Chapter III
Stability of the Equilibrium States

In this chapter, we first recall the known results on the eigenvalues of linear completely continuous fields. These results are basic for the studies of the asymptotically stability of equilibrium points and the bifurcation of invariant sets of the nonlinear evolution equations with dissipative structure. We also obtain an asymptotical property for the eigenvalues of the linear completely continuous fields with dissipative structure. In section 3.2, we consider the local asymptotical stability in the regularity sense, and give a global stability result for the abstract nonlinear evolution equations. We also discuss the local asymptotical stability for the abstract parabolic and hyperbolic differential equations possessing the variational structure. Finally, in section 3.3 we give some applications of the abstract results to the partial differential equations.

3.1. Eigenvalues of Linearized Operators

3.1.1. Motivation

We begin with a simple example given by

$$\left\{\begin{array}{l} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + w^2 + \lambda f(x), \quad x \in (0,1) \subset R \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = \phi(x) \end{array}\right. \tag{3.1.1}$$

where λ is a parameter, and $f \neq 0$ a given function. Let $u_\lambda(x)$ be a solution of the below stationary equation

$$\left\{\begin{array}{l} \frac{\partial^2 v}{\partial x^2} + v^2 + \lambda f(x) = 0, \quad x \in (0,1) \\ v(0) = v(1) = 0 \end{array}\right. \tag{3.1.2}$$

It is known that there exists $\Lambda \leq \infty$, as $0 \leq \lambda < \Lambda$ the steady states v_λ of (3.1.1) exist, and there exists a number $\lambda_0 (0 < \lambda_0 \leq \Lambda)$ such that v_λ is locally asymptotically stable, i.e. there is a neighborhood $U \subset L^2(0,1)$ of v_λ, for any

initial value $\phi \in U$, the solution $w(x,t,\phi)$ of (3.1.1) is convergent to v_λ in $L^2(0,1)$,

$$\lim_{t\to\infty} \|w - v_\lambda\|_{L^2} = 0$$

However, when $\lambda_0 < \lambda$, the stability will be lost.

Naturally, we concern the basic problems that from which values of λ the steady state v_λ will lose its stability, and if λ_0 is the point that v_λ loses its stability for $\lambda_0 < \lambda$, then what behaviors of the solution w of (3.1.1) near v_λ will occur for $\lambda - \lambda_0$ small enough. The problems are related with the stability of equilibrium states, static multiple solution bifurcation, and the dynamic attractor bifurcation of nonlinear evolution equations, which are the center topics of this chapter and the forthcoming chapters.

With the stability and bifurcation probelms in mind, we transform the equations (3.1.1) and (3.1.2) into some appropriate forms. Let $u = w - v_\lambda$, then from (3.1.1) and (3.1.2) we obtain

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2v_\lambda u + u^2 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \psi(x), \qquad\qquad \psi = \phi - v_\lambda \end{cases} \tag{3.1.3}$$

and

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + 2v_\lambda u + u^2 = 0 \\ u(0) = u(1) = 0 \end{cases} \tag{3.1.4}$$

In this case, the stability and bifurcation problems of (3.1.1) near v_λ is equivalent to these of (3.1.3) near $u = 0$. Now, we shall analyse the relation between the stability of (3.1.3) and the eigenvalues of the linearized operators in an abstract form. Let

$$\begin{cases} H_1 = W^{2,2}(0,1) \cap W_0^{1,2}(0,1) \\ H_{\frac{1}{2}} = W_0^{1,2}(0,1) \\ H = L^2(0,1) \end{cases}$$

We defined the mappings $L = -A + B : H_1 \to H$ and $G : H_1 \to H$ by

$$\begin{cases} Au = -\frac{\partial^2 u}{\partial x^2} \in H, & \forall u \in H_1 \\ Bu = 2v_\lambda u \in H, & \forall u \in H_1 \\ Gu = u^2 \in H, & \forall u \in H_1 \end{cases}$$

It is known that $A : H_1 \to H$ is a linear homeomorphism, $B : H_1 \to H$ is a linear compact operator, and $G : H_1 \to H$ is a compact operator with $Gu = 0(\|u\|_{H_{\frac{1}{2}}})$ near $u = 0$. Thus the equation (3.1.3) can be written as the abstract form

$$\begin{cases} \frac{du}{dt} = Lu + Gu \\ u(0) = \phi \end{cases} \tag{3.1.5}$$

here $L : H_1 \to H$ is the linearized operator of (3.1.5).

By the Sturm-Liouville theorem, there are an infinite eigenvalue sequence $\{\lambda_k\} \subset R$ with $\lambda_k \to +\infty$(as $k \to \infty$) and an eigenfunction sequence $\{\phi_k\} \subset H_1$ such that

$$L\phi_k = -\lambda_k I\phi_k$$

where $I : H_1 \to H$ is the inclusion embedding mapping, and $\lambda_k = \lambda_k(\lambda)$ are the continuous functions of λ. Therefore the equation (3.1.5) near $u = 0$ can be expressed by

$$\begin{cases} \frac{dx_k}{dt} = -\lambda_k x_k + o(\|u\|_{\frac{1}{2}}) \\ x_k(0) = \phi_k =< \phi, \phi_k >_H \end{cases} \tag{3.1.6}$$

where $u = \sum_{k=1}^{\infty} x_k\phi_k$. We know that if $0 < \lambda_1 < \lambda_2 < \cdots$, then

$$\|u\|^2_{\frac{1}{2}} =< -Lu, u >_H= \sum_{k=1}^{\infty} \lambda_k x_k^2$$

Thus, from (3.1.6) it follows that

$$\frac{d}{dt}\|u\|^2_H \leq -\frac{1}{2}\|u\|^2_{\frac{1}{2}} \leq -\frac{\lambda_1}{2}\|u\|^2_H, \tag{3.1.7}$$

for $\|u\|_{\frac{1}{2}}$ small enough. Hence, from (3.1.7) it follows that if there is a number $\lambda_0(0 < \lambda_0 < \Lambda)$ such that

$$\lambda_1(\lambda) = \begin{cases} < 0 & as\ 0 < \lambda < \lambda_0 \\ = 0 & as\ \lambda = \lambda_0 \\ > 0 & as\ \lambda > \lambda_0 \end{cases}$$

then $u = 0$ is locally asymptotically stable for $0 < \lambda < \lambda_0$, and will lose its stability as $\lambda_0 < \lambda$.

This example illustrates that in order to study the stability and bifurcation problems, it is necessary to investegate the eigenvalue problem of the linearized operators.

3.1.2. Eigenvalues of linear completely continuous fields

Let H, H_1 be the Hilbert spaces, and $H_1 \hookrightarrow H$ be a compact and dense embedding, $I : H_1 \to H$ be the inclusion embedding mapping which is compact.

Definition 3.1.1. A number $\lambda = \alpha + i\beta \in C$ is called a eigenvalue of a linear mapping $L : H_1 \to H$, if there exist $x, y \in H_1$ with $x \neq 0$ such that

$$Lz = \lambda Iz\ (z = x + iy)$$

where $Lz = Lx + iLy, Iz = Ix + iIy$, and the couple $\{x, y\}$(or $z = x + iy$) is called the eigenvector corresponding to λ.

It is clear that if $\lambda = \alpha + i\beta \in C$ is an eigenvalue of L, and the couple $\{x, y\}$ is the corresponding eigenvector, then

$$\begin{cases} Lx = \alpha Ix - \beta Iy \\ Ly = \beta Ix + \alpha Iy \end{cases}$$

Let $E \subset H_1$ be a linear subspace. We say that E is an invariable subspace of $L : H_1 \to H$, if $LE \subset IE$

Let $\lambda \in C$ be an eigenvalue of $L : H_1 \to H$. The space

$$E_\lambda = \{x, y \in H_1 | (L - \lambda I)^n z = 0, z = x + iy, n \in N\}$$

is called an eigen-subspace of L corresponding to λ. It is clear that an eigen-subspace E_λ is an invariable subspace of L.

Definition 3.1.2. A linear mapping $L : H_1 \to H$ is called to be symmetrical, if for any $x, y \in H_1$

$$< Lx, Iy >_H = < Ly, Ix >_H$$

Definition 3.1.3. Let $L : H_1 \to H$ be a linear bounded mapping. We say that L has a complete eigenvalue sequence $\{\lambda_k\} \subset C$, if each eigen-subspace E_{λ_k} is finite dimensional, and H_1 can be decomposed into the direct sum $H_1 = \bigoplus_{k=1}^{\infty} E_k$.

A linear bounded mapping $L = -A + B : H_1 \to H$ is called a completely continuous field, if

$$A : H_1 \to H \textit{ is a linear homeomorphism}$$
$$B : H_1 \to H \textit{ is a linear compact operator.}$$

In this chapter, the following assumption is basic that there exist real eigenvalue sequence $\{\beta_k\} \subset R$, and eigenvector sequence $\{\phi_k\} \subset H_1$ of A,

$$\begin{cases} A\phi_k = \beta_k I\phi_k \\ 0 < \beta_1 \leq \beta_2 \leq \cdots; \quad \beta_k \to +\infty (k \to \infty) \end{cases} \tag{3.1.8}$$

such that $\{\phi_k\}$ and $\{I\phi_k\}$ are the orthogonal base of H_1 and H respectively, i.e. A is symmetrical and positive definite.

Because a linear completely continuous field L is a Fredholm operator with zero index, the spectrum of L consists of eigenvalues.

Theorem 3.1.4. Let $L = -A + B : H_1 \to H$ be a linear completely continuous field which satisfies (3.1.8). Then we have the below conclusions.

i). Each eigenvalue $\lambda \in C$ of L is isolated, and the corresponding eigen-subspace E_λ is finite dimensional.

ii). If L has an infinite eigenvalue sequence $\{\lambda_k\} \subset C$, then $|\lambda_k| \to \infty$, and if $|Re\lambda_k| \to \infty$, then $Re\lambda_k \to -\infty$.

iii). If L is symmetrical, then L has a complete real eigenvalue sequence $\{\lambda_k\} \subset R$, and for the corresponding eigenvectors $\{\phi_k\}, \{I\phi_k\}$ are the orthogonal base of H.

Although this theorem is essencially known, for the sake of completion, here we shall give a proof. To this end, we need to introduce a lemma. For a Hilbert space H, one can define an almost complex Hilbert space $\widetilde{H}$, which is induced by H, as follows.

$$\widetilde{H} = C \bigotimes H = \{x + iy | x, y \in H\}$$

endowed with the inner product

$$< z_1, z_2 >_{\widetilde{H}} = < x_1, x_2 >_H + < y_1, y_2 >_H$$

for all $z_k = x_k + iy_k \in \widetilde{H}, k = 1, 2$. Let $\widetilde{H}, \widetilde{H_1}$ be the almost complex Hilbert spaces induced respectively by H and H_1, and $\widetilde{L} : \widetilde{H_1} \to \widetilde{H}$ induced by $L : H_1 \to H$ be defined as

$$\widetilde{L}z = Lx + iLy, \quad \forall z = x + iy \in \widetilde{H_1}$$

Lemma 3.1.5. Under the condition (3.1.8) there is a constant $\lambda_0 \in R$ such that

$$\widetilde{L} - (\alpha + i\beta)\widetilde{I} : \widetilde{H_1} \to \widetilde{H}$$

has a bounded inverse for all $\alpha > \lambda_0$ and $\beta \in R$.

