition can be verified in most cases when the degree $k > 2$.

When $k = 2$, one critical point in R suffices to satisfy the condition; and this critical point is always in R, necessarily; therefore the condition is satisfied.

When $k > 2$, we see that the number of limit points with uniform convergence can reach to *k* when the number of critical points of ψ(*z*) is not bigger than 2(*k* - 1).

Therefore it is *a priori* impossible that each one of the immediate domains of convergence to the limit points with uniform convergence, in the cases (since any arbitrary pair of these domains does not share any common point) they are simply connected (4), encloses at least *k* - 1 critical points in its interior, as soon as the number of existing limit points is bigger than 2.

It is *a priori* true in all cases that, if there are more than two limit points with uniform convergence then the *total domain of convergence is more extended than the immediate domain* , at least for one of these limit points.

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(1) S can be simply connected, like in the application 2°,3°,4°, or it may have an infinite order of connection like in the 1° application. If *z* describes R once and only once, its image z_1 describes the area S with K layers.
(2) I make here this restriction: because it may happen that , without including the number of critical points that I showed for the simply connected domains, R can satisfy the necessary and sufficient condition if R itself is multiply connected. This is a more delicate question that I'll fix later.
(3) The condition is *always satisfied by the immediate domain of convergence to the point at infinity*, in the case where φ(*z*) is *any polynomial with degree k*, since the point at infinity counts (*k* - 1) multiple critical points, around which the *k* branches, of the inverse function ψ(*z*), permute.
(4) One will see further an example, realizable by determining, for each limit point with uniform convergence, the critical points of ψ(*z*), whose images converge to a limit point, and by examining the permutations of the branches of ψ(*z*), around that points.

Up to present, this is a new circumstance seeming to be not realized by any author studying iterations.
In all their examples, that I have partially revived in the previous applications, the total domains of convergence always coincide with the immediate domains.

I've shown the reason about the subtraction of the number of critical points of $\psi(z)$ in the interior of the immediate domains.

For this reason, as I wrote (section 35) while I was dealing with the previous applications, these *are the simplest examples* that may occur during the iteration: the plane is just split by the set E′ in a *finite* number of regions R; these examples do not let anybody perceive that E′ may split the plane in an *infinite number of regions*: a more general case arising when the degree of the fraction to iterate is bigger than 2.

Again, these examples did not foresee that usually the total domain of convergence to a limit point with uniform convergence, consists of an *infinity of areas* where an arbitrary couple of regions does not share any common point; these areas are preimages of the area R and they are distinct from R that composes the immediate domain of convergence.

39. Firstly I show, as we consider any rational fraction of degree k, that the number of limit points with uniform convergence may be only $\leq k$.

Let us come back to the section 9 of this Memoir, dealing with the existence of the points of E $[z = \varphi_n(z), \mid \varphi'_n(z) \mid > 1]$.
Consider

$$z_1 = \varphi(z) = \frac{P(z)}{Q(z)}$$

be the fraction to iterate, when both P and Q are two polynomials of degree k.
The roots of $z = \varphi(z)$ are the roots $\zeta_1, \zeta_2, \ldots \zeta_{k+1}$, ([1]) of the equation

$$R = P - zQ = 0$$

of degree $(k+1)$.
Then one has demonstrated the relation

$$\sum_1^{k+1} \frac{Q(\zeta_i)}{R'(\zeta_i)} = -1$$

Then

$$\varphi'(\zeta i) = 1 + \frac{R'(\zeta_i)}{Q(\zeta_i)} \ .$$

([1]) We are dealing with the general case where all roots are distinct.

Therefore if the point M_i, whose affixe is $\varphi'(\zeta_i)$, is inside the circle, centered at O and with radius 1 [$|\varphi'_n(z)| > 1$], then the point N_i, with affixe $\frac{Q(\zeta_i)}{R'(\zeta_i)}$ is in the half-plane

$$\mathrm{Re}(z) = \text{retrieves the real part of } z < -\frac{1}{2} ,$$

and reciprocally.

The relation

$$\sum_1^{k+1} \frac{Q(\zeta_i)}{R'(\zeta_i)} = -1$$

proves that $(k + 1)$ points $N_1, N_2, \ldots, N_{k+1}$ have the point with affixe $-\frac{1}{k+1}$ as centre of gravity.

Immediately it follows that, among these points N_i, *there are k*, *but not more than k*, *points in the half-plane* $\mathrm{Re}(z) < -\frac{1}{2}$.

The point may be determined *a priori*; the $(k+1)^{th}$ will be given by the previous relation.

If we know *a priori* the $(k + 1)$ values of $\frac{Q(\zeta_i)}{R'(\zeta_i)}$, then one will generate $\frac{Q(z)}{R(z)}$ with the help of the following identity:

$$\frac{Q(z)}{R(z)} = \sum_1^{k+1} \frac{1}{z - \zeta_i} \frac{Q(\zeta_i)}{R'(\zeta_i)}$$

given arbitrarily the $(k + 1)$ points ζ_i.

$Q(z)$ and $R(z)$ will respectively be

$$Q(z) = a_0 z^k + \ldots.,$$
$$R(z) = -a_0 z^{k+1} + \ldots.$$

due to the identity

$$\sum_1^{k+1} \frac{Q(\zeta_i)}{R'(\zeta_i)} = -1$$

since we know Q and R, one gets $P(z)$ by

$$P(z) = R(z) + zQ(z)$$

And the fraction $z_1 = \varphi(z) = \frac{P(z)}{Q(z)}$ admits the points $\zeta_1, \zeta_2, \ldots, \zeta_k$ as the limit points with uniform convergence, since these points satisfy $|\varphi'(\zeta_i)| < 1$ due to the choice of the first k values $\frac{Q(\zeta_i)}{R'(\zeta_i)}$ in the half-plane $\mathrm{Re}(z) < -\frac{1}{2}$.

So we have found the way to generate any rational fraction $\varphi(z)$ of degree k, with k arbitrarily chosen *a priori* limit points ([1]) with uniform convergence $\zeta_1, \zeta_2, \ldots, \zeta_k$

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satisfying | $\varphi'(\zeta_i)$ | < 1. It is a very general fraction taking care of arbitrary aspects involved in this question. For such a fraction, (k - 1) of those limit points with uniform convergence may correspond to the same number of total domains of convergence, which are more extended than the immediate domain of convergence.

40. Let us suppose therefore that the immediate domain of convergence R (to a limit point ζ with uniform convergence) does not satisfy the required condition so that R is the same as the total domain of convergence D to ζ.
This means that, as z describes R, *there are some preimages of z not belonging to* R.

Let us imagine the set of points of R [the Riemann surface of $\psi(z)$, inverse of $\varphi(z)$], projecting in the interior of R: this set consists of many distinct parts, each one of them consists of one piece so that one can only go from part to another by a path whose all points are projected in the interior of R (2); one of those parts S, including the point ζ of the upper layer, consists of one piece with the part of the upper layer, projected in the interior of R; as z describes S, one of its preimages z_{-i}, which is a fixed point for $z = \zeta$, is a uniform function in z on S: it describes the simple area R with one only layer, while z describes the parts of R, projected on R and not branching with S in the interior of R: there are some preimages z_{-i} of z that won't fall in R *so that they describe some simple connected areas of the analytic plane*, *with one layer and without sharing any interior common point within both themselves and* R (read *Preliminaries*, § 6); we will define these areas R_{-1} to remark that they are described by preimages of rank 1 of *z while z describes* R. So, as z describes R, its preimages describe either R, either the areas R_{-1} without any common point with R.
This process may continue indefinitely. Then, as z describes every area R_{-1}, none of its preimages can fall in an area R_{-1} or in the area R (3); therefore the preimages describe a certain number of simple areas so that every area consists of one piece and it has one only layer; these areas will be defined R_{-2} and they won't share any interior common point within, neither any common points with R_{-1} and R.

As z describes R, its preimages of rank 2 describe R, R_{-1} and R_{-2}.
In the same way, as z describes the areas R_{-2}, its preimages describe some areas R_{-3}, without any common point with R, R_{-1}, R_{-2}, and so on …

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(1) So one sees that a fraction, with degree k, cannot have more than k limit points with uniform convergence. There is always a root of $z = \varphi(z)$, retrieving | $\varphi'(z)$ | > 1; this proves the existence of points belonging to E. One sees that it is easy to generate a fraction without any limit point with uniform convergence.
(2) On the contrary, R satisfies the necessary and sufficient condition so that every point, inside R, has all its preimages in R.
(3) It will force z to be in R: absurd .

As we keep on going indefinitely, one sees that *the total domain of convergence to ζ consists of the areas* R, R_{-1}, R_{-2}, …, *that's to say it is generated by an infinity of areas consisting of one piece* ([1]) *and with one only layer so that any arbitrary pair of areas does not share any common points*.

As *z* describes R, its preimages of rank *i* describe the areas R, R_{-1}, …, R_{-i}.
The R_{-i} are the preimages of $R_{-(i-1)}$; when *z* is in one area R_{-i}, then its image is in an area $R_{-(i+1)}$.
All boundary points of R belong to E′ ; and since every preimage of a point of E′ belongs to E'_1, then every boundary point of R_{-i} belongs to E′.

This is the case where E′ is a continuous line and it splits the plane in an infinity of regions, so that any pair of them does not share any common points: it is impossible to go from a point of any region to a point of another region by a path without intersecting points belonging to E′. Between there regions, there are R, R_{-1}, R_{-2}, ...

41. If ζ is the only limit point with uniform convergence, then none of R_{-1} shares any interior common points with its total or immediate domain of convergence; if, for a second limit point the same as ζ happens, that's to say if its immediate domain of convergence is not the same as the total domain, then one determines, starting from its immediate domain, a sequence preimage domains: R_{-1}, R_{-2}, … and so on.

It may occur that, for certain limit points with uniform convergence, the immediate domain R coincides with the total domain D.
We will see some examples further: it is clear that, for any polynomial φ(*z*) of degree *k*, since ∞ is a limit point with uniform convergence and so it is a critical point for the function ψ(*z*), inverse of φ(*z*) and since it's where the *k* branches of this function permute in its neighbourhood, then the immediate domain of the point at infinity is the same as its total domain; moreover, if there are at least two limit points at a finite distance with uniform convergence, then one may expect that their total domain consists of an infinity of areas R, R_{-1}, R_{-2}, … .

