Applied Mathematics E-Notes, 10(2010), 191-198 © ISSN 1607-2510
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Risk Process With Barrier And Random Income*

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Received 19 May 2010

Abstract

In this paper, we consider a risk process with random income and a constant barrier. We first derive an integral equation for the Gerber-Shiu function. Then we show that the Gerber-Shiu function satisfies a Volterra integral equation of the second kind when the individual premium income is exponentially distributed. Some explicit results are obtained for exponential claims.

1 Introduction

In the classical risk model, the premiums are assumed to be received by insurance companies at a constant rate over time. This hypothesis simplifies the study of many risk quantities of interest under such a frame work but it fails to capture the uncertainty of the customer's arrivals and the amount of premiums for different kinds of customers. To reflect the cash flows of the insurance company more realistically, some papers assumed that the insurer earns random premium income. Among them, Boikov [1] investigated the probability of ultimate ruin of an insurance portfolio, where the claim and the premium aggregate processes are both compound Poisson processes. Later, the same risk model was further studied by Yao et al. [12] and Labbé and Kristina [5]. Both of them managed to obtain some results about the Gerber-Shiu function. While Bao [2], Bao and Ye [3] and Yang and Zhang [6] made simpler assumption about the premium process and also managed to derive some results about the Gerber-Shiu function.

In this paper, we consider a modification of the risk model proposed by Boikov [1] in the presence of a constant dividend barrier. We recall that the barrier strategy has been first proposed by De Finetti [4] for a binomial model. Now barrier strategies for the classical risk model have been studied in detail by numerous authors, e.g. Lin et al. [8], Dickson and Waters [9], Landriault [10] and references therein.

The rest of the paper is organized as follows. In Section 2, we introduce the risk models and notations that are used throughout the article. In Section 3, we derive an integral equation for the Gerber-Shiu function. In Section 4, we consider the special case where the premium sizes are exponentially distributed. We show that the Gerber-Shiu

*Mathematics Subject Classifications: 91B30, 91B70.

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function satisfies a Volterra equation of the second kind, based on the Volterra equation we derive the general solution for the Gerber-Shiu function. We also obtain some explicit results for Gerber-Shiu function when claim sizes are exponentially distributed.

2 Risk Models and Notations

In the class of risk models studied by Boikov [1], it is assumed that the claim number process $\underline{N} = \{N(t),\, t \geq 0\}$ is a Poisson process with independent and identically distributed (i.i.d) exponential interclaim times $\{W_j, j = 1, 2, \cdots\}$. In this paper, we assume that $EW_j = 1/\mu$. The individual claim amounts $\{Y_j,\, j = 1, 2, \cdots\}$ are assumed to be a sequence of i.i.d. positive r.v.'s with the common absolutely continuous distribution function (d.f.) Q, continuous probability density function (p.d.f.) q and finite mean m_Y.

Denote the aggregate claim process by $\{S_1(t), t \geq 0\}$, i.e. $S_1(t) = \sum_{i=1}^{N(t)} Y_i$. While the premiums occur in time according to homogenous Poisson process $\{M(t), t \geq 0\}$ with intensity $\lambda > 0$. The premium sizes are given by the sequence of i.i.d positive random variables $X_1,\, X_2, \cdots$ with the common d.f.P, finite mean m_X and continuous p.d.f. p. Denote the aggregate premiums until time t by $S_2(t) = \sum_{i=1}^{M(t)} X_i$. We also assume that $\{M(t)\}$, $\{N(t)\}$, $\{X_i\}$ and $\{Y_i\}$ are mutually independent.

The insurer's surplus process without a barrier is $\{U(t),\, t \geq 0\}$ with $U(t) = u + S_2(t) - S_1(t)$ or $\mathrm{d}U(t) = \mathrm{d}S_2(t) - \mathrm{d}S_1(t)$. In the above, $u = U(0) \geq 0$ is the initial surplus. Let $\lambda m_X = (1+\theta)\mu m_Y$, where $\theta > 0$ is the relative security loading.

A barrier strategy considered in this paper assumes that there is a horizontal barrier of level $b > 0$ such that whenever the surplus exceeds the level b, the excess is paid out immediately as a dividend. Let $U_b(t)$ be the surplus process with initial surplus $U_b(0) = u$ under the barrier strategy above. Thus $U_b(t)$ can be expressed as

$$dU_b(t) = \begin{cases} \mathrm{d}S_2(t) - \mathrm{d}S_1(t), & U(t) \leq b; \\ -\mathrm{d}S_1(t), & U(t) > b. \end{cases} \tag{1}$$