Proof. It suffices to prove that $\widetilde{L} - (\alpha + i\beta)\widetilde{I} : \widetilde{H_1} \to \widetilde{H}$ are one to one for all $\alpha \in R$ large enough. Let $\alpha, \beta \in R$ and $z_0 \in \widetilde{H_1}$ satisfy

$$\widetilde{L}z_0 - (\alpha + i\beta)\widetilde{I}z_0 = 0 \tag{3.1.9}$$

By (3.1.8),$-\widetilde{A} - \alpha\widetilde{I} : \widetilde{H_1} \to \widetilde{H}$ have bounded inverse

$$(-\widetilde{A} - \alpha\widetilde{I})^{-1}z = -\sum_{k=1}^{\infty}(\alpha + \beta_k)^{-1}x_k\phi_k - i\sum_{k=1}^{\infty}(\alpha + \beta_k)^{-1}y_k\phi_k \tag{3.1.10}$$

for all $\alpha+\beta_k \neq 0 (k=1,2,\cdots)$, where $z=\sum_{k=1}^{\infty} x_k I\phi_k + i\sum_{k=1}^{\infty} y_k I\phi_k$. Hence we have

$$z_0 + (-\widetilde{A}-\alpha\widetilde{I})^{-1}\widetilde{B}z_0 = i\beta(-\widetilde{A}-\alpha\widetilde{I})^{-1}\widetilde{I}z_0 \tag{3.1.11}$$

We notice that for any linear bounded symmetrical operator $\widetilde{E}:\widetilde{H_1}\to\widetilde{H_1}$, the below equality holds

$$< i\widetilde{E}z, z >_{\widetilde{H_1}} = < -Ey + iEx, x+iy >_{\widetilde{H_1}}$$

$$= < Ex, y >_{H_1} - < Ey, x >_{H_1} = 0$$

Because $\{\phi_k\}$ is on orthogonal base of H_1, from (3.1.10) it follows that the mapping $(-\widetilde{A}-\alpha\widetilde{I})^{-1}\widetilde{I}:\widetilde{H_1}\to\widetilde{H_1}$ is symmetrical. Thus, by (3.1.11) we get

$$0 = \|z_0\|^2_{\widetilde{H_1}} + < (-\widetilde{A}-\alpha\widetilde{I})^{-1}\widetilde{B}z_0, z_0 >_{\widetilde{H_1}}$$

$$\geq \|z_0\|^2_{\widetilde{H_1}} - \|(-\widetilde{A}-\alpha\widetilde{I})^{-1}\|\ \|\widetilde{B}\|\ \|z_0\|^2_{\widetilde{H_1}} \tag{3.1.12}$$

On the other hand, from (3.1.10) one can see that

$$\lim_{\alpha\to+\infty} \|(-\widetilde{A}-\alpha\widetilde{I})^{-1}\| = 0 \tag{3.1.13}$$

Consequently, from (3.1.12) and (3.1.13) we infer that (3.1.9) only has the zero solution $z_0=0$ for all $\alpha\in R$ large enough. This proof is complete.

Proof of Theorem 3.1.4. It is clear that the eigenvalues of $L=-A+B: H_1\to H$ coincide with those of $\widetilde{L}=-\widetilde{A}+\widetilde{B}:\widetilde{H_1}\to\widetilde{H}$. By Lemma 3.1.5, we take $\alpha>0$ large enough such that $\widetilde{L}-\alpha\widetilde{I}:\widetilde{H_1}\to\widetilde{H}$ has bounded inverse. Hence the eigenvalue problem of $\widetilde{L}$ is equivalent to that of the linear compact operator $(\widetilde{L}-\alpha\widetilde{I})^{-1}\widetilde{I}:\widetilde{H_1}\to\widetilde{H_1}$, i.e. equivalent to

$$(\widetilde{L}-\alpha\widetilde{I})^{-1}\widetilde{I}z = \lambda z, z\in\widetilde{H_1}, \lambda\in C.$$

By the Riesz-Schauder theorem and Lemma 3.1.5, the claims i)-ii) of Theorem 3.1.4 is proved.

If L is symmetrical, then $L-\alpha I$ is also symmetrical, and if λ is an eigenvalue of L, then $\lambda+\alpha$ is an eigenvalue of $L-\alpha I$ thereby, without loss of generality, we assume that $L:H_1\to H$ is invertible, otherwise we may consider the operator $L-\alpha I$. Thus $\|x\|=\|Lx\|_H$ is a norm defined on H_1, which is equivalent to the H_1-norm. We take $< Lx, Lx >_H$ as the inner product of H_1, then under

the norm $\|Lx\|_H$, we have

$$\begin{aligned}
&< L^{-1}Ix, y >_{H_1} = < L(L^{-1}Ix), Ly >_H \\
&= < Ix, Ly >_H \\
&= < Lx, Iy >_H \quad (by\ the\ symmetry\ of\ L) \\
&= < Lx, L(L^{-1}Iy) >_H \\
&= < x, L^{-1}Iy >_{H_1}
\end{aligned}$$

Hence $L^{-1}I : H_1 \to H_1$ is symmetrical. Because $L^{-1}I : H_1 \to H_1$ is one to one and compact, by the eigenvalue theory of linear symmetrical compact operators, $L^{-1}I$ has a complete real eigenvalue sequence $\{\lambda_k^{-1}\}$ with $\lambda_k^{-1} \to 0 (k \to \infty)$, and the eigenvector sequence $\{\phi_k\}$ under the inner product $< x, y >_{H_1} = < Lx, Ly >_H$ is an orthogonal base of H_1. Therefore the claim iii) follows. The proof is complete.

3.1.3. Analytic semigroups of completely continuous fields

Let $L = -A + B : H_1 \to H$ be a completely continuous field. If we regard $H_1 \subset H$ as the domain of L, then L is an unbounded operator defined on H. Moreover, L is closed. To see this, let $\{x_n\} \subset D(L) = H_1$ which satisfies

$$\begin{cases} x_n \to x\ in\ H \\ Lx_n = y_n \to y\ in\ H \end{cases} \tag{3.1.14}$$

By Lemma 3.1.5, there is a $\alpha \in R$ such that $\alpha I - L : H_1 \to H$ is invertible. Hence we have

$$x_n = (\alpha I - L)^{-1}(\alpha x_n - y_n) \tag{3.1.15}$$

From (3.1.14) and (3.1.15) it follows that $x_n \to x$ in H_1, which implies the closedness of L.

According to the theory of semigroups of linear operators, a completely continuous field L with the condition (3.1.8) is the generator of a strong continuous semigroup. Furthermore, we shall show that L is the generator of an analytic semigroup.

We first introduce the fractional powers of the operators for the homeomorphism $A : H_1 \to H$ which satisfies (3.1.8)(cf. [Pa] and [Te]). Let

$$H_\alpha = \{x \in H |\ x = \sum_{i=1}^{\infty} x_i \phi_i,\ and\ \sum_{i=1}^{\infty} \beta_i^{2\alpha} x_i^2 < \infty\},\ (0 \le \alpha < \infty)$$

$$< x, y >_{H_\alpha} = \sum_{k=1}^{\infty} \beta_k^{2\alpha} x_k y_k$$

and $A^\alpha : H_\alpha \to H$ defined by

$$A^\alpha x = \sum_{k=1}^{\infty} \beta_k^\alpha x_k \phi_k$$

It is obvious that $A^\alpha \cdot A^\beta = A^{\alpha+\beta}$, and

$$< A^\alpha x, y >_H = < x, A^\alpha y >_H \qquad \forall x, y \in H_\alpha$$

The eigenvector sequence $\{\phi_k\}$ of A is the common orthogonal base of $H_\alpha (0 \leq \alpha < \infty), H_0 = H$. The operators $A^\alpha : H_\alpha \to H$ are invertible, and the inverses $A^{-\alpha} : H \to H_\alpha$ are as

$$A^{-\alpha} x = \sum_{k=1}^{\infty} \beta_k^{-\alpha} x_k \phi_k.$$

In addition, for any $0 \leq \alpha < \beta, H_\beta \hookrightarrow H_\alpha$ is a compact embedding. Hereafter, we always assume that there is a constant $0 < \gamma < 1$ such that

$$B : H_\gamma \to H \textit{ is bounded.} \tag{3.1.16}$$

Theorem 3.1.6. Let $L = -A + B : H_1 \to H$ be a completely continuous field which satisfies (3.1.8) and (3.1.16). Then L is the generator of an analytic semigroup.

Proof. We first prove that $-A : D(A) = H_1 \to H$ is the generator of an uniformly bounded analytic semigroup. By Theorem 5.2 of Ch.II in [Pa], it suffices to show that

$$\rho(-A) \supset \sum = \{\lambda \in C | \ |ang\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}\} \tag{3.1.17}$$

and

$$\|R(-A, \lambda)\| \leq \frac{M}{|\lambda|}, \ for \ \lambda \in \sum, \ \lambda \neq 0 \tag{3.1.18}$$

where $0 < \delta < \frac{\pi}{2}, M > 0, \rho(-A)$ is the resolvent set of $-A$, and $R(-A, \lambda) = \widetilde{I}(\lambda \widetilde{I} + \widetilde{A})^{-1} : \widetilde{H} \to \widetilde{H}$ is the resolvent of A(for the sake of simplicity, we always denote by $R(L, \lambda) = (\lambda id - L)^{-1}$). However, (3.1.17) and (3.1.18) follows from (3.1.8) and (3.1.10) respectively.

By the condition (3.1.16), $D(B) \subset D(A^\gamma)$ for some $0 < \gamma < 1$, thereby from Corollary 2.4 of Ch.III in [Pa] it follows that $L = -A + B$ is the generator of an analytic semigroup. This proof is complete.

Remark 3.1.7. The analytic semigroup $T(t)$ of a linear operator L has an important property (Cf.[Pa]), which amounts to say that if the spectrum

$$\delta = \sup\{Re\lambda|\ \lambda \in \sigma(L)\} < 0$$

then there are constants $M_\alpha, \mu > 0$ such that

$$\|L^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{-\mu t},\ for\ t > 0, 0 \leq \alpha < \infty. \tag{3.1.19}$$

3.1.4. A refined asymptotical property of eigenvalues.

For the eigenvalues of linear completely continuous fields we have the more refined asymptotical property as follows.

Theorem 3.1.8. Let $L = -A + B : H_1 \to H$ be a linear completely continuous field which satisfies (3.1.8) and (3.1.16), if L has infinite eigenvalues $\lambda_k \in C, \lambda_k = \alpha_k + i\rho_k$, then we have that $\alpha_k \to -\infty$, and for the index $0 < \gamma < 1$ defined in (3.1.16),

$$\lim_{k\to\infty} \frac{|\rho_k|}{|\alpha_k|^\delta} = 0 \qquad for\ \gamma < \delta \leq 1. \tag{3.1.20}$$

Proof. Let $z_k = x_k + iy_k$ be the eigenvector corresponding to λ_k. Then we have

$$\begin{cases} -Ax_k + Bx_k = \alpha_k x_k + \rho_k y_k \\ -Ay_k + By_k = -\rho_k x_k + \alpha_k y_k \end{cases} \tag{3.1.21}$$

From (3.1.21) we can obtain

$$\begin{aligned} -\alpha_k &= \frac{< Ax_k, A^{2\theta}x_k > + < Ay_k, A^{2\theta}y_k > - < Bx_k, A^{2\theta}x_k > - < By_k, A^{2\theta}y_k >}{< x_k, A^{2\theta}x_k > + < y_k, A^{2\theta}y_k >} \\ &= \frac{\|z_k\|^2_{\theta+\frac{1}{2}} - < Bx_k, A^{2\theta}x_k > - < By_k, A^{2\theta}y_k >}{\|z_k\|^2_\theta} \end{aligned}$$

$$\rho_k = \frac{< Bx_k, A^{2\theta}y_k > - < By_k, A^{2\theta}x_k >}{\|z_k\|^2_\theta}$$

where $0 < \theta \leq \frac{1}{2}$ is some determaned constant.