Let us remark again that, since we suppose (for example, sending a point inside R to infinity by a proper homographic mapping; there is no ambiguity about what we call R) that the point at infinity is not a *limit* point for the set of the areas R_{-i}, any arbitrary couple of areas R_{-i} (i = 1, 2, …, ∞) shall not share any interior common points; the same areas shall get smaller and smaller as the index *i* increases, so that, for example, one cannot find an interior circle in R_{-i} with a finite radius as *i* increases indefinitely.

([1]) If R is simply connected, then all R_{-1} will be simply connected: if each connected piece of the Riemann surface R, projecting in the interior of R, is simply connected.

42. *The case where the immediate domain of convergence to a limit point with uniform convergence is not simply connected.* - For showing that the total domain of convergence may be more extended than the immediate domain, I have *only* considered the case where it is *a priori* realized that the immediate domain R is simply connected.
One may also study the case where R is multiply connected.
In the same way the domain R have been considered as the limit (while $i \to \infty$) for the preimages areas (C_{-i}) of the small area (C) surrounding the limit point ζ, it is clear that R will be multiply connected if and only if, starting from a certain index *i*, the areas (C_{-i}) are multiply connected too.

Therefore it is a necessary and sufficient condition that a certain curve C_{-i}, with *one only contour*, encloses an area (C_{-i}) whose preimages $C_{-(i+1)}$ are bounded by *many contours*.

This requires that the portion (consisting of one piece) of R [the Riemann surface of $\psi(z)$], previously defined S_{-i} and projecting in the interior of (C_{-i}) and including the point ζ of the upper layer, is *multiply connected*: so, it is indispensable that S_{-i} is bounded by *more than only one contour*.

S_{-i} shall be bounded by many closed curves so that each portion is projected on the curve C_{-i}; each one of those curves can be traced onto many layers of R, such that it is impossible to link two points respectively belonging to two distinct curves by a path intersecting at least once the *interior of* S_{-i}.
Again, if one imagines that *z* describes the contour of (C_{-i}), then the algebraic function $\psi(z)$, inverse of $\varphi(z)$, permutes, on one hand, *p* of its determinations, on the other hand, the other *p'* determinations.

It is impossible, when *z* describes the boundary C_{-i}, to go from one of the first *p* determination to one of the second *p'* determinations, but, it is important that, if we start from a point of C_{-i} with one of the first *p* determinations, one can come back to the same point *with one of the other examined p'* determinations, on the condition to describe an interior path of (C_{-i}).
A simple example comes imagining that, since $\varphi(z)$ is of 2^{nd} degree, C_{-i} is a simple closed curve surrounding the two critical points of $\psi(z)$.

Then S_{-i} is a doubly connected Riemann surface with two layers, bounded by two distinct curves, so that each curve is both traced onto a layer of R and projected onto C_{-i}, since the two layers of S_{-i} are joint by a cross line linking the two critical points: one may suppose that this line is inside C_{-i} .

43. Let us come back to the general case. Supposing that the $2k$ - 2 critical points of $\psi(z)$ are simple (since k is the degree of φ), it is the result of a classic process, explained for example in the *Traité D' Analyse* de Picard, t. II, 2nd edition, p. 416 - 417, where it is supposed that each one of the k layers of R are joint to the previous one by one only *cross line*.
In these conditions, on the first p layers of R, C_{-i} intersects a closed curve Γ_{-i} and, on the other p' layer, it intersects another closed curve Γ'_{-i} ([1]).

Both Γ_{-i} and Γ'_{-i} are contours of S_{-i}; we can go from a point of Γ_{-i} to a point of Γ'_{-i}, exclusively by an *interior* path of S_{-i} and by no means following the border of S_{-i}.
This implies that the only cross line within the p^{th} layer and the $(p + 1)^{th}$ has its own two *interior* extremities inside S_{-i} [they are two proper critical points of $\psi(z)$] and, it is clear that, since the cross lines (in a certain measure) are arbitrary within known extremities, that the cross line may be always supposed to lie *entirely within the* p^{th} *and* $(p + 1)^{th}$ *interior layer of* S_{-i}.
From this very important consequence, it follows that, as we start from a point z of the plane, outside (C_{-i}), with one of the p determinations of $\psi(z)$ that are mapped to the first p layers of R, then one may only go to another determination of ψ by a closed circuit *intersecting the area* (C_{-i}) and by intersecting the cross line among p^{th} and $(p + 1)^{th}$ layer.
Therefore every closed circuit, not intersecting the area (C_{-i}), takes back to z one of the first p determinations of $\psi(z)$ of the system, where the initial determination belongs to.

In the same way, if one starts from z, outside (C_{-i}), with any other determination of $\psi(z)$ except one of the previous p determinations, then one can never come back to z with one of these p determinations, if the described circuit does not intersect (C_{-i}).

44. The conclusion is immediate.
If there is a limit point with uniform convergence ζ', distinct from ζ, then its immediate domain R′ cannot satisfy the necessary and sufficient condition for R′ to be the total domain, since R′ is outside R and R′ is outside (C_{-i}) too; then, if we start from an interior point z of R′ with one of the first p determinations of $\psi(z)$ and if we describe any closed interior circuit of R′, then one won't intersect (C_{-i}) and one won't come back to z with one of the $(k - p)$ determinations of $\psi(z)$, distinct from the first p determinations.

Therefore the total domain of convergence of ζ consists of *an infinity of distinct areas* so that any arbitrary pair of areas does not share any common point.

([1]) If $p + p' < k$, C_{-i} still intersects other curves in R, but now it is not worth to be worried about it.

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Therefore the hypothesis, about the multiply connected structure of the immediate domain of convergence to one of the limit points, induced me to consider this interesting conclusion: *the total domain of convergence of any other limit point consists of an infinity of areas*.

Then the fact, explained at the section 38, is GENERAL: *for any fraction of k degree*, with *more than two limit points with uniform convergence*, *there is at the most one only limit point whose immediate domain of convergence may be the same as the total domain*.

In fact, one of those limit points has a multiply connected immediate domain; then *all other limit points have a total domain* (consisting of an infinity of distinct areas) *that is bigger than the immediate domain of convergence*. So each limit point has a simply connected domain; then there is only one immediate domain including the (k - 1) critical points in its interior so that the necessary and sufficient condition (i.e.: the domain is the same as the total domain to the considered limit point) is verified.

45. Another interesting consequence comes out from the previous study.

If *a priori* one verifies that one of the limit points has *a total domain of convergence being the same as the immediate domain of convergence* [by determining if the critical points of $\psi(z)$ are inside the considered immediate domain], then one may affirm, if there are other points with uniform convergence, *that the immediate domains of convergence to each one of the other limit points are simply connected*.

46. Then, the *immediate domain* R *of convergence to a limit point* ζ *cannot be multiply connected without having an infinite order of connection* ([1]). In fact, one has seen that, if (C_{-i}) is the first preimage area of the area (C) determining in the Riemann surface R a new multiply connected Riemann surface (S_{-i}), so that S_{-i} includes the point ζ of the upper layer; then the only crossing line L within the p^{th} and $(p + 1)^{th}$ layer can be supposed to be *inside* (S_{-i}). Therefore the area ($C_{-(i+1)}$), preimage of (C_{-i}), is bounded by as many distinct boundary as those of (S_{-i}). Therefore the area ($C_{-(i+1)}$) has at least two distinct contours not traversing the cross line L and, since all contours are outside (C_{-i}), the area ($C_{-(i+1)}$) includes (C_{-i}) in its interior.

([1]) This is a property of the domain R, consisting of one piece and bounded by the set E′. This domain, if it is multiply connected, has an infinite order of connection. In fact, the existence of a closed curve Γ, whose all points are inside R and where both interior and exterior include some points of E′ (this means that R is not simply connected), implies *the existence of such a curve in the whole area*, *how small it is*, *surrounding a boundary point of* R; from which the enunciated property follows, since this area has always an iterated area including Γ in its interior and since the finite area is bounded by Γ [except for the trivial fraction $z_1 - a = (z - a)^k$, $z_1 - a = \frac{1}{(z-a)^k}$, for which this question does not exist].

Then E′ cannot consist of many continuous distinct lines without any common points (two by two), with an infinite order of connection.

If one considers the Riemann surface $S_{-(i+1)}$ consisting of points of R, which are projected in the interior of ($C_{-(i+1)}$) so that they are joint to ζ (of the upper layer) by a continuous path which is traced on R and projected in the interior of ($C_{-(i+1)}$), then one can see that each one of the contours of ($C_{-(i+1)}$) gives birth to at least two contours of $S_{-(i+1)}$, projected on the considered contour of ($C_{-(i+1)}$).

In fact, as z describes the contour of ($C_{-(i+1)}$), then $\psi(z)$ is supposed to start from one of the p determinations corresponding to the first p layers of R; then $\psi(z)$ won't exchange the original determination with another one of the p previous determinations, without going to the determination (corresponding to the $(p+1)^{th}$ layer), neither to its determinations; therefore ($S_{-(i+1)}$) has a first limit contour, projected on the considered contour of ($C_{-(i+1)}$) and traced on all or some of first p layers of R.

Since R has some interior points in the $(p+1)^{th}$ layer and, more generally, in the p' layers following the first p layers, then the same reasoning, like the one applied on the boundary of ($C_{-(i+1)}$) by replacing the p' determinations of $\psi(z)$ (which follow the already considered p) with these p determinations, proves that $S_{-(i+1)}$ has a second limit contour, which is projected on the same contour of ($C_{-(i+1)}$) and traced entirely or on a part of p' layers of R following the first p layers.

The conclusion is that $S_{-(i+1)}$ has at least 2^2 contours like ($C_{-(i+2)}$).
Applying this same reasoning again would lead us to see that ($S_{-(i+p)}$) has at least 2^{p+1} contours, such as ($C_{-(i+p+1)}$); all those contours are outside, two by two ([1]); ($C_{-(i+p+1)}$) is a connected area bounded by all those boundaries and including ζ in its interior.

Therefore R, the limit of (C_{-n}) for $n \to \infty$, has an infinite order of connection.
We have already shown some analogous examples (*Read* Fatou, *Comptes Rendus*, October, 15th , 1906).
In the example of $z_1 = 2z^k + 1$, the domain of convergence to the point at infinity has an infinite order of connection: it is the limit of the points of the perfect discontinuous set E′: this is a case where the previously examined contours converge to zero in all their dimensions as p diverges to infinity; so there is only one point with uniform convergence.