By (3.1.16) we derive that

$$| < Bx_k, A^{2\theta}y_k > | \leq \|B\|_\gamma \|x_k\|_\gamma \|y_k\|_{2\theta}$$

$$| < By_k, A^{2\theta}x_k > | \leq \|B\|_\gamma \|y_k\|_\gamma \|x_k\|_{2\theta}$$

thereby

$$|\rho_k| \leq \frac{\|B\|_\gamma \|z_k\|_\gamma \|z_k\|_{2\theta}}{\|z_k\|_\theta^2} \tag{3.1.22}$$

$$|\alpha_k| \geq \frac{\|z_k\|^2_{\frac{1}{2}+\theta} - \|B\|_\gamma \|z_k\|_\gamma \|z_k\|_{2\theta}}{\|z_k\|_\theta^2} \tag{3.1.23}$$

$$|\alpha_k\| \leq \frac{\|z_k\|^2_{\frac{1}{2}+\theta} + \|B\|_\gamma \|z_k\|_\gamma \|z_k\|_{2\theta}}{\|z_k\|_\theta^2} \tag{3.1.24}$$

By the claim ii) of Theorem 3.1.4, from (3.1.22) and (3.1.24) it follows that either

$$\frac{\|z_k\|^2_{\frac{1}{2}+\theta} + \|B\|_\gamma \|z_k\|_\gamma \|z_k\|_{2\theta}}{\|z_k\|_\theta^2} \to \infty \ (k \to \infty) \tag{3.1.25}$$

$$\frac{\|B\|_\gamma \|z_k\|_\gamma \|z_k\|_{2\theta}}{\|z_k\|_\theta^2} \to \infty \quad (k \to \infty) \tag{3.1.26}$$

Because $0 < \gamma < 1$, we can take $\theta (0 < \theta \leq \frac{1}{2})$ such as

$$\begin{cases} \frac{1}{2} + \theta \geq \max\{\gamma, 2\theta\} \\ \frac{1}{2} + \theta > \min\{\gamma, 2\theta\} \end{cases} \tag{3.1.27}$$

We can take that $\|z_k\|_{\frac{1}{2}+\theta} = 1, \forall k \in N$. Then from (3.1.25)-(3.1.27) it follows that $\|z_k\|_\theta \to 0$, which implies that $z_k \rightharpoonup 0$ in $H_{\frac{1}{2}+\theta}$. Because $H_\alpha \hookrightarrow H_\beta$ is compact for all $\alpha > \beta$, by (3.1.27) we infer that

$$\|z_k\|_\gamma \cdot \|z_k\|_{2\theta} \to 0, \quad k \to \infty \tag{3.1.28}$$

Therefore, from (3.1.23) it follows that $|\alpha_k| \to \infty$, which means that $\alpha_k \to -\infty$, and from (3.1.22)(3.1.23) and (3.1.28) we deduce that there exists a $k > 0$ large enough such that

$$\frac{|\rho_k|}{|\alpha_k|^\gamma} \leq \frac{\|B\|_\gamma \|z_k\|_\gamma \cdot \|z_k\|_{2\theta}}{\|z_k\|_\theta^{2-2\gamma} |\|z_k\|^2_{\frac{1}{2}+\theta} - \|B\|_\gamma \|z_k\|_\gamma \|z_k\|_{2\theta}|^\gamma} \tag{3.1.29}$$

for all $k \geq K$.

If $\gamma > \frac{1}{2}$, we take $\theta = \gamma - \frac{1}{2}$, and otherwise take $\theta = \gamma$. Then from (3.1.28) and (3.1.29) we get

$$\frac{|\rho_k|}{|\alpha_k|^\gamma} \leq \frac{2\|B\|_\gamma \|z_k\|_{2\theta}}{\|z_k\|^{2\theta}_{\frac{1}{2}+\theta} \|z_k\|^{1-2\theta}_\theta} \tag{3.1.30}$$

for all $k \in N$ large enough.

By the intepolation theory of spaces (Cf.[Te]),

$$\begin{cases} H_{2\theta} = [H_{\frac{1}{2}+\theta}, H_\theta]_\epsilon \\ (1-\epsilon)(\theta + \frac{1}{2}) + \epsilon\theta = 2\theta, \ i.e. \ \epsilon = 1 - 2\theta \end{cases}$$

The intepolation inequality provides that

$$\|z_k\|_{2\theta} \leq c_\theta \|z_k\|^{2\theta}_{\frac{1}{2}+\theta} \|z_k\|^{1-2\theta}_\theta \tag{3.1.31}$$

where $c_\theta > 0$ is a constant independent of k. Hence from (3.1.30) and (3.1.31) it follows that

$$\frac{|\rho_k|}{|\alpha_k|^\gamma} \leq \frac{2\|B\|_\gamma}{c_\theta} < \infty,$$

which implies that (3.1.20) holds true. The proof is complete.

3.2. Asymptotic Stability of Abstract Nonlinear Evolution Equations

3.2.1. Local stability of nonlinear completely continuous fields

We consider the stability problems of equilibrium points of the below abstract nonlinear evolution equation

$$\begin{cases} \frac{du}{dt} = Lu + Gu \\ u(0) = \phi \end{cases} \tag{3.2.1}$$

where $L = -A + B : H_1 \to H$ is a linear completely continuous field and $G : H_1 \to H$ is a nonlinear continuous operator with $G(u) = o(\|u\|_\alpha)$ for some $0 \leq \alpha \leq 1$.

Definition 3.2.1. We say that $u = 0$ is a locally asymptotically stable equilibrium point of (3.2.1) in $H_\alpha (0 \leq \alpha < \infty)$, if there is a neigbborhood

$U \subset H_\alpha$ of $u = 0$, for any $\phi \in U$ the global solutions $u(t, \phi)$ of (3.2.1) exist in H_α, which satisfy

$$\lim_{t\to\infty} \|u(t, \phi)\|_\alpha = 0$$

And $u = 0$ is exponentially decayed if there are constants $c, \mu > 0$ such that

$$\|u(t, \phi)\|_\alpha \le ce^{-\mu t} \quad \forall t \ge 0$$

If $U = H_\alpha$, then $u = 0$ is called to be globally asymptotically stable in H_α.

The following is the local stability theorem of completely continuous fields, which is well known. In order to compare the result with the later theorem of local stability in the regularity sense, here we give the proof.

Theorem 3.2.2. Let $L = -A + B : H_1 \to H$ be a linear completely continuous field which satisfies (3.1.8) and (3.1.16). Suppose that the eigenvalues $Re\lambda_k < 0$ of L, and there are constants $c > 0, k > 1$ and $0 < \theta < 1$ such that

$$\|Gu\|_0 \le c\|u\|_\theta^k \quad \forall u \in H_1 \tag{3.2.2}$$

then the solutions of (3.2.1) exist in a neighborhood $U \subset H_\theta$ of $u = 0$ for $0 < t < \infty$, and $u = 0$ is a locally asymptotically stable equilibrium point of (3.2.1) in H_θ, which is exponentially decayed.

Proof. Because the eigenvalues of L satisfy $Re\lambda_k < 0, \|L^\alpha u\|_H (0 \le \alpha < \infty)$ is a norm defined on H_α, which is equivalent to the H_α-norm. Let $T(t)$ be the analytic semigroup of L, then the solution of (3.2.1) satisfies

$$u = T(t)\phi + \int_0^t T(t-\tau)Gud\tau$$

From (3.1.19) and (3.2.2) it follows that

$$\|L^\theta u\| \le \|T\| \cdot \|L^\theta \phi\| + \int_0^t \|L^\theta T(t-\tau)\| \cdot \|Gu\| d\tau$$

$$\le \|\phi\|_\theta e^{-\mu t} + c\int_0^t e^{-\mu(t-\tau)}(t-\tau)^{-\theta}\|u\|_\theta^k d\tau \tag{3.2.3}$$

Let $\|u\|_\theta = e^{-\mu t}z(t)$, then from (3.2.3) we get

$$z(t) \le c\|\phi\|_\theta + c\int_0^t (t-\tau)^{-\theta} e^{-\mu(k-1)\tau} z^k(\tau) d\tau$$

which implies that

$$z(t) \le c\|\phi\|_\theta + c_\theta[\sup_{0\le\tau<t} z(\tau)]^k, \quad (0 < \theta < 1)$$

Thus we infer that $z(t)$ is bounded for ϕ in a neighborhood $U \subset H_\theta$ of $u = 0$, and therefore $u = 0$ is locally asymptotically stable, which is exponentially decayed. This proof is complete.

3.2.3. Local stability in the regularity sense

In the above paragraph, we introduce the local stability in the space $H_\theta, 0 < \theta < 1$. In this subsection we shall discuss the local stability of (3.2.1) in the spaces $H_\theta, 1 \leq \theta < \infty$, provided some appropriate conditions. To this end, we need to introduce the following lemmas.

Lemma 3.2.3. Let $\{L_n\}$ be the generators of analytic semigroups $\{T_n(t)\}$. If there exist constants $M, \delta > 0$ and $\frac{\pi}{2} > \delta > 0$ such that for all $n \in N$,

$$\rho(L_n) \supset \sum = \{\lambda \in C|\ arg|\lambda + \delta| < \frac{\pi}{2} + \delta\} \tag{3.2.4}$$

and

$$\|R(L_n, \lambda)\| \leq \frac{M}{|\lambda|} \quad for\ \lambda \in \sum \tag{3.2.5}$$

then there are constants M_1 and $0 < \mu < \sigma$ such that

$$\|T_n(t)\| \leq M_1 e^{-\mu t}, \quad \forall n \in N$$

where $\rho(L)$ and $R(L, \lambda)$ are respectively the resolvent set and resolvent of L.

The proof of this lemma is exactly the same as that of Theorem 4.3 of Ch.IV in [Pa]. Notice that the condition (3.2.4) implies that

$$\sup_{n \in N} \sup_{\lambda \in \sigma(L_n)} Re\lambda \leq -\sigma < 0$$

Denote by $P_n^\alpha : H_\alpha \to H^n$ the projective operators where $H^n = span\{\phi_1, \cdots, \phi_n\}$, generally, without confusion we denote p_n^α by p_n. The following lemma is well known.

Lemma 3.2.4. Let $B : H_\alpha \to H (0 \leq \alpha < \infty)$ be a linear compact operator. Then we have

$$\lim_{n \to \infty} \|B - P_n B\|_\alpha = 0$$
$$\lim_{n \to \infty} \|B - BP_n\|_\alpha = 0$$
$$\lim_{n \to \infty} \|B - P_n B P_n\|_\alpha = 0$$

Notice that in this lemma the condition that $\{\phi_k\}$ is a common orthogonal base of H_α and H is essential.

From Lemma 3.2.3 and Lemma 3.2.4 we can get the below lemma.

Lemma 3.2.5. Let $L == -A + B : H_1 \to H$ be a linear completely continuous field, which satisfies (3.1.8) and (3.1.16). If the eigenvalues of L satisfy $Re\lambda_k < 0$, then there are constants $k, M, \mu > 0$ such that the analytic semigroups $T_n(t)$ of $L_n = P_n L P_n$ satisfy

$$\|T_n(t)\| \leq Me^{-\mu t}, \qquad for\ all\ n \geq K.$$

Proof. We first check the condition (3.2.4), which amounts to say that the eigenvalues of the operators $L_n = P_n L P_n$ for all $n > 0$ sufficiently large are in the domain $C/\sum$ as shown in Fig. 3.1 as follows

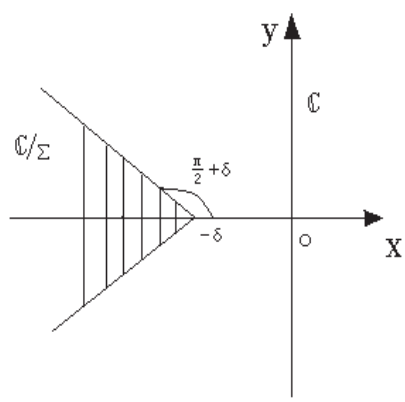

Fig. 3.1

Because $L = -A+B$ is the generator of an analytic semigroup (by Theorem 3.1.6) and the eigenvalues of L obeys that $Re\lambda_k < 0$, then by Theorem 5.2 of Ch.II in [Pa], we have

$$\rho(L) \supset \sum \tag{3.2.6}$$

and

$$\|R(L, \lambda)\| \leq \frac{M}{|\lambda|} \qquad for\ \lambda \in \sum. \tag{3.2.7}$$

We know that under the common orthogonal base $\{\phi_k\}$ of $H_\alpha (0 \leq \alpha < \infty)$,

the operator $-A + B_n (B_n = P_n B P_n)$ and $L_n = P_n L P_n$ respectively corresponde the matrices as follows

$$-A + B_n = \begin{pmatrix} -\beta_1 + b_{11} & \cdots & b_{1n} & & \\ \vdots & \ddots & \vdots & & 0 \\ b_{n1} & \cdots & -\beta_n + b_{nn} & & \\ & 0 & & -\beta_{n+1} & \\ & & & & \ddots \end{pmatrix}$$

$$L_n = \begin{pmatrix} -\beta_1 + b_{11} & \cdots & b_{1n} & \\ \vdots & & \vdots & 0 \\ b_{n1} & \cdots & -\beta_n + b_{nn} & \\ & 0 & & 0 \end{pmatrix}$$

It is clear that $-A + B_n$ and L_n have respectively the below eigenvalues

$$\lambda_1^{(n)}, \cdots, \lambda_n^{(n)}, -\beta_{n+1}, -\beta_{n+2}, \cdots$$

and

$$\lambda_1^{(n)}, \cdots, \lambda_n^{(n)}, 0, 0, \cdots$$

i.e. they have the same n eigenvalues $\{\lambda_k^{(n)} | 1 \leq k \leq n\}$.

We claim that there is a number $N > 0$ sufficiently large such that for all $n \geq N$ the eigenvalues $\{\lambda_k^{(n)} | 1 \leq k \leq n\}$ of L_n are in the domain $C/\sum$.

If the claim is not true, then there is a subsequence of $\{\lambda_k^{(n)} | 1 \leq k \leq n, n \in N\}$, we denote if by $\{\lambda^k = \alpha_k + i\rho_k\}$, such that the following cases occur.

Case 1. $\{\lambda^k\}$ is bounded and $Re\lambda^k \geq \gamma > -\delta$.

Case 2. $Re\lambda^k \to +\infty$.

Case 3. $\lim_{n\to\infty} \frac{|\rho_k|}{|\alpha_k|} = \infty$.

If the case 1 occurs, then there is a convergent subsequence of $\{\lambda^k\}$ which convergence to $\lambda_0 \in C$, and $Re\lambda_0 > -\sigma$. On the other hand, by Lemma 3.2.4,

$$\| - A + B_n - L\|_1 = \|B_n - B\|_1 \to 0 \tag{3.2.8}$$

which implies that λ_0 is an eigenvalue of L, and one reads a contradiction with (3.2.6).

In the same fashion as used in Lemma 3.1.5, one can deduce by (3.2.8) that the case 2 never appears. Finally similar to the proof of Theorem 3.1.8, we can prove that the case 3 likewise is not true. Consequently, the condition (3.2.4) holds true.

Next, we verify the condition (3.2.5). We see that

$$\begin{aligned} R(L_n,\lambda) &= (\lambda id + A_n - B_n)^{-1} \\ &= R(-A_n,\lambda)(id - B_n R(-A_n,\lambda))^{-1} \end{aligned}$$

where $A_n = P_n A P_n$. It follows that

$$\|R(L_n,\lambda)\| \leq \frac{\|R(-A_n,\lambda)\|}{1 - \|B_n R(-A_n,\lambda)\|}, \tag{3.2.9}$$

for $\|B_n R(-A_n,\lambda)\| < 1$. Due to

$$\|A_n^{-1}\| = \|P_n A^{-1} P_n\| \leq \|A^{-1}\|,$$

for all $|\lambda + \beta_k| \geq$ some constant $\alpha > 0$ we have

$$\|R(-A_n,\lambda)\| \leq \begin{cases} M, & for\ \frac{|\lambda|}{2} \leq \|A^{-1}\| \\ \frac{1}{|\lambda|(1-|\lambda|^{-1}\|A^{-1}\|)} \leq \frac{M}{|\lambda|}, & for\ \frac{|\lambda|}{2} > \|\lambda\|^{-1} \end{cases} \tag{3.2.10}$$

As H^n is the invariable subspace of $R(A,\lambda)$, than

$$B_n R(-A_n,\lambda) = P_n B P_n (\lambda id = A_n)^{-1}$$

$$= P_n B P_n (\lambda id + A)^{-1} = B_n R(-A,\lambda)$$

In addition, $BR(-A,\lambda) : H \to H$ is compact for all $\lambda \in \rho(A)$, from Lemma 3.2.4 and (3.1.13) it follows that

$$\|B_n R(-A_n,\lambda) - BR(-A,\lambda)\|$$

$$= \|B_n R(-A,\lambda) - BR(-A,\lambda)\| \to 0 \quad (n \to \infty) \tag{3.2.11}$$

uniformly for $\sup_k |\lambda + \beta_k| \geq$ some constant $\alpha > 0$.

On the other hand, from the first step one can see that

$$dist(\sum, \sigma(L_n)) \geq \ some\ constant\ \alpha > 0,\ for\ all\ n \geq N$$

where $\sigma(L_n)$ is the set of eigenvalues of L_n. Applying the fact that $\|BR(-A,\lambda)\| \to 0$ as $|\lambda + \lambda_k| \to \infty$, from (3.2.9)-(3.2.11) it follows that there is a number $N > 0$ large enough, such that for all $n \geq N$ we have

$$\|R(L_n,\lambda) \leq \begin{cases} M, & for\ \lambda \in \sum\ and\ |\lambda| < K \\ \frac{M}{|\lambda|}, & for\ \lambda \in \sum\ and\ |\lambda| \geq K \end{cases}$$

where $K > 0$ is some constant. Thus the condition (3.2.5) is checked. The proof is complete.

Now we return to discuss the local stability in $H_\theta, \theta \geq 1$. We say that a mapping $K : H_\mu \to H_\gamma (0 \leq \mu, \gamma)$ is σ-regular ($\sigma \geq \gamma$), if for all $0 \leq \beta + \gamma \leq \sigma$ and $0 \leq \delta \leq \min\{\gamma, \mu\}, K : H_{\mu+\beta} \to H_{\gamma+\beta}$ and $K : H_{\mu-\delta} \to H_{\gamma-\delta}$ are continuous.

If K is a differential operator, and $H_0 = L^2(\Omega), H_1 = H^m(\Omega)(m \geq 1), K : H_1 \to H_0$ is the σ-regularity which means that the coificients of K belong to the space $H^{m\sigma}(\Omega)$.

Suppose that the eigenvalues β_k of A have the asymptotic behavior such as

$$\beta_k \sim ck^p \quad as\ k \to \infty \tag{3.2.12}$$

for some constants $c, p > 0$. It is known that for may elliptic operators, the index $p = 2m/n$, where $2m$ is the order of the operator and n is the space dimension (see [Te]). In the following we give and prove the local stability theorem in $H_\theta, \theta \geq 1$.

Theorem 3.2.6. Let $L = -A + B : H_1 \to H$ be a linear completely continuous field which satisfies (3.1.8)(3.1.16) and (3.2.12). Assume that $B : H_1 \to H_\alpha$ is bounded for $\alpha > \frac{1+p}{4p}, B, G : H_1 \to H$ are $(1+\theta)$-regular ($1 \leq \theta < \infty$), and in a neighborhood of $u = 0$

$$\|A^{\theta-\frac{1}{2}}Gu\| \leq c\|u\|_\theta^k, \quad \forall u \in H_{\theta+\frac{1}{2}} \tag{3.2.13}$$

where $k > 1$ and $c > 0$ are some constants. If the eigenvalues $Re\lambda_k < 0$ of L, and the global solutions of (3.2.1) exist in $H_{2\theta}$ for $\phi \in U$ a neighborhood of $u = 0$ in $H_{2\theta}$, then $u = 0$ is a locally asymptotically stable equilibrium point of (3.2.1) in H_θ, which is exponentially decayed.

Proof. Let $u \in H_{1+\theta}, u = \sum_{k=1}^\infty x_k\phi_k$, where the eigenvector sequence $\{\phi_k\}$ of A is taken to be nomal in H, and

$$\begin{cases} u = w + v \\ w = P_n u = \sum_{k=1}^n x_k\phi_k \\ v = P_n^c u = (id - P_n)u = \sum_{k=n+1}^\infty x_k\phi_k \end{cases}$$

Then the equation (3.2.1) can be written as

$$\frac{dw}{dt} = L_n w + B_n v + G_n(w + v) \tag{3.2.14}$$

$$\frac{dv}{dt} = -Av + B_n^c v + B_n^c w + G_n^c(w + v) \tag{3.2.15}$$

where $B_n = P_n B, B_n^c = P_n^c B$ and so on.

We take the inner product on the both sides of (3.2.15) by $A^{2\theta}v$, and noticing that

$$< \frac{dv}{dt}, A^{2\theta}v >=< \frac{dA^{\theta}v}{dt}, A^{\theta}v >= \frac{1}{2}\frac{d}{dt}\|A^{\theta}v\|^2$$

from (3.2.15) it follows that

$$\begin{aligned}
\frac{d}{dt}\|A^{\theta}v\|^2 &= -2 < Av, A^{2\theta}v > +2 < B_n^c v, A^{2\theta}v > \\
&\quad +2 < B_n^c w, A^{2\theta}v > +2 < G_n^c(w+v), A^{2\theta}v > \\
&= -2\|A^{\frac{1}{2}+\theta}v\|^2 + 2 < A^{\theta-\frac{1}{2}}B_n^c v, A^{\frac{1}{2}+\theta}v > + \\
&\quad +2 < A^{\theta-\frac{1}{2}}B_n^c w, A^{\theta+\frac{1}{2}}v > +2 < A^{\theta-\frac{1}{2}}G_n^c(w+v), A^{\frac{1}{2}+\theta}v > \\
&\leq -\frac{3}{2}\|A^{\frac{1}{2}+\theta}v\|^2 + c\|A^{\theta-\frac{1}{2}}B_n^c v\|^2 + c\|A^{\theta-\frac{1}{2}}B_n^c w\|^2 + c\|A^{\theta-\frac{1}{2}}Gu\|^2
\end{aligned}$$

where $c > 0$ is a constant independent of n. We notice that $B_n^c w = P_n^c B P_n w$, and $A^{\theta-\frac{1}{2}}BP_n : H \to H$ is bounded (as it is finite dimentional), thereby

$$\|A^{\theta-\frac{1}{2}}B_n^c w\|^2 \leq \|A^{\theta-\frac{1}{2}}B_n^c P_n\|\|w\|^2$$

By (3.2.13) it follows that

$$\|A^{\theta-\frac{1}{2}}G(w+v)\|^2 \leq c\|A^{\theta}w\|^{2k} + c\|A^{\theta}v\|^{2k}$$