Further, I will give another example with more points with uniform convergence: one of those points is bounded by an infinity of curves (not more by only a discontinuous set of points) and it has an immediate domain where the order of connection is infinite.

([1]) In the plane of z, there may be one including all the areas, but on the Riemann sphere this does not happen.

47. *Examples*. – I will give three kinds of examples:

1. firstly, in a particular example, I study the way of generation and grouping of the domains R, R_{-1}, … that we saw in the previous study;

2. after, I will show that the application of the Newton's method, for finding the roots of a polynomial of any degree > 2, led me to the previous circumstances explaining the failure of Cayley when he splitting the plane in different regions converging to the root of polynomial where the application of the Newton's method takes to ([1]) as one starts from any point of these examined regions. I succeeded very well, and I will show that equations of 2[nd] degree split the extended plane into two regions so that each one of them includes only a root, where the approximation of the Newton's method led me to, by starting from any point of that region.
We will see that the solution cannot be so easy as for equations with degree > 2: the total domain of convergence to a root may consist of an infinity of separated areas;

3. finally, I will give the example of an immediate multiply connected domain which is not only bounded by a perfect discontinuous set of *points*.

48. *First example*. – Let us consider the polynomial

$$z_1 = \frac{-z^3+3z}{2} = \varphi(z)$$

We have three limit points with uniform convergence :

1. $z = 1$, $\varphi'(1) = 0$.
2. $z = -1$, $\varphi'(-1) = 0$.
3. $z = \infty$, like all polynomials.

The equation $z = \varphi(z) = \frac{-z^3+3z}{2}$ still admits the root $z = 0$ so that $\varphi'(0) = \frac{3}{2}$: this root belongs to the set E. From the calculus of $\varphi'(z) = \frac{3(1-z^2)}{2}$, it follows that the points, where $\varphi'(z)$ vanishes, are $z = 1$ and $z = -1$; they are images of themselves.
The critical points of $\psi(z)$, inverse of $\varphi(z)$, are therefore three: 1. The points $z = 1$; around it, two branches of $\psi(z)$ permute: they assume the value 1 for $z = 1$ [because $\varphi'(1) = 0$ and $\varphi''(1) \neq 0$]; 2. The points $z = -1$; around it, two branches of $\psi(z)$ permute: they assume the value -1 for $z = -1$ [because $\varphi'(-1) = 0$ and $\varphi''(-1) \neq 0$]; 3. The point $z = \infty$; around it, three branches of $\psi(z)$ permute: they assume an infinite values for $z = \infty$.

([1]) Read KOENIGS, *Annales de l'Ecole normale*, 1884, Supplement, p.40 and 41. – CAYLEY, *C.R. Acad, Sc*., t. CX, 1890, p.215-218. He said : «I hope to apply this theory to the case of a cubic equation, but calculations are much more hard». We'll see this question later.

One may affirm that *there are no more limit points with uniform convergence, neither any periodic cycle, except three cited above.*

Since each one of the three limit points with uniform convergence (∞ , 1 , -1) has in their own immediate domain a critical point of $\psi(z)$ (∞ , 1 , -1) and, since there are only three critical points, then there is no any other 4^{th} limit point with uniform convergence neither periodic cycles.

Let us try to generate the immediate domains of convergence of each one of the points ∞, 1, -1. It suffices to consider 1 and ∞, because the domain of (-1) is symmetric to the domain of the point $z = 1$, in respect of the origin; in fact the polynomial $\varphi(z) = \frac{-z^3+3z}{2}$ is odd: $\varphi(z) = -\varphi(-z)$.

Therefore, two symmetric points, in respect of O, correspond to two symmetric images in respect of O. For the point $z = \infty$, one may start from the circle C, centered at O and with radius $\rho = |z| = 3$.

If z is any point satisfying $|z| > 3$, then one will get

$$|z_1| \geq \frac{|z|^3 - 3|z|}{2} \geq 3|z|$$

so the domain (C), bounded by $|z| = 3$ and including the point at infinity (1), is entirely in the interior of the immediate domain of convergence to ∞.

Let us search for the preimages of the domain (C) to find the immediate domain of convergence to ∞.

If z describes the interior of the domain (C), then its three preimages describe the interior of a domain (C_{-1}) including the point at infinity, such as (C) is bounded by one only simple algebraic curve C_{-1}, which is the preimage of C.
If z describes C_{-1} once in the positive direction, then z_1 describes C three times in the positive direction.
It happens since $z = \infty$ is a critical point: the three branches of $\psi(z)$, inverse of $\varphi(z)$, permute around it and it is the only critical point in C.

The curve C_{-1} is simply defined by the equation

$$\left|\frac{-z^3+3z}{2}\right| = 3$$

This curve surrounds the two points –1 and +1 and it splits them from the circle C (1).

(1) The domain (C) is the exterior of the circle C and the point at infinity is included in (C).

As z describes the interior of the domain (C_{-1}), its three preimages describe the interior of a domain (C_{-2}) including (C_{-1}) and bounded by a simply algebraic curve C_{-2}, preimage of C_{-1}. C_{-2} surrounds the (-1) and (+1) and it separates these points from the curve C_{-1}.
This process may keep on going indefinitely times.

All the preimage areas (C), (C_{-1}), (C_{-2}) … are some simply connected areas, so that each area is included in the preimage area and it includes the point at infinity.
(C_{-i}) is bounded by a simple algebraic curve C_{-i}, surrounding the points (-1) and (+1), so that it separates the points from the curves $C_{-(i-1)}$, …, C.

The limit of (C_{-i}) for $i \to \infty$ is a simply connected domain R_∞, including the point at infinity and leaving in its *exterior* the points (–1) and (+1) (such as some small enough circles centered at those two points), so that any simple closed curve inside R_∞ surrounds the two points (–1) and (+1) or it surrounds no point by leaving them at its *exterior*, splitting (–1) from (+1) ([2]).

At present, I do not insist on the nature of the boundary of R_∞: we will see further that there is *a continuous curve with some multiple points being everywhere dense on such curve*.
What immediately results from the previous studies is that this boundary is continuous and it separates each point inside R_∞ from both the points (–1) and (+1).

Since the point at infinity is the critical point, so that the three branches of $\psi(z)$ permute around it, the area R_∞ coincides with all its preimages: R_∞ is at the same time the immediate domain and the total domain of convergence to the point at infinity.

It is good to remark that every point of the imaginary axis (except the origin) is inside the domain R_∞; it suffices to assure that if one puts $z = i\lambda$, $z_1 = i\lambda_1$, since λ real, λ_1 is real too, so one will get

$$\lambda_1 = \frac{\lambda^3 + 3\lambda}{2} \ ;$$

therefore every point of the imaginary axis has its images on the same axis; and those images converge uniformly to the point at infinity, because one get always $|\lambda_1| > \frac{3}{2}|\lambda|$, given $\lambda \neq 0$.

([1]) That's to say, it is impossible to move to the point 1, for example, to a point of C, without traversing C_{-1}.
([2]) This property comes out from the fact that every closed interior curve inside R is, starting from a certain rank, inside all domains (C_{-i}); for each one of those domains, this property is still evident.

The origin is a boundary point ([1]) of R_∞: it is a *multiple point* of this boundary, because one may reach to the point O, remaining in R_∞ ([2]), from both the negative and positive direction of the imaginary axe; it is impossible to link a point, infinitesimally near to O and lying on the positive imaginary half-axis, to another point, infinitesimally near to O but lying on the negative imaginary half-axis, by an infinitesimally small path without escaping from R_∞.
(In fact, this path intersects the real axis in a point, infinitesimally near to O; and one sees that the points of the real axis, in the neighbourhood of O, are exterior from R_∞).

The entire imaginary axis, except the origin, since it belongs to R_∞, is the same as all preimage curves of this axis (it is confused with all its images).
The preimages of rank 1 are: the same axis itself and an hyperbola coming out from

$$\mathrm{Re}\left[\frac{-z^3+3z}{2}\right]=0$$

by fixing $z = x + iy$. This hyperbola is the equation:

$$x^2 - 3y^2 - 3 = 0$$

It intersects the two points $x = \pm\sqrt{3}$ of the real axis which are the two preimages, distinct from O, of the origin. These two points are the boundary points of R_∞; all other points of the hyperbola are inside R_∞. The preimages of that hyperbola are interior curves of R_∞. The imaginary axis and the real axis are axes of symmetry for R_∞, since the relation $z_1=\frac{-z^3+3z}{2}$ maps two symmetric points, in respect of Ox or Oy, to two other symmetric points, in respect of with Ox and Oy.
Therefore (referring to the theory of the images in respect of an algebraic curve), both all preimage curves of the imaginary axis or of the real axis are symmetry axes of R_∞, since two points of R_∞, symmetric in respect of Oy or with Ox, are mapped, by the relation $z_1 = \psi(z)$, inverse of $\varphi(z)$, to two symmetric preimages in respect of the considered preimage of Ox or of Oy. All preimage curves of the imaginary axis intersect the real axis with a right angle (the real axis coincides with one of its own preimage curves) like the imaginary axis itself, in points which are both all real preimages of the origin and the boundary points of R_∞.
There are algebraic curves whose degrees constantly grow. All of them intersect the point at infinity that coincides with all its preimages.
In the extended plane (or on the Riemann sphere) they shall be considered like closed curves whose the point at infinity is a multiple point.

([1]) It is a point of E, $\varphi(0) = 0$, $|\varphi'(0)| > 1$.
([2]) Therefore the origin is a boundary point of R_∞, *accessible from the interior of* R_∞ (*read* Preliminaries, § 4).

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So the hyperbola $x^2 - 3y^2 - 3 = 0$ shall be considered as a closed curve where the point at infinity is a double point, where the tangent lines of the double points (asymptotics), joint with the imaginary axis (which is an preimage curve of the imaginary axis) are regularly star-like disposed around the point at infinity; the tangent lines (the asymptotics of the hyperbola) intersect the imaginary axis forming angles of 60° degrees.
The result is that the imaginary axis, where preimages come from, intersect the critical point $z = \infty$ of $z_{-1} = \psi(z)$, where the *three branches of* $\psi(z)$ *permute around it*.
It is useless to insist on it anymore.
Let us add that *all preimages of the origin* are boundary points of R_∞ and they are *everywhere dense on the boundary of* R_∞.