In addition

$$\begin{aligned}
\|A^{\theta-\frac{1}{2}}B_n^c v\|^2 &= \|A^{\theta-\frac{1}{2}}B_n^c A^{-(\frac{1}{2}+\theta)} \cdot A^{\frac{1}{2}+\theta}v\|^2 \\
&\leq \|P_n^c A^{\theta-\frac{1}{2}}BA^{-(\frac{1}{2}+\theta)}\|^2\|A^{\frac{1}{2}+\theta}v\|^2
\end{aligned}$$

Due to the $(1+\theta)$-regularity of $B, A^{\theta-\frac{1}{2}}BA^{-(\frac{1}{2}+\theta)} : H \to H_\alpha$ is bounded, and by the compactness of $H_\alpha \to H, A^{\theta-\frac{1}{2}}BA^{-(\frac{1}{2}+\theta)} : H \to H$ is compact. By Lemma 3.2.4 we have

$$\|P_n^c A^{\theta-\frac{1}{2}}BA^{-(\frac{1}{2}+\theta)}\|^2 \to 0 \quad as\ n \to \infty.$$

Thus, for all $n \geq N(N$ is sufficiently large), we obtain

$$\frac{d}{dt}\|A^{\theta}v\|^2 \leq -\|A^{\frac{1}{2}+\theta}v\|^2 + c\|A^{\theta-\frac{1}{2}}B_n^c P_n\|^2\|w\|^2$$

$$+c\|A^{\theta}w\|^{2k} + c\|A^{\theta}v\|^{2k} \tag{3.2.16}$$

On the other hand, from (3.2.14) it follows that

$$w = T_n(t)\phi_n + \int_0^t T_n(t-\tau)(B_n v + G_n(v+w))d\tau$$

where $T_n(t)$ is the analytic semigroup of L_n and $\phi_n = P_n\phi$. Hence we obtain that

$$\|w\|^2 =< T_n(t)\phi_n, w > + \int_0^t < T_n(t-\tau)(B_n v + G_n u), w > d\tau$$

$$\leq \frac{1}{2}\|w\|^2 + c\|T_n(t)\phi_n\|^2 + \int_0^t \|T_n(t-\tau)\|^2[\|B_n v\|^2 + \|G_n u\|^2]d\tau$$

From Lemma 3.2.5 and (3.2.13) one derives

$$\begin{aligned}\|w\|^2 &\leq c\|\phi_n\|e^{-\mu t} + c\int_0^t e^{-\mu(t-\tau)}[\|A^\theta v\|^{2k} + \|A^\theta w\|^{2k} + \\ &\quad +\|B_n A^{-(1-\alpha)} \cdot A^{-(\theta+\alpha-1)} \cdot A^\theta v\|^2]d\tau \\ &\leq c\|\phi_n\|e^{-\mu t} + c\int_0^t e^{-\mu(t-\tau)}[\|A^\theta v\|^{2k} + \|A^\theta w\|^{2k} + \\ &\quad +\|BA^{-(1-\alpha)}\|^2\|A^{-(\theta+\alpha-1)}A^\theta v\|^2]d\tau\end{aligned}$$

However,

$$\|A^{-(\theta+\alpha-1)}A^\theta v\|^2 = \|P_n^c A^{-(\theta+\alpha-1)}A^\theta v\|^2 \leq \beta_n^{-2(\theta+\alpha-1)}\|A^\theta v\|^2$$

and $BA^{-(1-\alpha)} : H \to H$ is bounded. Therefore we get

$$\|w\|^2 \leq c\|\phi_n\|e^{-\mu t} + c\int_0^t e^{-\mu(t-\tau)}\times$$

$$\times[\beta_n^{-2(\theta+\alpha-1)}\|A^\theta v\|^2 + \|A^\theta v\|^{2k} + \|A^\theta w\|^{2k}]d\tau \tag{3.2.17}$$

Because $A^{\theta-\frac{1}{2}}B : H_{\frac{1}{2}+\theta} \to H_\alpha$ is bounded, and

$$B\phi_k = \sum_{i=1}^{\infty} b_{ik}\phi_i$$

$$b_{ik} = \frac{< B\phi_k, \phi_i >_{H_\alpha}}{< \phi_i, \phi_i >_{H_\alpha}} =< B\phi_k, \phi_i >_H$$

then

$$\|A^{\theta-\frac{1}{2}}B\phi_k\|_\alpha^2 = \sum_{i=1}^{\infty} b_{ik}^2 \cdot \beta_i^{2(\alpha+\theta-\frac{1}{2})}$$

$$\leq c\|\phi_k\|_{\frac{1}{2}+\theta} = c\beta_k^{2\theta+1}, \qquad (by\ \|\phi_k\|_0 = 1).$$

Hence we have

$$\sum_{i=1}^{\infty} b_{ik}^2 \beta_i^{2(\alpha+\theta-\frac{1}{2})} \leq c\beta_k^{2\theta+1} \tag{3.2.18}$$

In addition, we know that

$$\|A^{\theta-\frac{1}{2}} B_n^c P_n\|^2 \leq \sum_{k=1}^{n} \sum_{i=n+1}^{\infty} b_{ik}^2 \beta_i^{2\theta-1}$$

From (3.2.18) we get

$$\sum_{k=1}^{n} \sum_{i=n+1}^{\infty} b_{ik}^2 \beta_i^{2\theta-1} \leq \sum_{k=1}^{n} \frac{\sum_{i=n+1}^{\infty} b_{ik}^2 \beta^{2\alpha+2\theta-1} - i}{\beta_{n+1}^{2\alpha}} \leq \sum_{k=1}^{n} \frac{c\beta_k^{2\theta+1}}{\beta_n^{2\alpha}}$$

thereby one derives

$$\|A^{\theta-\frac{1}{2}} B_n^c P_n\|^2 \leq \sum_{k=1}^{n} \frac{c\beta_k^{2\theta+1}}{\beta_{n+1}^{2\alpha}} \leq cn \cdot \beta_n^{2\theta+1-2\alpha}$$

From (3.2.16) we obtain

$$\frac{d}{dt}\|v\|_\theta^2 \leq -\beta_n \|v\|_\theta^2 + cn\beta_n^{2\theta+1-2\alpha}\|w\|^2 + \tag{3.2.19}$$

$$+c\|w\|_\theta^{2k} + c\|v\|_\theta^{2k}$$

Denote by $x = \|w\|^2, y = \|v\|_\theta^2$, and note that $\|w\|_\theta \leq \beta_n^\theta \|w\|$ then (3.2.17) and (3.2.19) can be expressed as

$$x \leq c\|\phi\|e^{-\mu t} + c\int_0^t e^{-\mu(t-\tau)}(\beta_n^{-2(\theta+\alpha-1)} y + y^k + \beta_n^{2\theta k} x^k) d\tau \tag{3.2.20}$$

$$\frac{dy}{dt} \leq -\beta_n y + cn\beta_n^{-(2\alpha-2\theta-1)} x + cy^k + c\beta^{2\theta k} - nx^k \tag{3.2.21}$$

By the comparision principle, for any $x \leq \widetilde{x}$ from (3.2.21) it follows that

$$y(t) \leq \widetilde{y}(t) \quad \forall t \geq 0 \tag{3.2.22}$$

where $\widetilde{y}(t)$ satisfies

$$\begin{cases} \frac{d\widetilde{y}}{dt} = -\beta_n \widetilde{y} + cn\beta_n^{2\theta+1-2\alpha}\widetilde{x} + c\widetilde{y}^k + c\beta_n^{2\theta k}\widetilde{x}^k \\ \widetilde{y}(0) = y(0) \end{cases}$$

Let

$$\widetilde{x} = c\|\phi\|e^{-\mu t} + c\int_0^t e^{-\mu(t-\tau)}(\beta^{-2(\alpha+\theta-1)}\widetilde{y} + \widetilde{y}^k + \beta_n^{2\theta k}\widetilde{x}^k)d\tau$$

Then from (3.2.20) and (3.2.22) it follows that

$$x(t) \le \widetilde{x}(t), y(t) \le \widetilde{y}(t), \quad \forall t \ge 0 \tag{3.2.23}$$

where $\widetilde{x}(t)$ and $\widetilde{y}(t)$ satisfy

$$\begin{cases} \frac{d\widetilde{x}}{dt} = -\mu\widetilde{x} + c\beta_n^{-2(\theta+\alpha-1)}\widetilde{y} + c\widetilde{y}^k + c\beta_n^{2\theta k}\widetilde{x}^k \\ \frac{dy}{dt} = cn\beta_n^{2\theta+1-2\alpha}\widetilde{x} - \beta_n\widetilde{y} + c\widetilde{y}^k + c\beta_n^{2\theta k}\widetilde{x}^k \\ \widetilde{x}(0) = x(0), \widetilde{y}(0) = y(0) \end{cases} \tag{3.2.24}$$

The eigenvalues of (3.2.24) are as follows

$$\begin{aligned} \lambda_\pm &= -\frac{(\mu-\beta_n)}{2} \pm \frac{\sqrt{(\beta_n-\mu)^2 + 4cn\beta^{-4\alpha+3}}}{2} \\ &= \frac{1}{2}[-(\mu+\beta_n) \pm (\beta_n-\mu)(1 + \frac{4cn\beta_n^3)}{\beta_n^3}\beta_n^{4\alpha}(\beta_n-\mu)^2)^{\frac{1}{2}}] \end{aligned}$$

By (3.2.12) we get

$$\lambda_{\min} \sim -\mu + cn^{-(4p\alpha-p-1)}, \ as\ n \to \infty$$

which implies, by $\alpha > \frac{p+1}{4p}$, that $\lambda_{\min} < 0$ as $n \ge N$ sufficiently large. Consequently the solutions of (3.2.24) is exponentially decayed in a neighborhood $U \subset R^2$ of $(\widetilde{x}, \widetilde{y}) = 0$, and this theorem follows from (3.2.23). The proof is complete.

For the symmetry linear completely continuous field, we have the same result as the above theorem.

Theorem 3.2.7. Let $L = -A + B : H_1 \to H$ be a symmetry linear completely continuous field which satisfies (3.1.8). Suppose that L and G are $(1+\theta)$-regular ($\theta \ge 1$), and G satisfies (3.2.13). If the eigenvalues of L satisfy $\lambda_k < 0$, and the global solutions of (3.2.1) exist in $H_{2\theta}$ for $\phi \in U$ a neighborhood of $u = 0$ in $H_{2\theta}$, then $u = 0$ is a locally asymptotically stable equilibrum point of (3.2.1) in H_θ, which is exponentially decayed.

Proof. Because the eigenvalues $\lambda_k < 0$ of L, for the operator $L_1 = -L$, the norm $\|L_1^\alpha u\|$ is equivalent to the $H_\alpha-$norm. We take the inner product on the both side of (3.2.1) by $L_1^{2\theta}u$, then we have

$$\frac{d}{dt}\|L_1^\theta u\|^2 = -2\|L^{\theta+\frac{1}{2}}u\|^2 + < L_1^{\theta-\frac{1}{2}}Gu, L_1^{\theta+\frac{1}{2}}u >$$

$$\leq -\|L_1^{\theta+\frac{1}{2}}u\|^2 + c\|L_1^{\theta-\frac{1}{2}}Gu\|$$

By (3.2.13) and

$$\mu\|L_1^{\theta}u\| \leq \|L_1^{\theta+\frac{1}{2}}u\|^2 \ for\ some\ \mu > 0$$

we obtain

$$\frac{d}{dt}\|L_1^{\theta}u\|^2 \leq -\mu\|L_1^{\theta}u\| + c\|L_1^{\theta}u\|^{2k}$$

which implies that this theorem holds true. The proof is complete.

3.2.3. Global stability

A linear operator $B : H_1 \to H$ is termed to be have a duality $B^* : H_1 \to H$, if

$$< Bx, Iy >=< B^*y, Ix > \ for\ all\ x, y \in H_1.$$

Without confnsion, hereafer we always denote $< Bx, y >$ instead of $< Bx, Iy >$ for $x, y \in H_1$. If $B : H_1 \to H$ has a duality B^*, then B can be decomposed into a sum

$$B = B_1 + B_2, \quad B_1 = \frac{1}{2}(B + B^*), \quad B_2 = \frac{1}{2}(B - B^*) \tag{3.2.25}$$

where B_1 is symmetric and B_2 is antisymmetic, namely

$$< B_2x, y >= - < B_2y, x > \ for\ all\ x, y \in H_1.$$

Obviously, if B is antisymmetric, then

$$< Bx, x >= 0, \quad \forall x \in H_1.$$

For a linear completely continuous field $L = -A + B : H_1 \to H$ which satisfies (3.1.8), we assume that $B : H_1 \to H$ has a duality thereby has the decomposition (3.2.25), and the eigenvalues $\{-\lambda_k\}$ of the symmetric part $-A+B_1$ of L satisfy

$$0 = \lambda_1 = \cdots = \lambda_k < \lambda_{k+1} \leq \lambda_{k+2} \leq \cdots \ (k \geq 1) \tag{3.2.26}$$

Denote by

$$\begin{aligned}
&E_0 = \{u \in H| - Au + B_1u = 0\}\\
&E_1 = E_0^{\perp} = \{u \in H| < u; v >= 0, \forall v \in E_0\}\\
&P_i : H \to E_i (i = 0, 1)\ the\ projective\ operators.
\end{aligned}$$

By the assumption (3.2.26), $\dim E_0 = k$.

Theorem 3.2.8. Under the condition (3.2.26), if $G : H_1 \to H$ is orthogonal, i.e.

$$< Gu, u >= 0 \quad \forall u \in H_1 \tag{3.2.27}$$

and $P_1(G + B_2)u \neq 0$ for $u \in E_0$ and $u \neq 0$, then $u = 0$ is a globally asymptotically stable equilibrum point of (3.2.1) in H.

Proof. By Theorem 3.1.4, the eigenvectors $\{\phi_k\}$ of $-A + B_1$ is an orthogonal base of H, and by (3.2.26)

$$E_0 = \{w \in H | w = \sum_{i=1}^{k} x_i \phi_i\}$$
$$E_1 = \{v \in H | v = \sum_{j=k+1}^{\infty} x_j \phi_j\}.$$

Therefore, for any $u \in H, u$ can be decomposed into

$$u = w + v, w \in H_0 \; and \; v \in H_1.$$

From (3.2.1) and (3.2.27) it follows that

$$\begin{aligned} \frac{d\|u\|^2}{dt} &= < Lu + Gu, u >=< (-A + B_1)u, u > \\ &= < (-A + B_1)v, v > \\ &\leq -\lambda_{k+1}\|v\|^2 \end{aligned}$$

which implies that

$$\|u\|^2 \leq \|u(0)\|^2 - \lambda_{k+1} \int_0^t \|v\|^2 d\tau \tag{3.2.28}$$

Due to $\|u\|^2 = \|v\|^2 + \|w\|^2$, we get

$$\|v\|^2 \leq \|u(0)\|^2 - \lambda_{k+1} \int_0^t \|v(\tau)\|^2 d\tau$$

By the Gronwell inequality we obtain

$$\|v\|^2 \leq \|u(0)\|^2 e^{-\lambda_{k+1} t} \tag{3.2.29}$$

On the other hand, the equation (3.2.1) can be written as

$$\begin{cases} \frac{dw}{dt} = P_0 B_2 w + P_0 B_2 v + P_0 G(w + v) \\ \frac{dv}{dt} = (-A + B_1)v + P_1 B_2 v + P_1 B_2 w + P_1 G(w + v) \end{cases} \tag{3.2.30}$$

Because $P_1Gw + P_1B_2w \neq 0$ for all $w \in E_0$ with $w \neq 0$, then from (3.2.30) and (3.2.28) it follows that for any $\phi \in H$, the solutions $u(t, \phi)$ of (3.2.1) are exactly decreasing, i.e.

$$\|u(t_2, \phi)\| < \|u(t_1, \phi)\|, \quad \forall t_1 < t_2 \tag{3.2.31}$$

By (3.2.29), the ω-limit set $\omega(L + G)$ of the flows of (3.2.1) exists, and $\omega(L + G) \subset E_0$. If the solutions

$$\lim_{t\to\infty} \|u(t)\| = \lim_{t\to\infty} \|w(t)\| = \delta > 0$$

then

$$\omega(L + G) \subset S_\delta = \{w \in E_0 | \|w\| = \delta\}$$

However, $\omega(L + G)$ is an invariant set of the flows of (3.2.1), namely for any $\phi \in \omega(L + G)$, the solution

$$u(t, \phi) \subset \omega(L + G) \subset S_\delta \quad \forall t \geq 0$$

which is contrary to (3.2.31). The proof is complete.

It is well known that if L is a symmetric linear completely continuous field, then as the odd multiple first eigenvalues of L transit from negative to positive, the global stability of (3.2.1) will be lost because the nonzero equilibrum solutions of (3.2.1) will be bifurcated from zero equilibrum state.

The following examples show that for the general linear completely continuous fields, the bifurcation problems of global stability is complex.

Example 3.2.9. We consider the system given by

$$\begin{cases} \frac{dx}{dt} = \lambda x - y + y^3 \\ \frac{dy}{dt} = x - y - xy^2 \end{cases} \tag{3.2.32}$$

It is clear the the bifurcation of global stability of (3.2.32) is equivalent to that of the below infinite system, which can be transform into the form of (3.2.1) satisfying (3.1.8).

$$\begin{cases} \frac{dx_1}{dt} = \lambda x_1 - x_2 + x_2^3 \\ \frac{dx_2}{dt} = x_1 - x - 2 - x_1x_2^2 \\ \frac{dx_k}{dt} = -kx_k, (k \geq 3) \end{cases}$$

It is easy to verify by Theorem 3.2.8 that as $\lambda \leq 0$ the system (3.2.32) is globally asymptotically stable, and as $0 < \lambda$ the global stability of (3.2.32) is lost because there are four nonzero equilibrum points which are bifurcated from infinite for $0 < \lambda$,

$$\left(-\frac{(1+\sqrt{\lambda})^{\frac{1}{2}}}{\sqrt{\lambda}}, (1+\sqrt{\lambda})^{\frac{1}{2}}\right), \left(\frac{(1-\sqrt{\lambda})^{\frac{1}{2}}}{\sqrt{\lambda}}, (1-\sqrt{\lambda})^{\frac{1}{2}}\right)$$

$$\left(\frac{(1+\sqrt{\lambda})^{\frac{1}{2}}}{\sqrt{\lambda}}, -(1+\sqrt{\lambda})^{\frac{1}{2}}\right), \left(-\frac{(1-\sqrt{\lambda})^{\frac{1}{2}}}{\sqrt{\lambda}}, -(1-\sqrt{\lambda})^{\frac{1}{2}}\right)$$

namely, $\lambda = 0$ is the bifurcation point of global stability.

Example 3.2.10. Consider the example given by

$$\left\{ \begin{array}{l} \frac{dx}{dt} = \lambda x - y - y^3 \\ \frac{dy}{dt} = x - y + xy^2 \end{array} \right. \tag{3.2.33}$$

It is easy to check that (3.2.33) has no nonzero equilibrum point for $\lambda \leq 1$, and one can show that if (3.2.33) has no limit circle for $\lambda < 1$, then it is globally asymptotically stable.

Similarily, we can also obtain the below global stability theorem.

Theorem 3.2.11. Under the condition (3.2.26), if $G : H_1 \to H$ is negative definite, namely

$$< Gu, u >< 0 \quad \forall u \in H_1 \ with \ u \neq 0$$

then $u = 0$ is a globally asymptotically stable equilibrum point of (3.2.1) in H.

3.2.4. Parabolic differential operators with the variational structure

Let X be a separable and reflexive Banach space, H be a Hilbert space. Let $X \hookrightarrow H$ be dense and compact, $G : X \to X^*$ be a mapping with $G(0) = 0$.

Here we consider the stability of equilibrum point $u = 0$ of the following equation

$$\left\{ \begin{array}{l} \frac{du}{dt} + Gu = 0 \\ u(0) = \phi \end{array} \right. \tag{3.2.34}$$

Suppose that $G = DF$, and there are constants $r, \alpha > 0$, as $\|u\|_X < r$ we have

$$\left\{ \begin{array}{l} F(u) \geq \alpha \|u\|_X^p, p > 0 \\ < Gu, u > \to 0 \Leftrightarrow \|u\|_X \to 0 \end{array} \right. \tag{3.2.35}$$

Then we have the below local stability theorem.

Theorem 3.2.12. Under the condition (3.2.35), if $G : X \to X^*$ is weakly continuous, or coeceivly continuous, then there is a constant $\rho > 0$, when $\phi \in X$ with $\|\phi\| < \rho$, (3.2.34) has a global solution $u \in W^{1,2}((0,\infty), H) \cap L^\infty((0,\infty), X)$, and $u = 0$ is locally asymptotically stable in X.

Proof. The solution u of (3.2.32) satisfies

$$\int_0^t [< \frac{du}{dt}, v >_H + < Gu, v >]dt = 0 \tag{3.2.36}$$

for all $v \in L^\infty((0,\infty), X)$. Taking $v = u$ we get

$$\|u(t)\|_H^2 = \|\phi\|_H^2 - 2\int_0^t < Gu, u > dt \tag{3.2.37}$$

Again putting $v = \frac{du}{dt}$ in (3.2.36) we obtain

$$F(u(t)) = F(\phi) - \int_0^t \|u_t\|_H^2 dt \tag{3.2.38}$$

From (3.2.35) and (3.2.38) it follows that

$$\alpha\|u(t)\|_X^p \leq F(\phi) - \int_0^t \|u_t\|_H^2 dt, \qquad as\ \|u\|_X < r$$

which implies by (3.2.35) that there is a constant $\rho(0 < \rho \leq r)$ as $\|\phi\|_X < \rho$ the solution $u(t,\phi)$ of (3.2.34) satisfies

$$\|u(t,\phi)\|_X < \rho, \quad \forall t \geq 0 \tag{3.2.39}$$

By using the Galerkin method as in Section 2.2, from (3.2.39) we can infer that (3.2.34) has a global solution in $W^{1,2}((0,\infty), H) \cap L^\infty((0,\infty), X)$.

Now we cosider the stability of (3.2.34). If

$$\lim_{t\to\infty} F(u(t)) = 0$$

then by (3.2.35) we have

$$\lim_{t\to\infty} \|u(t)\|_X = 0$$

thereby Theorem 3.2.12 holds. Otherwise we have

$$\lim_{t\to\infty} F(u(t)) = c > 0$$

which implies that there is a constant $\delta > 0$ such that

$$\|u(t)\|_X \geq \delta, \quad \forall t \geq 0$$

Consequently, by (3.2.35) there exists $k > 0$ such that

$$< Gu, u > \geq k > 0 \quad \forall t \geq 0$$

and from (3.2.37) it leads to

$$\|u(t)\|_H^2 \leq \|\phi\|_H - 2kt, \quad \forall t \geq 0$$

Thus we read a contradiction. The proof is complete.