All of them, like the origin itself, are *multiple points of that boundary* ([1]); here's is something important I just remarked above.
At the beginning of this Memoir, one has seen that the boundary points of a domain like R_∞ are point of E' ; on the other hand, every point of E', since it is the limit of the preimages of any arbitrary point of the plane (except the point at infinity) and in particular of the preimages of an interior point of R_∞, shall be a boundary point of R_∞; since all the preimages of an interior point of R_∞ lie in the interior of R_∞, then every limit point for its preimages shall be the limit of the interior point of R_∞.

Therefore the boundary of R_∞ *exhausts all the set* E' *that the following rational fraction has given birth to*

$$z_1 = \frac{-z^3 + 3z}{2}$$

It is the source for apparent paradoxes coming out from our naïve intuition of the boundary of a domain under the shape of a simple curve as soon as the nature of the domains of convergence to the point (-1) and (+1) is clarified.
Then all those paradoxes are perfectly explained by remarking, as I did previously and insistently, that the boundary of R_∞ encloses *some multiple points*, everywhere dense on the boundary itself. If one studies the real axis, then one sees well that only two half straight lines (of the real axis) belong to R_∞: the first one, going from $(+\sqrt{5})$ to $(+\infty)$, the second one, going from $(-\sqrt{5})$ to $(-\infty)$. Then the points $(+\sqrt{5})$ and $(-\sqrt{5})$ a periodic cycle of rank 2.
They belong to E, because they satisfy

$$z = \varphi_2(z) \quad \text{with} \quad |\varphi'_2(z)| > 1$$

([1]) And, like the origin, they are boundary points of R_∞, *accessible from the interior of* R_∞: for example, one attains them in following a curve, preimage of the imaginary axis.

They are boundary points of R_∞. As soon as $|z| > \sqrt{5}$, in fact one gets $|z_1| > |z|$ and the images of a point of the half-axis $(+\sqrt{5}, +\infty)$, for example, converge uniformly on that axis to $+\infty$.

49. Let us go immediately to examine the point $z = 1$ for determining its immediate domain of convergence R_1.
We will start from a small circle Γ, centered at $z = 1$, where the preimages encloses a bigger and bigger domain including the point $z = 1$ and converging to R_1.

The circle Γ behaves so that, if z describes its interior, then the image z_1 shall remain in a *completely interior* area of Γ.
If ρ_0 is the radius of Γ, one shall get, as soon as $|z - 1| \leq \rho_0$,

$$|z_1 - 1| < K|z - 1|,$$

as K ranges from 0 to 1.
Now, it is easy to get the following relation

$$z_1 - 1 = \frac{-(z-1)^3 - 3(z-1)^2}{2} .$$

If $|z - 1| = \rho$, then one gets

$$\rho_1 = |z_1 - 1| < \frac{\rho^3 + 3\rho^2}{2}$$

For verifying $\rho_1 < \rho$, it suffices that $3\rho + \rho^2 < 2$, that's to say

$$0 \leq \rho \leq \frac{-3+\sqrt{17}}{2} .$$

Therefore, if one chooses $\rho_0 > 0$ and $< \frac{-3+\sqrt{17}}{2}$, for example $\rho_0 = \frac{1}{2}$, one is assured that for $|z - 1| < \rho_0$, one will get

$$|z_1 - 1| < K|z - 1|,$$

as K ranges from 0 to 1.
Let us imagine that z describes the interior (Γ) of Γ; then the branch of $\psi(z)$, inverse function of $\varphi(z)$, showing a simple critical point at $z = 1$, describes the interior of an area (Γ_{-1}), which is preimage of (Γ) and including (Γ) in its interior.
Γ_{-1} is the limit curve of (Γ_{-1}) ([1]).
If z is in (Γ), then there are two and only two preimages in (Γ_{-1}), permuting when z makes a circular tour around $z = 1$.

([1]) Γ_{-1} is evidently a simple closed algebraic curve, surrounding $z = 1$. It will intersect again the real axis, as Γ_{-1} itself.

If one imagines two superimposed layers where each layer branches in the other one at $z = 1$, since the area Σ (simply connected and with two layers) is intersected in those two layers twice by Γ and since it includes $z = 1$ and it is bounded by Γ, then the area (Γ_{-1}) is a simple area of the analytic plane: it is described by the point $z_{-1} = \psi(z)$ as z describes the area Σ [in fact, $\psi(z)$ is uniform on Σ and it takes only once the values in Σ].
Since Γ does not intersect the imaginary axis O*y* again, then Γ_{-1} won't intersect the imaginary axis O*y* anymore.

It is the place to remark again that, when z describes (Γ), the third preimage of z describes a simple area (with only one layer) surrounding the point $z = -2$ (an the preimage of $z = 1$) and bounded by a simple closed algebraic curve surrounding $z = -2$ and *without any common point with* O*y*.

The behaviour of the this process is easy: as z_{-1} describes the interior of (Γ_{-1}), two preimages z_{-2} describe the interior of a simple area (Γ_{-2}), bounded by a closed algebraic curve Γ_{-2}, without any common point with O*y* and surrounding Γ_{-1}, then the third preimage describes a simple area, bounded by a closed algebraic curve, without any common point with O*y* and surrounding the point $z = -2$.

There is just the need to remark that all the areas (Γ)(Γ_{-1})(Γ_{-2}) … are inside a circle C with radius $\rho = 3$ and centered at O, used like a initial point in the construction of R_∞.

All areas are in the interior region of that circle, bounded by O*y* and including $z = 1$, that's to say in the right half of the area, bounded by the circle.

All regions, described by the third preimage of z, as z describes one of those (Γ_{-i}), lie in the left half of the area, bounded by C; they never share any common point with O*y* and with the areas (Γ_{-i}).

All the areas (Γ_{-i}) are simply connected; their limit, for $i \to \infty$, which is the immediate domain of convergence R_1 to $z = 1$, is a *simply connected area* including $z = 1$ and lying in the right half of the area, bounded by C.

One immediately realizes that, since the portion of the real axis, lying between the origin and the point with $x = \sqrt{3}$, belongs to R_1, then the points O and $\sqrt{3}$ are boundary points for R_1 [$+\sqrt{3}$ is the preimage of the origin].
O is the only point where the boundary of R_1 touches the imaginary axis O*y*.

50. The domain R'_1, symmetric of R_1 in respect of the origin, is the immediate domain of convergence to $z = -1$. It includes the segment (O, $-\sqrt{3}$) of the real axis.
The boundary of R_1, as the boundary of R'_1, belongs to E′. On that boundaries, the preimages of the origin are everywhere dense, but since the origin, considered as the

boundary point of R_∞, is the multiple point of that boundary, then it does not seem to be *a priori* a multiple point of the boundary of R_1 or of R'_1.

The curves C_{-i}, surrounding the origin, whose limit is fixed at the boundary of R_∞, tend to admit the origin (and all its preimages) as a double point, that's to say there are two distinct arcs of C_{-i}: they are symmetric in respect of the origin and separated from each other by a finite length curve, finally not infinitely near *on the curve*: both two arcs converge to O as *I* increases indefinitely, while it does not seem to be *a priori* that on (Γ_{-i}), both two distinct arcs (not infinitely near on that curve), converge to O, since the (Γ_{-i}) do not *surround* the origin and they do not reach to O*y* (1).

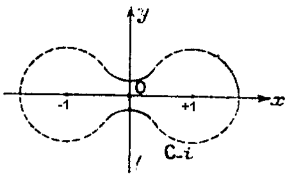
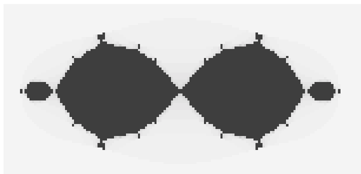


Fig. 17


A current drawing of the same Julia set.

R_1 and R'_1 intersect only in their common boundary point O. R_1 and R'_1 intersect the domain R_∞ in O, so that all their boundary points, since all that points belong to E′, belong to the boundary of R_∞ (2).

It is immediately necessary to determine the total domains of convergence D_1 and D'_1, to the point $z = 1$ and $z = -1$.

(1) The following note demonstrates rigorously that O, like every point of the boundary of R_1 (or of R'_1), which is accessible from the interior of R_1 (or of R'_1), is a *simple point* for that boundary. Let us consider O for example: if there is a multiple point for the boundary of R_1, then there is a simple closed line L (Jordan line) intersecting O, so that every point of L, distinct from O, is *inside* R_1; and it cannot be reduced to the only point O by a continuous movement without intersecting at least a boundary point of R_1, distinct from O (the definition of the multiple points of a boundary); that's to say, that line includes in *its interior* (one knows well that a Jordan line splits the plane in an interior and exterior regions) at least a point Q on the boundary of R_1. This point Q is a point of E′: therefore it is a boundary point of R_∞ because E′ is the same as the boundary of R_∞. Therefore, in *every neighbourhood* of Q, there are *interior* points of R_∞. Therefore we will find an infinity of points (inside R_∞) in the *interior* region of L. Randomly I take a point Q′.
Let us take then a point Q″, *inside* R_∞ and surely *exterior from* L; for example, an *exterior* point of the circle C [$| Z | > 3$], then it is impossible to join Q′ with Q″ (both *inside* R_∞) by a simple line without intersecting L (in an interior point of R_1, therefore exterior from R_∞) or without intersecting O (the boundary point of R_∞). This is absurd since Q′, Q″ (both interior points of R_∞, a domain consisting of one piece) can be joined by a simple line, whose all points are *inside* R_∞. Therefore the contradiction demonstrates that O is a simple point of the boundary of R_1: the same applies to all accessible points of that boundary. The previous reasoning will appear further again.
(2) There is an apparent paradox: the boundary of R_1 is a part of the boundary of R_∞; that part, the boundary of R_∞ includes some everywhere dense *multiple points* : these are all the preimages of the origin: they lie on the boundary of R_1.

51. If z is in R_1, then two of its preimages are in R_1, the third one describes a simply connected area which I call $R_1^{(-1)}$, surrounding the point z = -2 and including in its interior a small segment of the real axis, starting from $z = -\sqrt{3}$; this segment is the real preimage of the segment (0, $+\sqrt{3}$) of the same axis. $R_1^{(-1)}$ cannot reach to the axis Oy from no points, because R_1 reaches only from O, whose preimages are O, ($+\sqrt{3}$), ($-\sqrt{3}$): it is entirely located at the left of Oy, it does not share any common point with R_1, it intersects R'_1 only in the point $z = -\sqrt{3}$. $R_1^{(-1)}$ is inside the circle C.