3.2.5. Hyperbolic differential operators

In this paragraph, we discuss the stability of equilibrum points of the equation given by

$$\begin{cases} \frac{d^2u}{dt^2} + k\frac{du}{dt} + Gu = 0, k > 0 \\ u(0) = \phi, u_t(0) = \psi \end{cases} \tag{3.2.40}$$

where $G = DF$ which satisfies

$$\|u\|_X \to \infty \Leftrightarrow F(u) \to \infty \tag{3.2.41}$$

$$\|u\|_X \to 0 \Leftrightarrow \begin{cases} F(u) \to 0 \\ < Gu, u > \to 0 \end{cases} \tag{3.2.42}$$

Theorem 3.2.13. Under the assumptions (3.2.41) and (3.2.42), if $G : X \to X^*$ is weakly continuous, then for any $(\phi, \psi) \in X \times H$, the global solution $u(t, \phi, \psi)$ of (3.2.40) exists in $W^{1,\infty}((0,\infty), H) \cap W^{1,2}((0,\infty), H) \cap L^\infty((0,\infty), X)$, and $u = 0$ is globally asymptotically stable, namely

$$\lim_{t\to\infty} [\|u\|_X + \|u_t\|_H] = 0 \quad \forall (\phi, \psi) \in X \times H. \tag{3.2.43}$$

Proof. By using the same method as used in Theorem 2.3.3, one can derive the global existence of (3.2.40) in $W^{1,\infty}((0,\infty), H) \cap W^{1,2}((0,\infty), H) \cap L^\infty((0,\infty), X)$, and the solution $u(t)$ satisfies

$$\frac{1}{2}\|u_t\|_H^2 + F(u) = \frac{1}{2}\|\psi\|_H^2 + F(\phi) - k\int_0^t \|u_t\|_H^2 dt \tag{3.2.44}$$

and

$$\int_0^t [< Gu, v > + k < \frac{du}{dt}, v >_H - < \frac{du}{dt}, \frac{dv}{dt} >_H] dt$$

$$+ < \frac{du(t)}{dt}, v(t) >_H - < \psi, v(0) >_H = 0 \tag{3.2.45}$$

for all $v \in W^{1,\infty}((0,\infty),H) \cap W^{1,2}((0,\infty),H) \cap L^{\infty}((0,\infty),X)$. Inserting $v = \frac{k}{2}u$ in (3.2.45) we get

$$\frac{k}{2}\int_0^t \|u_t\|_H^2 dt = \frac{k^2}{4} < u,u >_H + \frac{k}{2}\int_0^t < Gu,u > dt$$

$$+\frac{k}{2} < u_t,u >_H - \frac{k}{2} < \psi,\phi >_H - \frac{k^2}{4} < \phi,\phi >_H \tag{3.2.46}$$

By the Young inequality

$$| < u_t,u >_H | \leq \frac{k}{2} < u,u >_H + \frac{1}{2k} < u_t,u_t >_H$$

from (3.2.44) and (3.2.46) it follows that

$$\frac{1}{2}\|u_t\|_H^2 + F(u) \leq -\frac{k}{2}\int_0^t [\|u_t\|_H^2 + < Gu,u >]dt +$$

$$+\frac{1}{4}\|u_t\|_H^2 + \frac{k^2}{4}\|\phi\|_H^2 + \frac{1}{2}\|\psi\|_H^2 + F(\phi) + \frac{k}{2} < \phi,\psi >_H$$

Hence

$$\frac{1}{4}\|u_t\|_H^2 + F(u) \leq c - \frac{k}{2}\int_0^t [\|u_t\|_H^2 + < Gu,u >]dt \tag{3.2.47}$$

By (3.2.44), $\frac{1}{2}\|u_t\|^2 + F(u)$ is a decreasing function on $t \in [0,\infty)$. If

$$\frac{1}{2}\|u_t\|_H^2 + F(u) \to 0, \quad t \to \infty$$

then by (3.2.42) we infer (3.2.43), and otherwise there is a constant $\alpha > 0$ such that

$$\|u_t\|_H + \|u\|_X \geq \alpha > 0 \quad \forall t \geq 0$$

which means that (by (3.2.42))

$$\|u_t\|_H^2 + < Gu,u > \geq \alpha_1 > 0 \quad \forall t \geq 0$$

and one reads a contradiction with (3.2.47). This proof is complete.

If the conditions (3.2.41) and (3.2.42) are replaced by the following local condition, i.e. there are a constant $\delta > 0$ such that as $\|u\|_X < \delta, F(u) > 0$ and

$$\begin{cases} < G(u),u > \to 0 \\ F(u) \to 0, \end{cases} \Leftrightarrow \|u\|_X \to 0 \tag{3.2.48}$$

then we can obtain the below local stability theorem.

Theorem 3.2.14. Under the condition (3.2.45), if $G : X \to X^*$ is weakly continuous, then there is a constant $\epsilon > 0$, as $(\phi, \psi) \in X \times H$ and $\|\phi\|_X + \|\psi\|_H < \epsilon$, the global solution of (3.2.40) exists in $W^{1,\infty}((0,\infty), H) \cap W^{1,2}((0,\infty), H) \cap L^\infty((0,\infty), X)$, and $u = 0$ is locally asymptotically stable.

Proof. From (3.2.44) and (3.2.48) it follows that there is a constant $\epsilon > 0$, when $\|\phi\|_X + \|psi\|_H < \epsilon$ the solution $u(t, \phi, \psi)$ of (3.2.40) satisfies

$$\|u(t, \phi, \psi)\|_X < \delta, \quad \forall t \geq 0$$

Then, in the same fashion as used in Theorem 3.2.13, we can obtain this theorem. The proof is complete.

3.3. Applications to PDE

3.3.1. 2D Navier-Stokes equations

In this paragraph, we discuss the global stability of the two dimensional Navier-Stokes equations

$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = \mu \triangle v - \nabla P + \lambda f(x), \quad x \in \Omega \subset R^2 \\ divv = 0 \\ v|_{\partial\Omega} = 0, \quad v(x, 0) = \phi(x) \end{cases} \tag{3.3.1}$$

where $\mu > 0$ is the kinematic viscosity, $\lambda > 0$ a parameter, and $f(x)$ represent a density of force. Let v_λ be the equilibrum solution of (3.3.1) which satisfies

$$\begin{cases} -\mu \triangle v + (v \cdot \nabla)v + \nabla p = \lambda f(x) \\ divv = 0 \\ v|_{\partial\Omega} = 0 \end{cases} \tag{3.3.2}$$

Denote by

$$u \cdot \nabla v = \{u_i \frac{\partial v_i}{\partial x_1}, u_i \frac{\partial v_i}{\partial x_2}\}$$

We say that $\lambda_0 > 0$ is a critical parameter of (3.3.1) if either $\lambda_0 = +\infty$, or the equation

$$(3.3.3) \qquad \begin{cases} -\mu\triangle u + \frac{1}{2}(u\cdot\nabla)v_\lambda + \frac{1}{2}u\cdot\nabla v_\lambda - \nabla p = 0 \\ divu = 0 \\ u|_{\partial\Omega} = \end{cases}$$

has no nonzero solution for any $0 \le \lambda < \lambda_0$, and has the nonzero solutions for $\lambda = \lambda_0$. Because the equation (3.3.3) can be transformed into an operator equation of symmetric completely continuous field, the difinition of critical parameter λ_0 of (3.3.1) makes sense.

As λ_0 is the critical parameter of (3.3.1) it is known that the solution set of (3.3.3) with $\lambda = \lambda_0$ is a finite dimensional space. From Theorem 3.2.8 we can derive the following theorem.

Theorem 3.3.1. Let $\lambda_0 > 0$ be a critical parameter of (3.3.1). If $\lambda_0 < \infty$ and for any nonzero solution $u(x)$ of (3.3.) with $\lambda = \lambda_0$ we have

$$(3.3.4) \qquad (u\cdot\nabla)u + (v_{\lambda_0}\cdot\nabla)u + \frac{1}{2}(u\cdot\nabla)v_{\lambda_0} - \frac{1}{2}u\cdot\nabla v_{\lambda_0} \neq 0,\ mod\ \nabla p$$

and which is not a solution of (3.3.3), then the equilibrum solution v_λ of (3.3.1) is globally asymptotically stable for all $0 \le \lambda \le \lambda_0$. If $\lambda_0 = +\infty$, then v_λ is globally asymptotically stable for all $0 \le \lambda < \infty$.

Proof. Let $u = v - v_\lambda$. Then from (3.3.1) and (3.3.2) it follows that u obeys the below equation

$$(3.3.5) \qquad \begin{cases} \frac{\partial u}{\partial t} = \mu\triangle u - (v_\lambda\cdot\nabla)u - (u\cdot\nabla)v_\lambda - (u\cdot\nabla)u - \nabla p \\ divu = 0 \\ u|_{\partial\Omega} = 0, \quad u(x,0) = \psi(x) \end{cases}$$

Hence the global stability of v_λ for (3.3.1) is equivalent to that of $u = 0$ for (3.3.5).

It is known that $v_\lambda \in H^2(\Omega)$ provided $f \in L^2(\Omega)$, and $L^2(\Omega)$ can be decomposed into the below direct sum

$$\begin{cases} L^2(\Omega) = D(\Omega)\bigoplus G(\Omega), \quad D(\Omega)\perp G(\Omega) \\ D(\Omega) = \{u \in L^2(\Omega)|divu = 0, \quad u\cdot n|_{\partial\Omega} = 0\} \\ G(\Omega) = \{u \in L^2(\Omega)|\ u = \nabla\phi, \quad \phi \in H^1(\Omega)\} \end{cases}$$

Denote by

$$D^2(\Omega) = \{u \in H^2(\Omega)|\ divu = 0, \quad u|_{\partial\Omega} = 0\}$$

We define the mappings $A, B, G : D^2(\Omega) \to D(\Omega)$ by

$$< Au, v >= \mu \int_\Omega \triangle u \cdot v dx$$
$$< Bu, v >= - \int_\Omega [(v_\lambda \cdot \nabla)u + (u \cdot \nabla)v_\lambda] \cdot v dx$$
$$< Gu, v >= - \int_\Omega (u \cdot \nabla)u \cdot v dx$$

for all $v \in D(\Omega)$. Evidently, if $v_\lambda \in D^2(\Omega)$, then the operator B has the duality $B^* : D^2(\Omega) \to D(\Omega)$ which defined as

$$< B^* u, v >= \int_\Omega [(v_\lambda \cdot \nabla)u - u \cdot \nabla v_\lambda] v dx$$

Thus the equation (3.3.5) is transformed into the form

$$\begin{cases} \frac{du}{dt} = Au + Bu + Gu \\ u(0) = \psi \end{cases}$$

It is clear that

$$< Gu, u >= 0, \quad \forall u \in D^2(\Omega)$$

The condition that $\lambda_0 > 0$ is a critical parameter implies that the condition (3.2.26) is fulfilled, and the condition (3.3.4) means that $P_1(G + B_2)u \neq 0$ for $u \in E_0$ with $u \neq 0$. Therefore from Theorem 3.2.8 this theorem follows. The proof is complete.

Remark 3.3.2. It is easy to verify that there is an open set $F \subset L^2(\Omega)$ such that as $f \in F$ the conditions in Theorem 3.3.1 are fulfilled. Moreover, it is natural to conjecture that the set F is also dense in $L^2(\Omega)$, i.e. we have the following conjecture.

Theorem 3.3.3. There exists an open and dense set $F \subset L^2(\Omega)$ such that for any $f \in F$ the equilibrum solution v_λ of (3.3.1) is globally asymptotically stable for $0 \leq \lambda \leq \lambda_0$ as $\lambda_0 < \infty$, and $0 \leq \lambda < \infty$ as $\lambda_0 = \infty$, here $\lambda_0 > 0$ is the critical parameter of (3.3.1).