As z describes $R_1^{(-1)}$, each one of its three preimage points is an analytic function of z in $R_1^{(-1)}$ and it describes a simple area with one only simply connected layer: those three areas are completely exterior each other: they are outside $R_1^{(-1)}$, to R_1 and to R'_1 and they are called as the areas $R_1^{(-2)}$; each one of them includes a small segment of the real positive axis (the real preimage of the segment of the negative real axis) intersecting $R_1^{(-1)}$; this segment is separated from the segment $(0,+\sqrt{3})$ by the symmetric segment in respect of O belonging to the negative segment intersecting $R'^{(-1)}_1$, the separating segment is in the *interior* of $R'^{(-1)}_1$, preimage of R'_1, like $R_1^{(-1)}$ is the preimage of R_1.

This process can keep on going indefinitely.

Progressively, the areas $R_1^{(-3)}$ are described by the preimages of z as z describes the areas $R_1^{(-2)}$.

The areas $R_1^{(-3)}$ are simply connected and they do not trespass on $R_1^{(-2)}$, nor $R_1^{(-1)}$, nor on R_1.The areas $R'^{(-2)}_1$, $R'^{(-3)}_1$, … are symmetric of $R_1^{(-2)}$, $R_1^{(-3)}$, … in respect of O.

All the defined areas are (two by two) exterior and they cannot intersect anywhere else but in the boundary points.

The set R_1,$R'^{(-1)}_1$, $R'^{(-2)}_1$, … generates the total domain of convergence to (+1), and the ensemble R_1,$R'^{(-1)}_1$, $R'^{(-2)}_1$, … generate the total domain of convergence to (-1).

Every boundary point of $R_1^{(-i)}$, or $R'^{(-i)}_1$, … is a point of E′ and it is a boundary point of R_∞, so that R_∞ touches all $R_1^{(-i)}$ and $R'^{(-i)}_1$ in all its boundary points.

As well, one sees what's the strange type the continuous E′ belongs to, while bounding R_∞ and all $R_1^{(-i)}$, and $R'^{(-i)}_1$; E′ includes some everywhere dense and multiple points on itself and it splits the plane in an infinity of regions.

The real axis, for example, intersects E′ in an infinity of points that are all the real preimages of the origin; it is necessary to include the set of two points ($-\sqrt{5}, +\sqrt{5}$), since they are the limit points of those real preimages.

52. If one follows the real axis Ox to the right direction, then one firstly traverses the R_1 along the segment ($0, +\sqrt{3}$), then $R'^{(-1)}_1$, after $R_1^{(-2)}$, $R'^{(-3)}_1$, $R_1^{(-4)}$ … and so on.
Since an evident rule is followed, the described segments constantly descend to ($+\sqrt{5}$): their limit point.

Then, after trespassing ($+\sqrt{5}$), one gets in R_∞. If one follows Ox' (the real axis in the left direction), then one intersects R'_1,$R_1^{(-1)}$, $R'^{(-2)}_1$, $R_1^{(-3)}$,$R'^{(-4)}_1$…
In the same way, it is easy to see that the preimages of the real axis intersect an infinity of regions belonging *alternatively* to the total domains of $z = 1$ and of $z = -1$; finally they enter in R_∞.

The regions $R_1^{(-i)}$, $R'^{(-i)}_1$ converge to zero in all their dimensions as z increases indefinitely.

Then one immediately realizes that, in the exterior of the circles Γ and Γ′, respectively centered at (+1) and (-1), with the same radius $\rho=\frac{1}{2}$, the value $\left|\frac{dz_1}{dz}\right| = \frac{3}{2}\left|1-z^2\right|$ is > M, where M is a number > 1.

In fact, on Γ and Γ′, one gets

$$\left|\frac{dz_1}{dz}\right| = \frac{3}{2}\left|1-z^2\right|\left|1+z\right| \geq \frac{3}{2}\frac{1}{2}\frac{3}{2} = \frac{9}{8}$$

and, evidently the curve

$$\left|\frac{dz_1}{dz}\right| = 1 = \frac{3}{2}\left|1-z\right|\left|1+z\right|$$

will be an algebraic curve consisting of two Cassinian ovals, surrounding respectively the points (-1) and (+1), where $\left|\frac{dz_1}{dz}\right| = 0$.

Therefore, on Γ and Γ′ and on their exterior, one gets

$$\left|\frac{dz_1}{dz}\right| \geq \frac{9}{8} > 1$$

One the contrary, none of $R_1^{(-i)}$ ($i = 1, 2, \ldots, \infty$) and $R'^{(-i)}_1$ encloses the critical point of $\psi(z)$. Therefore in the $R_1^{(-i)}$, as in $R'^{(-i)}_1$, $\psi(z)$ is holomorphic.

All those $R_1^{(-i)}$ and $R'^{(-i)}_1$ are exterior from the circles Γ and Γ′, lying inside $R_1^{(-i)}$ and $R'^{(-i)}_1$.

Therefore in each $R_1^{(-i)}$, one gets

$$\left|\frac{d\psi_1(z)}{dz}\right| < \frac{8}{9}$$

as in every $R'^{(-i)}_1$. Therefore, if S is the area ([1]), covered by $R_1^{(-1)}$, then every covered area by each one of $R_1^{(-2)}$ is $<\left(\frac{8}{9}\right)^2 S$, and then, in general, every region, covered by each one of $R_1^{(-i)}$, is $<\left(\frac{8}{9}\right)^{2(i-1)} S$.

Therefore it converges to zero as *i* increases indefinitely ([2]).

Every point of E′ is the boundary point of R_∞, it is the limit point of the regions $R_1^{(-i)}$ and of $R'^{(-i)}_1$, whose index *i* increases indefinitely.

Every point of E′ is the boundary point of $R_1^{(-i)}$ and $R'^{(-i)}_1$, or it is a boundary point of $R_1^{(-i)}$ and the limit point of the region $R'^{(-i)}_1$, whose index *i* increases indefinitely; or, finally, it is a boundary point of a region $R'^{(-i)}_1$ and the limit of a region $R_1^{(-i)}$, whose index *i* increases indefinitely.

The result is that a point of E′ is the limit for its preimages of any arbitrary point of R_∞, or R_1, or R'_1.

53. It is really difficult to depict exactly what this continuous E′ can be.

But it is possible to have an idea about it by generating a process, that I will explain soon: it shows that it is possible, for example, that a simply connected area R_∞ is bounded by a continuous line E′ including everywhere dense and multiple points.

The continuous E′ splits the plane in an infinity of regions, so that each one of them touches R_∞ for all its boundary; then the boundary of every small region is a simple curve.

It follows that it is evidently a scheme helping the intuition and the explanation of the paradox consisting of the existence of the boundary of R_∞, which is a continuous line intersecting the boundary of each one of the small previous regions in *all the points* of that last boundary, but *nevertheless coinciding with it*.

([1]) It is not absurd to talk about the area of $R_1^{(-1)}$, since R_1, for example, which is the limit, for $i \to \infty$, of the quarrable areas, each one is included in the following area; (Γ), (Γ_{-1}), …, (Γ_{-i}), … is quarrable. Therefore it is in the same as all $R_1^{(-i)}$, of all $R'^{(-i)}_1$ and of R_∞.

([2]) It is shown that, if L is the *biggest* chord of $R_1^{(-1)}$, then the biggest chord of $R_1^{(-i)}$ will be $\left(\frac{8}{9}\right)^{i-1} L$, and , it converges to zero as *i*.

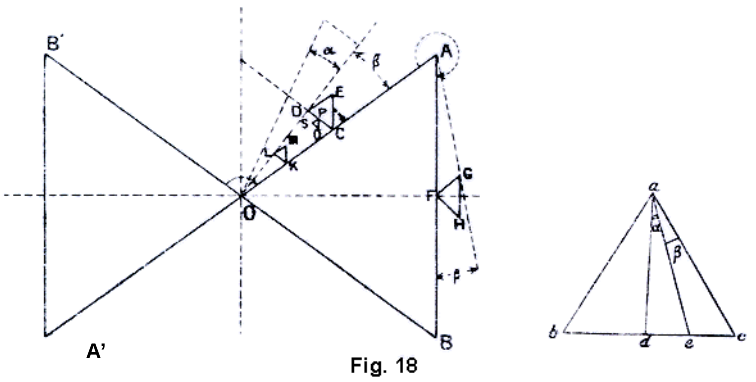


Fig. 18

I put the idea, behind the process I'm going to explain, from the nice Memoir of Helge von Koch, written in Swedish for *Acta Mathematica*, inserted in *Acta* of 1906 (t. XXX) under the title *Une méthode géometrique élémentaire…*

54. Let us start from two equilateral triangles, equal and opposed by a vertex (see pic.18). Let a be their side and OAB and OA′B′ the two triangles. The set of the two triangles makes a closed polygonal line P_1 (regular and with side a) OABOA′B′O, splitting the plane in three regions:

1. R_1, interior of OAB;
2. R'_1, interior of OA′B′;
3. a region including ∞ and bounded by the whole polygonal line OABOA′B′O.

O is a double point for the boundary of this last region. In the middle of each one of sides of the previous polygonal line P_1, let us set a vertex of an equilateral triangle with a side $\frac{a}{8}$, with parallel sides to the sides of the polygonal line P_1; each one of those triangles is in the region, bounded by P_1 and including ∞. CDE, FGH are two of those triangles. The set of P_1 and of the new built triangles makes a polygonal closed line P_2 whose vertexes follow in this order OCDECAFGHFB … .
P_2 splits the plane in 3 + 6 = 9 regions: R_1, R'_1, the interior of six triangles, which have just been constructed, finally the region including ∞ and bounded by P_2; for this last region, the boundary, which is the whole P_2, has some multiple (double) points in O, C, F, … It is useful to add that if one describes P_2, with the previous region (including ∞) at the left of those, then the angles, made by two consecutive angles of the line P_2, (the angles are counted to the interior of the region) are 120°, or 60°, or 300° (= 360° - 60°). (Examples: the angles O,C,A.).

55. In the middle of each side of the line P_2, let us set a vertex of an equilateral triangle whose the side is $\frac{1}{8}$ *of the side considered on the line* P_2 and whose sides are parallel to those sides of the line P_2, since each one of the new triangles lies in the region, bounded by P_2 and including the point at infinity (KLM, PQS are two of those triangles). The set of P_2 and of the new triangles makes a closed polygonal line P_3, which enjoys (for its angles) the same properties as P_2, but it determines in the plane a number of regions that is equal to the amount of regions, determined by P_2, increased with the number of sides of P_2. It is easy to see that *two of the new built triangles cannot share any common point* and that *they are all inside the region bounded by* P_2 *and including the point at infinity*.

For the generation of P_2, we remove this difficulty that already came out before.

Let us take the triangle CDE, built on the side OA of P_1. Since $CD = \frac{a}{8} = \frac{CO}{4}$, it is visible that the angle $C\hat{O}D < 30°$. (One generates, aside the figure, an equilateral triangle *abc*; $cd = \frac{1}{2}cb$ and $ce = \frac{1}{4}cb$, it is visible that $C\hat{O}D = c\hat{a}e = \beta < 30°$.)

Therefore, the triangle CDE cannot trespass on the analogous triangle, built on OB′. A difficult passage for P_2. For P_3, the matter is just to show that two triangles, built on two consecutive sides of P_2, since they make an angle of 60° within (counted interiorly in the region, bounded by P_2 and including ∞), cannot trespass on each other. For example, the triangle KLM and PQS, built on the side OC and CD of P_2.

It is easy to see that

$$L\hat{O}K = M\hat{C}K = \beta \qquad \text{due to} \qquad KL = \frac{OC}{8} = \frac{OK}{4}$$

In the same way

$$Q\hat{C}P = \beta \qquad \text{due to} \qquad PQ = \frac{CD}{8} = \frac{CP}{4}$$

Therefore the triangle KLM is interior for the angle $M\hat{C}O = \beta$ and the triangle PQS is interior for the angle $P\hat{C}Q = \beta$.

These two angles are symmetric in respect of the bisector of the angle $O\hat{C}D$, they are separated by that bisector and by an angle, equal to 2α ($\alpha = 30° - \beta$, *see* figure), split by the bisector.

The two triangles cannot trespass on each other. In relation to P_2, we have seen that the two triangles, built on the two consecutive sides of P_2 and making an angle of 120° within (such as the sides coming from O), cannot trespass on each other.

It is clear, on the other hand, that the triangles, built on the sides of P_2, so that DE or GH cannot trespass on other triangles.

All this may be resumed by the following simple remark: the triangle, built on the sides of the polygonal line P_1 or P_2 is entirely *inside* an isosceles triangle, whose base is that side itself and whose angle at the base is β.

Therefore it is very important that the angle β has been chosen < 30° so that the isosceles triangles (with the base angle β and built over two consecutive sides of P_2 as base) make an angle of 60° within; surely, these two triangles are reciprocally exterior and they share only the point of articulation of the two considered sides ([1]).

It is been proved that P_3 bounds a simply connected region, consisting of one piece and including the point at infinity; each one of the vertexes of that region is accessible from the interior of the region by every path lying in an angle of at least 2α, split by the bisector of the two sides of P_3, coming out from the considered vertex.

56. The process may keep on going indefinitely. In the middle of every side of P_i, we put the vertex of an equilateral triangle, whose side is $\frac{1}{8}$ of the considered P_i and whose sides are parallel to the sides of P_i, finally lying in the region, consisting of one piece and including the point at infinity and bounded by P_i.
The new triangles, joined with P_i, retrieve a closed polygonal line P_{i+1}.

It is demonstrated that the construction is progressively and always possible so that the each one of new triangles (built at every step) trespasses on each other or on those triangles that have been built at the previous steps of the entire process of construction.

The reason is immediate: if one considers the sides of P_3, then the new triangles, built on P_3, are inside the isosceles triangles (with the angle β at the base), built on that sides as bases.

The triangles, built on each one of the sides of P_3, are exteriors two by two; and, as a consequence, the new triangles, coming out from the step to go from P_3 to generate P_4, are still exterior two by two; and they do not trespass on the triangles of P_3.

On following, it is verified that the isosceles triangles, built on the sides of P_4 (with the angle β at base) as bases, are still exterior two by two ([1]).

Those triangles are in the region of the plane, consisting of one piece and bounded by P_4 and including ∞; on the following, the process may keep on going on P_4, as on P_3, to generate P_5.

([1]) For this purpose, I took the side of every new triangle, with its side = $\frac{1}{8}$ of the side of P_2, on which the triangle is built; given an angle β < 30°, I can replace the value of the denominator by any other number > 5 (because for 4 the angle β = 30°). I took 8 for simplicity of explanation.
([2]) Since the isosceles triangles are *inside* those triangles, built on the sides of P_2, it suffices to examine what happens in the interior of the other isosceles triangles, built on the sides of P_3; up there the verification is immediate.

57. Let us imagine that the process really keeps on going indefinitely.
The closed polygonal line P_i will have as limit (for $i \to \infty$) the closed continuous line ε' with double points everywhere dense on itself ([1]).

That points are like O, K, C, P, F, ..., that are evidently dense on every side of any P_i.
It happens that the length of the sides of the new triangles, added to P_i for obtaining P_{i+1}, converges to zero as the index i increases indefinitely.

The portion of the continuous line ε', enclosed within two consecutive vertexes of P_i, is entirely in the interior of the isosceles triangle with the base angle β, built on the side of P_i by joining the two vertexes to obtain the base. Then ε' splits the plane in a countable infinity of regions, which may be listed as following:

1. R_1 and R'_1.
2. The interior of the triangles to go from P_2 to P_3 ;
3. The interior of the triangles to go from P_3 to P_4 , … ;
4. Finally, a simply connected region R_∞, consisting of one piece and including the point at infinity and bounded by ε: it is the limit of the region including the point at infinity and bounded by P_i, for $i \to \infty$.

Every vertex of a line P_i (whatever i is) is a point of ε'; it is a *accessible point* ([2]) from the interior of R_∞ by all paths kept in an angle 2α that has the same bisector as the bisector of the considered angle of the line P_i (this bisector goes to the interior of R_∞). ε' is the same as the *derived* set of the countable set, consisting of all vertexes of the all lines P_i (as $i = 1, 2, \ldots, \infty$).
On ε', every point, that is a vertex of a line P_i where two sides of P_i make an angle of 60° within, is a *double* point of the boundary of R_∞. So O is a double point.
The double points of ε' are everywhere dense on ε'.
(Those vertexes of P_i play the role of the preimages of the origin in the iteration of $z_1 = \frac{-z^3+3z}{2}$) .

58. Therefore the continuous line ε', we are going to generate, enjoys all the properties of the set E′, related to the rational function

$$z_1 = \frac{-z^3+3z}{2} .$$

([1]) It is clear that ε' has all its points in the interior of the isosceles triangles with the base angle β and built on the sides of the two starting equilateral triangles, used like bases.
([2]) The meaning of the expression «boundary point of an *accessible* domain *from the interior of that domain*» is clear (*read*, for example, Preliminaries).

I'm going to correspond some letters to the elements which, in the iteration of $z_1 = \frac{-z^3+3z}{2}$ and in the previous construction, play the same role: so ε′ and E′, R_1 and R_1, R'_1 and R'_1, R_∞ and R_∞.
The triangles, added by the polygonal lines P_2, P_3, … added to the line P_1, may be analogous to the domains $R_1^{(-1)}$, $R'^{(-1)}_1$, $R_1^{(-2)}$, $R'^{(-2)}_1$, …

All the points ([1]) of the boundary of R_1 are points of the boundary ε′ of R_∞, so that ε′ does not coincide, in the considered region of the plane, with the boundary of R_1; ε′ includes that last boundary, but, on that last boundary, (it is necessary to remark because it is very important) ε′ has everywhere dense *double points*.
I do not need to insist on the novelty of circumstances coming out from this simple following example

$$z_1 = \frac{-z^3+3z}{2}.$$

This example shows what precaution is necessary to use, because no *a priori* suppositions can be made in problems of analysis and function theory.
The keenest concepts, about the boundaries of simply connected plane domains that recently have been treated by some interesting Memoirs ([2]), arise naturally and, after looking at the present circumstances, one may say, as I remarked before, that these examples of iterations, which have been dealt with up to now, are *the most simple ones*.

59. *Second example*. – In the last pages, I wrote in a general way what happens in the case of the application of the Newton's method to any algebraic equation.

If $f(z) = 0$ is any algebraic equation, since $f(z)$ is a polynomial, then it is said that the Newton's method, for finding find a root ζ of $f(z) = 0$, consists in proceeding by approximations as we start from a sufficiently near value to ζ; the image of z is defined by the rational relation

$$z_1 = z - \frac{f(z)}{f'(z)} \;;$$

the images of z are the approximations; their limit is a root ζ of $f(z) = 0$. Since $f(z)$ is any polynomial, let us examine therefore the iteration defined by

$$z_1 = z - \frac{f(z)}{f'(z)} = \varphi(z)\;;$$

([1]) It is evident that every point of the boundary of R_1 is simple on that boundary R_1, since R_1 is the interior of an equilateral triangle.
([2]) *Read* LINDELÖF, *Sur un principe general de l'Analyse…* (A*cta Soc.Sc. Fennicæ*, 1915), where a complete bibliography may be found. – *Read* MONTEL, *Sur la représentation conforme* (*Comptes Rendus* , June 4th 1917).

if *f* is of degree *k*, φ(*z*) is a rational function with degree *k*. If

$$f(z) = a_0 z^k + ...$$
$$f'(z) = k a_0 z^{k-1} + ...$$

and

$$z_1 = \varphi(z) = \frac{(k-1)a_0 z^k + ...}{k a_0 z^{k-1}}$$

The point $z = \infty$ is an infinite root of $z = \varphi(z)$: it is a double point of the transformation $z_1 = \varphi(z)$ of the plane. If one comes back to the origin by an auxiliary homographic mapping $z = \frac{1}{Z}, z_1 = \frac{1}{Z_1}$, then the transformed substitution of $z_1 = \varphi(z)$ is

$$Z_1 = \frac{k a_0 Z + ...}{(k-1)a_0 + ...}$$

so that both numerator and denominator have been ordered by the increasing powers of Z. One sees that Z = 0 corresponds to Z_1 = 0. Then, at the origin, $\frac{dZ_1}{dZ} = \frac{k}{k-1}$, therefore $\left|\frac{dz_1}{dz}\right|_0 > 1$. Therefore the origin is a point of the set E, in respect of $Z_1 = \Phi(Z)$.