3.3.2. Stability in the higher differentiability

Let us consider the local asymptotical stability in the higher differentiable spaces of the below equations.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a(x)u + b(x)\frac{\partial u}{\partial x} + f(x, u, \frac{\partial u}{\partial x}), 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \phi \end{cases} \tag{3.3.6}$$

where $a, b \in c^\infty[0,1], f \in c^\infty([0,1] \times R \times R)$, and

$$f(x, \beta, \beta) = o(|\beta|^k) \quad near\ \beta = 0 \tag{3.3.7}$$

$k > 1$ is an integer. Let $\{\lambda_k\} \subset C$ be the complex eigenvalue sequence as follows

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} + a(x)w + b(x)\frac{\partial w}{\partial x} = \lambda_k w, \\ w(0) = w(1) = 0 \end{cases} \tag{3.3.8}$$

where $w = u + iv$ are the complex functions.

Applying Theorem 3.2.6 we can derive the following theorem.

Theorem 3.3.4. Under the condition (3.3.7), if the eigenvalues of (3.3.8) satisfy that $Re\lambda_n < 0, \forall n \geq 1$, and for some integer $m \geq 1$ the global solutions of (3.3.6) exist in $H^{2m}(\Omega)$ for $\phi \in U \subset H^{2m}(\Omega)$ a neighborhood of $u = 0$, then the equilibrum solution $u = 0$ of (3.3.6) is locally as asymptotically stable in $H^m(\Omega) \cap H_0^1(\Omega)(\Omega = (0,1))$, and which is exponentially decayed.

Proof. Let $H_1 = H^2(\Omega) \cap H_0^1(\Omega)$ and $H = L^2(\Omega)$. It is known that $H_\theta = H^{2\theta}(\Omega) \cap H_0^1(\Omega)(\theta \geq \frac{1}{2})$. We define the operator $L = -A + B$ and $G : H_1 \to H$ respectively by

$$Au = -\frac{\partial^2 u}{\partial x^2} \in H, \forall u \in H_1$$

$$Bu = a(x)u + b(x)\frac{\partial u}{\partial x} \in H, \forall u \in H_1$$

$$Gu = f(x, u, \frac{\partial u}{\partial x}) \in H, \forall u \in H_1$$

Thus the equation (3.3.6) can be written as the abstract form

$$\begin{cases} \frac{du}{dt} = Lu + Gu, u \in H_1, \\ u(0) = \phi \end{cases}$$

It is well known that the operator $L : H_1 \to H$ is a linear completely continuous field which satisfies the conditions (3.1.8) and (3.1.16) with $\gamma = \frac{1}{2}$. The exponent in (3.2.12) is $p = 2$, and $B : H_1 \to H_{\frac{1}{2}}(\frac{1}{2} > \frac{1+p}{4p} = \frac{3}{8})$ is bounded.

Because the functions a, b, f are c^∞ in their arguments,

$$B, G : H_m = H^{2m}(\Omega) \cap H_0^1(\Omega) \to H^{2m-1}(\Omega) \cap H_0^1(\Omega)$$

are continuous for all $m \geq 1$, which implies that B, G are m-regular. By the condition (3.3.7), $f(x, z, y)$ has the Taylor expansion near $(x, y) = 0$ as follows

$$f(x, z, y) = \sum_{i+j=k} \alpha_{ij} z^i y^j + o((z^2 + y^2)^{\frac{k}{2}})$$

where $\alpha_{ij} \in c^{\infty}[0,1]$. Hence for any integer $m \geq 1$ we have that

$$\|f(x,u,\frac{\partial u}{\partial x})\|_{H^{2m}} \leq c\|u\|^{k}_{H^{2m+1}}$$

for all $u \in U$ a neighborhood of $u = 0$ in $H^{2m+1}(\Omega)$, which implies that the condition (3.2.13) is satisfied, and this theorem follows from Theorem 3.2.6. The proof is complete.

3.3.3. Quasilinear parabolic equations

We discuss the locally asymptotically stable of the below quasilinear parabolic equations

$$\text{(3.3.9)} \qquad \begin{cases} \frac{\partial u}{\partial t} = -D_i A_i(x,u,\nabla u) + A_0(x,u,\nabla u), x \in \Omega \subset R^n \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where $A_i(x,0,0) = 0 (1 \leq i \leq n)$ and $A_0(x,0,0) = 0$.

Suppose that there is a function $f(x,z,\xi) \in c^1(\Omega \times R \times R^n)$ such that

$$\text{(3.3.10)} \qquad \begin{cases} A_i(x,z,\xi) = \frac{\partial}{\partial \xi_i} f(x,z,\xi), 1 \leq i \leq n \\ A_0(x,z,\xi) = \frac{\partial}{\partial z} f(x,z,\xi) \end{cases}$$

and

$$\text{(3.3.11)} \qquad \begin{cases} [A_i(x,z,\xi_1) - A_i(x,z,\xi_2)][\xi_{1i} - \xi_{2i}] \geq \alpha_1 |\xi_1 - \xi_2|^p \\ \alpha > 0, p > 1 \textit{ are constants} \end{cases}$$

$$\text{(3.2.12)} \qquad \begin{cases} \int_\Omega f(x,u,\nabla u)dx \geq \alpha_2 \|u\|^p_{W^{1,p}} + 0(\|u\|^p_{W^{1,p}}) \\ \int_\Omega [A_i(x,u,\nabla u)D_i u + A_0(x,u,\nabla u)u]dx \\ \geq \alpha_3 \|u\|^p_{W^{1,p}} + 0(\|u\|^p_{W^{1,p}}) \end{cases}$$

$$\text{(3.2.13)} \qquad \begin{cases} A_j(x,z,\xi)(0 \leq j \leq n) \textit{ satisfy the exponent growth} \\ \textit{condition } (2.4.53) \textit{ in Ch.II.} \end{cases}$$

Applying Theorem 3.2.10 we can obtain the below theorem.

Theorem 3.3.5. Under the conditions (3.3.10)-(3.3.13), there is a constant $p > 0$, when $\phi \in X$ with $\|\phi\|_X < \rho$, (3.2.9) has a global solution $u \in W^{1,2}((0,\infty), L^2(\Omega)) \cap L^\infty((0,\infty), W_0^{1,p}(\Omega))$, and the equilibrum solution $u = 0$ of (3.2.9) is locally asymptotically stable in $W_0^{1,p}(\Omega)$.

Proof. The conditions (3.3.11) and (3.3.13) means that the operator $G : W_0^{1,p}(\Omega) \to W_0^{1,-p}(\Omega)$ is coeceivly continuous, which defined by

$$< Gu, v >= \int_\Omega [A_i(x, u, \nabla u)D_i v + A_i(x, u, \nabla u)v]dx$$

and the condition (3.3.10) implies that G has the variational structure. Finally, from (3.2.12) one derives the condition (3.2.33) in Theorem 3.2.10. Consequently this theorem is verify. The proof is complete.

3.3.4. Nonlinear wave equations

In the final paragraph, we consider the global and local asymptotical stability of the nonlinear wave equations with a damping term.

First, let us investigate the global asymptotical stability of the equations given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} - \triangle u + f(x, u) = 0, \quad x \in \Omega \subset R^n \\ u|_{\partial\Omega} = 0 \\ u(x, 0) = \phi, u_t(x, 0) = \psi \end{cases} \tag{3.3.14}$$

where $k > 0$ and $f(x, 0) = 0$. Suppose that

$$\int_0^z f(x, z,)dz \geq c_1|z|^p - c_2, \quad p > 1 \tag{3.3.15}$$

$$f(x, z) = o(|z|) \quad near\ z = 0 \tag{3.3.16}$$

$$|f(x, z)| \leq c(|z|^q + 1), \quad q = \max\{p - 1, \frac{n + 2}{n - 2}\}. \tag{3.3.17}$$

Theorem 3.3.6. Under the conditions (3.3.15)-(3.3.17), the equilibrum solution $u = 0$ of (3.3.14) is globally asymptotically stable for (u, u_t) in $[H_0^1(\Omega) \cap L^q(\Omega)] \times L^2(\Omega)$.

Proof. We shall apply Theorem 3.2.13 to prove this theorem. Let $H = L^2(\Omega), X = H_0^1(\Omega) \cap L^q(\Omega), q = \max\{p - 1, \frac{n+2}{n-2}\}$, and $G : X \to X^*$ defined by

$$< Gu, v >= \int_\Omega [\nabla u \cdot \nabla v + f(x, u)v]dx, \quad v \in X$$

Obviously, G is a gradient operator, i.e. $G = DF$ and $F : X \to R$ defined by

$$\begin{cases} F(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 + f_1(x, u)]dx \\ f_1(z) = \int_0^z f(x, s)ds \end{cases}$$

By (3.3.17) it is easy to see that $G : X \to X^*$ is weakly continuous and from (3.3.15) and (3.3.16) it follows that the conditions (3.2.41) and (3.2.42) in Theorem 3.2.13 are fulfilled. Hence this theorem holds true. The proof is complete.

Next, let us consider the local asymptotical stability on the critical state of the equations given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} - \triangle u - \lambda_1 u + f(x,u) = 0, \quad x \in \Omega \subset R^n \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, u_t(x,0) = \psi \end{cases} \tag{3.3.18}$$

where $k > 0$ and λ_1 is the first eigenvalue of $-\triangle$. Suppose that

$$\begin{cases} f(x,z) = o(|z|) \\ |f(x,z)| \leq c(|z|^p + 1), p = \frac{n+2}{n-2} \end{cases} \tag{3.3.19}$$

and there is a constant $\delta > 0$, as $\|u\|_{H^1} < \delta$ we have

$$\begin{cases} \int_\Omega f(x,u)udx \geq c_1 \int_\Omega |u|^q dx \ for \ some \ q > 2 \\ \int_\Omega f_1(x,u)dx \geq c_2 \int_\Omega |u|^q dx \\ f_1(x,z) = \int_0^z f(x,s)ds \end{cases} \tag{3.3.20}$$

Theorem 3.3.7. Under the assumptions (3.3.19) and (3.3.20), the equilibrum solution $u = 0$ of (3.3.18) is locally asymptotically stable for (u, u_t) in $H_0^1(\Omega) \times L^2(\Omega)$.

Proof. We prove this theorem by using Theorem 3.2.14. Let $X = H_0^1(\Omega)$, and $G : X \to X^*$ defined by

$$< Gu, v >= \int_\Omega [\nabla u \nabla v - \lambda_1 u \cdot v + f(x,u)v]dx$$

And $G = DF, F : X \to R$ defined as

$$F(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 - \frac{1}{2}\lambda_1 |u|^2 + f_1(x,u)]dx$$

It suffices to check the condition (3.2.48). By (3.3.20) we have

$$\begin{cases} < Gu, u >\geq c_1 \int_\Omega |u|^q dx \\ F(u) \geq c_2 \int_\Omega |u|^q dx \end{cases} \quad \forall u \in H_0^1(\Omega) \ with \ \|u\|_{H^1} < \delta$$

which implies that (3.2.48) holds true. The proof is complete.

Remark 3.3.8. If the condition (3.3.20) is relaxed as

$$\begin{cases} f(x,z) \geq c|z|^{q-1} + o(|z|^{q-1}), q > 2 \\ f_1(x,z) \geq c|z|^q + o(|z|^q), \end{cases}$$

then the conclusion of Theorem 3.3.7 still holds true.

Remark 3.3.9. The local stability problem on the critical state of nonlinear wave equations is very important for the discusion of dynamic attractor bifurcation in Ch V.