Let us come back to $z_1 = \varphi(z)$: the infinity is a point of E for that substitution.
The equation $z = \varphi(z)$ has *k* roots at a finite distance; these are the *k* roots of *f*(*z*) = 0. In each one of the roots ζ, one gets

$$\frac{dz_1}{dz} = 1 - \frac{(f')^2 - f f''}{(f')^2} = \frac{f f''}{(f')^2},$$

and since *f*(*z*) = 0, one finally gets

$$\left(\frac{dz_1}{dz}\right)_\zeta = 0 .$$

Therefore the *k* roots are *k* limit points with uniform convergence and so the *k* critical points of the inverse algebraic function of

$$z_1 = \varphi(z) = z - \frac{f(z)}{f'(z)}$$

and so *k* points satisfying $\left(\frac{dz_1}{dz}\right) = 0$. Since the point at infinity is an ordinary point for the inverse of φ(*z*), one sees that, as well as the *k* roots of *f*(*z*), the only points of the extended plane, that are critical for the inverse of φ(*z*), are the images of (*k* - 2) points where *f''*(*z*) = 0. Then

$$\frac{d^2 z_1}{dz^2} = \frac{f''}{f'} + \frac{f f'''}{(f')^2} - 2\frac{f(f'')^2}{(f')^3} ,$$

and, in a root ζ of *f*(*z*) = 0, one gets

$$\frac{d^2 z_1}{dz^2} = \frac{f''(\zeta)}{f'(\zeta)} .$$

Therefore $\frac{d^2 z_1}{dz^2} \neq 0$ if, as it may be supposed in the general case, each one of the roots retrieves $f''(\zeta) \neq 0$. In a point where $f''(z) = 0$, one gets

$$\frac{d^2 z_1}{dz^2} = \frac{f f'''}{(f')^2}$$

and in the general case $\frac{d^2 z_1}{dz^2} \neq 0$ in that point.

Therefore in the general case the critical points of $\psi(z)$, the inverse of $\varphi(z)$, are:

1. The k roots of $f(z) = 0$ are some simple critical points, where the two branches of $\psi(z)$ permute around;
2. The images of rank 1 of (k - 2) roots of $f''(z) = 0$, which are simple critical points where the two branches of $\psi(z)$ permute around. This means that if $f(z)$ is any algebraic equation of degree k, then there are $2k$ - 2 distinct critical points for the algebraic function $\psi(z)$, the inverse of

$$z_1 = \varphi(z) = z - \frac{f(z)}{f'(z)} .$$

60. *a*. Firstly, let us get rid of the simplest case: when $f(z)$ is of the 2^{nd} degree.

If one define its two roots ζ_1 and ζ_2, then there are only two critical points of the inverse of $z = \varphi(z)$; in this case [because $f''(z)$ is a constant, since $f(z)$ has 2^{nd} degree], there are two limit points with uniform convergence.

It is a simple case, as it has been treated in the 3° application. Therefore the plane shall be split in two regions R_1 and R_2, separated by a continuous line; each one of those regions includes one of the limit points and it is the total domain of convergence (R_1 includes ζ_1, R_2 includes ζ_2) for that point.

For discerning those results, it is enough to remember that if one has

$$\frac{f'(z)}{f(z)} = \frac{1}{z-\zeta_1} + \frac{1}{z-\zeta_2} \quad ,$$

following, the iterated relation becomes

(1) $$\frac{1}{z-z_1} = \frac{1}{z-\zeta_1} + \frac{1}{z-\zeta_2}$$

since z_1 is the image of z. The conclusion is: *if z describes the perpendicular line in the middle of the segment $\zeta_1 \zeta_2$, then z_1 describes it too*.

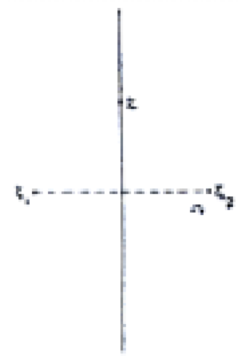


Fig. 19.

The simplest case is to apply an auxiliary homographic mapping, acting like a simple displacement of the plane z to take both ζ_1 and ζ_2 on the real axis, so that they are symmetric in respect of the imaginary axis; it is clear that one has the same relation as (1) between the transformed of the points z, z_1, ζ_1, ζ_2 .

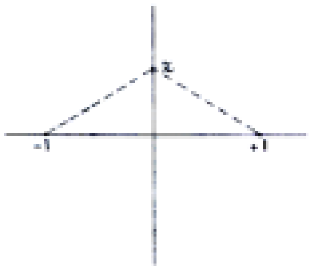


Fig. 30.

Therefore nothing may stop us to suppose that ζ_1 is real and > 0 and that $\zeta_2 = - \zeta_1$.
We do not restrict the generality of this case, if we suppose $\zeta_1 = 1$ (it suffices to apply an auxiliary homothethy to get to this value). Therefore, the problem is just to study

$$\frac{1}{z-z_1} = \frac{1}{z-1} + \frac{1}{z+1}$$

Then, if z is an imaginary value (i.e.: lies on the imaginary axis), then $\frac{1}{z-1}$ and $\frac{1}{z+1}$ are two symmetric points in relation to Oy.

Therefore their sum will be a point of Oy. Therefore $\frac{1}{z-z_1}$ lies on the imaginary axis, like z and so like z_1.

Therefore the perpendicular line in the middle of $\zeta_1\zeta_2$ is preserved by the Newton's relation

$$z_1 = z - \frac{f(z)}{f'(z)}$$

It is still easy to prove that each one of the half-planes, determined by the relation, is transformed in itself.

Then we come back to the case of the fractions in the fundamental circle, which Fatou considered in its Note, published on May 21th 1917 in *Comptes Rendus*.
The simplest case consists of coming back to the expression

$$z_1 = z - \frac{f(z)}{f'(z)}$$

where one supposes that $f(z) = z^2 - 1$; an auxiliary homographic relation mapping both ζ_1 and ζ_2 to the points (-1) and (+1) respectively, allows to come back to the expression above. Given

$$z_1 = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and, for that expression, it is clear that each of the half-planes, determining the perpendicular line in the middle of $\zeta_1\zeta_2$, is mapped to itself by the substitution

$$z_1 = \varphi(z) = z - \frac{f(z)}{f'(z)} \ ;$$

each half-plane is the total domain of convergence to the limit point (ζ_1 or ζ_2), included by the half-plane itself. On the common boundary of the two domains, which is the common perpendicular line in the middle of $\zeta_1\zeta_2$, the two root points of the equation $z = \varphi_n(z)$ ($n = 1, 2, \ldots, \infty$), satisfying $|\varphi'_n(z)| > 1$ (the set E) are everywhere dense.

61. *b*. If one takes a general polynomial $f(z)$ of degree k, then the fraction

$$z_1 = \varphi(z) = z - \frac{f(z)}{f'(z)}$$

admits the distinct k roots of $f(z) = 0$ as limit points with uniform convergence.
The function $\psi(z)$, the inverse of $\varphi(z)$, admits the k roots of $f(z) = 0$ and those $(k - 2)$ roots of $f''(z)$ as critical points which are in general distinct from the roots of other previous functions. It follows that *at least* (k - 1) *roots of* $f(z)$ *have a total domain of convergence consisting of an infinity of areas* like those areas considered in the first example.

62. If, for example, one considers the polynomial $f(z) = z(z^2 - 1)$ whose root points are 0, -1, +1, then one gets

$$z_1 = z - \frac{z(z^2-1)}{3z^2-1} = \frac{2z^3}{3z^2-1}$$

and if one transforms this substitution by an auxiliary homographic mapping

$$z = \frac{1}{Z}, \qquad z_1 = \frac{1}{Z_1},$$

the result is

$$Z_1 = \frac{3Z - Z^3}{2},$$

that's to say the *first example we dealt with* : so we won't insist anymore on it.

Therefore in general the splitting of the plane in regions, whose points take straight to the same determined root of $f(z) = 0$, is an impracticable problem since there are at least (k - 1) roots whose domain of convergence consists of an infinity of areas: so we would be obliged to split the plane in an infinity of regions.

That's why Cayley failed after applying the Newton's method to the equations with degree $k \geq 3$.

63. Finally, let us deal with the case of $f(z) = z^3 - a^3$. Then

$$z_1 = \frac{2z^3 + a^3}{3z^2} = \varphi(z)$$

The three roots of f are a, $a\omega$ and $a\omega^2$ ($\omega^3 = 1$): three limit points with uniform convergence and so three critical points of the inverse of $\varphi(z)$.
The fourth critical point of $\psi(z)$ is evidently the point at infinity, because the point at infinity has two preimages at the origin. This is a point of E′. Therefore, there′s no periodic cycles or any other limit point with uniform convergence.
The immediate domain of convergence to each one of the root points a, $a\omega$ and $a\omega^2$ won't include in its interior any other critical point of $\psi(z)$ but the point itself that permutes only two branches of $\psi(z)$. Therefore, each one of the total domains of convergence to a, $a\omega$ and $a\omega^2$ consists of an *infinity of pieces*, like the previous example (a domain of convergence to –1 and one to +1).

Each one of the total domains generates the other two domains by two rotations of 120° around the origin because z_1 and $z_1\omega$ are the consecutives of z and $z\omega$, respectively.
The continuous line E′, generating the set of the boundaries of the previous total domains (evidently, every total domain has the whole set E′ as boundary, because every point of E′ is the limit for the preimages of any arbitrary point belonging to each one of those domains), is preserved by a rotation of 120° around the origin.
By an simple homographic mapping ($z = aZ$), one comes back to $a^3 = 1$.
Then, without restricting the generality of this case, one takes

$$z_1 = \frac{2z^3 + 1}{3z^2} .$$

The three limit points are 1, ω, ω^2.

A priori, one realizes that the positive part of the real axis belongs to the immediate domain to (+1), since both O and ∞ are boundary points. The real axis is a symmetry axis for the considered domain. Now, we have some further informations about the domains of the points 1, ω, ω^2.

64. *Third example.* – In conformity with the considerations, explained in § 42-47 of this Memoir, let us show that the immediate domain of convergence to a limit point can be bounded by an infinite set of *curves*, exterior two by two, and not only by points as we showed in the examples $z_1 = 2z^k + 1$, previously mentioned.

Immediately, I explain an example about it by the following process: I take a polynomial $z_1 = \varphi(z)$, where $\varphi'(z) = 0$ has only two distinct roots with a finite distance, 0 and a.
I take a *real value* and I arrange it so that $z = a$ is a point belonging to the immediate domain of the point ∞; for any polynomial, the point ∞ is related to an immediate domain of convergence R_∞ confounding with the total domain.
If a belongs to R_∞, then it is clear that R_∞ is not simply connected, because R_∞ includes two critical points of $\psi(z)$, ∞ and the image of a; a contour, described around only these two critical points ([1]), does not permute any determination of $\psi(z)$ with *all others* determinations of $\psi(z)$.

In fact, the Riemann surface R, related to the inverse function $\psi(z)$ of a polynomial with degree k, may be generated by taking some lines linking the critical points of $\psi(z)$ and lying at a finite distance to the point R_∞, behaving like crossing lines between the different layers.
Now, some considerations can allow us to realize what I'm going to explain.

If I immediately impose that the point O is a limit point with uniform convergence by assuming that $\varphi(0) = 0$ and $\varphi'(0) = 0$, then this point has an immediate domain of convergence R_0, simply connected and bounded by a continuous line belonging to the boundary of R_∞.

It is clear that the domain of convergence to zero consists of an infinity of preimage regions of R_0: all of them are simply connected, exterior from R_∞ and each one of them is bounded by a continuous line.
The set of all these continuous lines and of all their limit points generates the boundary of R_∞.

So R_∞ is bounded by an infinity a continuous lines (and their limit points), that would not generate a continuum as they would be joint.

Between two points belonging to two distinct continuous lines, the boundary of R_∞ is badly linked; one may trace, in the domain R_∞, an infinity of closed curves, exterior two by two and including those continuous lines in their interior (and in their exterior).

([1]) There is no possible ambiguity about the sense of this expression.

The generation of R_∞, starting from a small domain surrounding the point ∞, brings all in evidence, like for the example:

$$z_1 = 2z^k + 1 \ ;$$

but in this example the limit curves of the preimages of that small domain converge to zero in all their dimensions and those curves are reduced to a set of points whose the derived set is the perfect and discontinuous set E′.

65. The accomplishment of the example is easy.
Simply I take

$$\varphi'(z) = Az^2 (z - a)^2 .$$

since A is a real positive number to be determined and a is a positive arbitrary, but fixed, number.
Then, I take

$$\varphi(z) = A\left(\frac{z^5}{5} - 2a\frac{z^4}{4} + a^2\frac{z^3}{3}\right)$$

to get

$$\varphi(0) = 0, \qquad \text{with} \qquad \varphi'(0) = \varphi''(0) = 0$$

The point O is a limit point with uniform convergence.

This may happen since the images of a converge to infinity.
The curve $z_1 = \varphi(z)$ has this behaviour since both z and z_1 are considered the rectangular coordinates of a point.

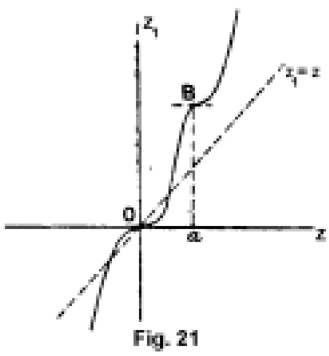


Fig. 21

It suffices that the portion of the curve, corresponding to the variation of z in (a;+∞), is all entire *above the bisector* $z_1 = z$, so that one is assured that the images of a have abscissas increasing regularly to +∞.

Now, it's easy: it is necessary to choose A to be enough big to have, for $z \geq a > 0$,

$$\frac{z^5}{5} - 2a\frac{z^4}{4} + \frac{a^2z^3}{3} > \frac{z}{A}$$

Now, since the curve

$$Z_1 = \frac{z^5}{5} - 2a\frac{z^4}{4} + a^2\frac{z^3}{3}$$

has a *well determined* shape, analogous with the previous one, then it is always possible to choose $\frac{1}{A}$ to be both small enough and positive so that the straight line, with an angular coefficient $\frac{1}{A}$ and coming out from the origin, is entirely above the portion of this curve

$$Z_1 = \frac{z^5}{5} - 2a\frac{z^4}{4} + a^2\frac{z^3}{3}$$

which corresponds to the variation of z in $(a, +\infty)$.

66. After choosing A, then

$$z_1 = \varphi(z) = A\left[\frac{z^5}{5} - 2a\frac{z^4}{4} + a^2\frac{z^3}{3}\right]$$

has 0 and ∞ as limit points with uniform convergence.

There are no other points $z = \varphi(z)$, satisfying $|\varphi'(z)| < 1$, nor any periodic cycle.
The equation $\varphi'(z) = 0$ has two distinct roots: 0 and a.

$\psi(z)$, the inverse of $\varphi(z)$, has two critical points, 0 and $\frac{Aa^5}{30}$ at a finite distance.

Around O, three determinations of $\psi(z)$ permute; around the point $z = \frac{Aa^5}{30}$, three determinations permute.

Three upper layers (1, 2, 3) may be imagined to branch around the origin $z = 0$ and the third, fourth and fifth branching in $z = \frac{Aa^5}{30}$. Five layers are branching in $z = \infty$. The cross lines may be traced from 0 to $-\infty$ and from $z = \frac{Aa^5}{30}$ to $+\infty$ on the real axis. If we go around $z = \frac{Aa^5}{30}$, then one moves from the third to fourth, from fourth to fifth, from firth to third, an so on. If one starts from a point of the first layer by describing in the negative sense a circuit surrounding the two critical points, then one goes on the fifth layer, etc.

67. Since the five layers branch around the point ∞, the domain R_∞ is at the same time the immediate domain and the total domain of convergence to infinity.
R_∞ consists of one piece and it is infinitely connected.

If we starting from a curve c, delimiting the area (c) around the point at infinity and if we take the preimages, then one of those areas delimits a region (C_{-i}) including the

infinity and $z = \frac{Aa^5}{30}$; the preimages of $(C_{-(i+1)})$ … have a bigger and bigger order of connection.
All the areas (C_{-p}) leave zero at their exterior.

R_0, the domain of convergence to zero, includes the only critical point 0 of $\psi(z)$.
R_0 is simply connected. R_∞ leaves R_0 at its exterior, but all the boundary points of R_0 are boundary points of R_∞ too.

The result is that all those points are points of E′ and since R_∞ consists of one piece and it is the total domain of convergence to infinity, then the boundary of R_∞ *is the same as the set* E′, because each point of E′, which is the limit for its preimages of a point of R_∞ (all those preimages are inside R_∞), shall be a boundary point of R_∞.

68. R_0 intersects the Riemann surface R of $\psi(z)$:

1. A simply connected piece S_0, enclosing the portions (branching within themselves around 0) of three upper layers projecting in the interior of R_0;
2. Two other simply connected pieces (the same as R_0) that are parts of the fourth and fifth layers projecting on R_0;

When z describes R_0, three of its preimages describe R_0; the other two layers, corresponding to the fourth and the fifth determination of $\psi(z)$, describe a simply connected area, outside R_0: these two areas do not share any *common point in the interior or on the boundary* nor within themselves, nor with R_0.

These areas are called $R_0^{(-1)}$. One may trace a simple contour, surrounding R_0, either by leaving in its exterior the two areas $R_0^{(-1)}$ and a contour surrounding one of the areas $R_0^{(-1)}$, either by leaving the other one and R_0 at its exterior.

As one keeps on going, one determines the areas $R_0^{(-2)}$, the preimages of $R_0^{(-1)}$, etc.
All the areas $R_0^{(-i)}$ are simply connected: none of them includes a critical point of $\psi(z)$; all areas are at a finite distance.

They conglomerate within themselves ([1]), so that each boundary point of an area $R_0^{(-i)}$, (since it is a point of E′) is a limit point of the preimages of the continuous boundary lines of the areas $R_0^{(-i+1)}$, $R_0^{(-i+2)}$, …: all exterior areas of the examined area $R_0^{(-i)}$.

It is not useful to insist anymore on a generating process that we already know.

So the domain R_∞ seems a pierced plane: an infinity of holes at a finite distance and exterior two by two; their boundary curves conglomerate to generate a perfect set, since every point of a boundary curve is a limit point of an infinity of smaller and smaller boundary curves in the exterior of the first curve.

The total domain of convergence to the origin is the set of the interior areas in all that holes.

69. *Fourth example.* – We have seen (§ 62-63), about the example $z_1 = \frac{2z^3+1}{3z^2}$ taken from the Newton's method, that a critical point of the function ψ(*z*), the inverse of φ(*z*), is a image of a point, satisfying φ′(*z*) = 0, may belong to the set E′.

Since we write that a polynomial of second degree, where the second image of φ′(*z*) = 0 is a double point, one falls on this example

$$z_1 = az^2 - \frac{2}{a},$$

where *a* is any arbitrary parameter; as we fix *az* = Z, then we get

$$Z_1 = Z^2 - 2$$

and this is an interesting example.

One see immediately that Z = 2 is a point of E, *its preimage* (-2) *is a critical point of* ψ(*Z*) *and it is a point of* E′. One sees that *all the preimages* of the point Z = 2 are reals and ranging from –2 and +2. Every point of the segment (-2, +2) has some images *that never escape from that segment*.

Therefore the segment (-2, +2) does not belong to the domain of the point at infinity. Since Z = 2 belongs to E, its preimage are dense everywhere on E′ which is therefore the derived set of the set of the preimages of Z = 2.

Therefore E′ has all its points on the segment (-2,+2).
Every point of the plane, lying out of the segment, has therefore some images converging uniformly to infinity.

The domain R_∞ consists of the entire plane but the segment (-2,+2).
In conclusion the segment (-2,+2), the complete boundary of R_∞, is the set E′ itself.

The preimages of Z = 2, like every point belonging to that segment, are everywhere dense on that segment.

This example shows that E′ can be a continuous line with two distinct extreme points.

Up to now, I've given only examples of the sets E′, which are ashaped as *continuous closed lines*.

([1]) The points of a perfect discontinuous set are grouping by an analogous process of the following one.