Let $T_b = \inf\{t : U_b(t) < 0 | U_b(0) = u\}$ be the ruin time associated to the surplus process $U_b(t)$ with $T_b = \infty$ if $U_b(t) \geq 0$ for $t \geq 0$ (i.e. ruin does not occur). Let $\omega(x_1, x_2)$, $x_1 \geq 0$, $x_2 > 0$ be a nonnegative bounded function. For $\delta \geq 0$, the Gerber-Shiu function $m_b(u)$ is defined as

$$m_b(u) = E[e^{-\delta T_b}\omega(U_b(T_b-), |U_b(T_b)|)I(T_b < +\infty)|U_b(0) = u], \tag{2}$$

where $U_b(T_b-)$ is the surplus just prior to ruin, $|U_b(T_b)|$ is the deficit at ruin, $I(\cdot)$ is the indicator function. When $\omega(x_1, x_2) = 1$, (2) is the Laplace transform of the time of ruin T_b, denoted by $\phi_b(u) = E[e^{-\delta T_b} I(T_b < \infty)|U_b(0) = u]$. When $\omega(x_1, x_2) = 1$, $\delta = 0$, (2) is the ruin probability $\psi_b(u) = P(T_b < \infty|U_b(0) = u)$. Note that for b finite, ruin will occur almost surely, which implies that the indicator function $I(T_b < \infty)$ can be dropped from the definition of $m_b(u)$.

3 Integral Equation

In this section, our goal is to derive an integral equation for the Gerber-Shiu function $m_b(u)$.

THEOREM 1. For $0 \leq u \leq b$, the Gerber-Shiu function $m_b(u)$ satisfies the following integral equation

$$\begin{aligned}(\lambda+\mu+\delta)m_b(u) &= \lambda\int_u^b m_b(x)p(x-u)\mathrm{d}x + \lambda m_b(b)\overline{P}(b-u) \\ &\quad +\mu\int_0^u m_b(u-y)q(y)\mathrm{d}y + \mu\omega(u), \end{aligned} \tag{3}$$

where $\omega(u) = \int_u^\infty \omega(u, y-u)q(y)\mathrm{d}y$.

PROOF. We consider all possible events over an infinitesimal interval $(0, \mathrm{d}t)$ and obtain

$$\begin{aligned} m_b(u) &= (1-\lambda\mathrm{d}t)(1-\mu\mathrm{d}t)e^{-\delta\mathrm{d}t}m_b(u) \\ &\quad +\lambda\mathrm{d}t(1-\mu dt)e^{-\delta\mathrm{d}t}[\int_0^{b-u} m_b(u+x)p(x)dx + m_b(b)\int_{b-u}^\infty p(x)\mathrm{d}x] \\ &\quad +(1-\lambda\mathrm{d}t)\mu\mathrm{d}te^{-\delta dt}[\int_0^u m_b(u-y)q(y)\mathrm{d}y + \int_u^\infty \omega(u,y)q(y)\mathrm{d}y], \end{aligned}$$

for $0 \leq u < b$. Letting $\mathrm{d}t \to 0$ and rearranging it, we obtain (3).

Similarly, for $u = b$

$$(\mu+\delta)m_b(b) = \mu\int_0^b m_b(b-y)q(y)\mathrm{d}y + \mu\omega(b). \tag{4}$$

This illustrates that (3) still holds for $u = b$.

REMARKS: 1. When $b \to \infty$, (3) becomes (4.2) of Labbé and Sendova [5]; (2.1) of Yao et al. [12].

2. For $\delta > 0$, (4) is a defective equation. Using Theorem 2.1 of Lin and Willmot [7], we have

$$m_b(b) = \frac{\mu}{\delta}\int_0^b \omega(b-x)\mathrm{d}V(x) + \frac{\mu}{\mu+\delta}\omega(b), \tag{5}$$

where $\overline{V}(u) = \frac{\delta}{\mu+\delta}\sum_{n=1}^{+\infty}\left(\frac{\mu}{\mu+\delta}\right)^n\overline{Q^{*n}}(u), u \geq 0$ and $\overline{Q^{*n}}(u)$ is the tail of the n-fold convolution of $Q(u)$ with itself. Throughout the paper, "$*$" denotes the operation of convolution.

3. When $\delta > 0$, and $\omega(x_1, x_2) = 1$, the Gerber-Shiu function simplifies to the Laplace transform of the time of ruin $\phi_b(u)$, and (4) simplifies to

$$\phi_b(b) = \frac{\mu}{\mu+\delta}\int_0^b \phi_b(b-y)q(y)\mathrm{d}y + \frac{\mu}{\mu+\delta}\int_b^{+\infty} q(y)\mathrm{d}y,$$

which has a compound geometric representation:

$$\phi_b(b) = \frac{\delta}{\mu+\delta}\sum_{j=1}^{+\infty}\left(\frac{\mu}{\mu+\delta}\right)^j \overline{Q^{*j}}(b), \qquad b \geq 0. \tag{6}$$

4. When $\delta = 0$, (4) simplifies to the proper renewal equation

$$m_b(b) = \int_0^b m_b(b-y)q(y)\mathrm{d}y + \omega(b),$$

which is equivalent to

$$m_b(b) = \sum_{n=0}^{+\infty} Q^{*n} * \omega(b). \tag{7}$$

5. When $\delta = 0$, $\omega(x,y) = 1$, then $\omega(u) = \overline{Q}(u)$ and (4) is the ruin probability $\psi_b(b)$:

$$\psi_\delta(b) = \int_0^b \overline{Q}(b-y)\mathrm{d}R(y) + \overline{Q}(b) = \int_0^b (1-Q(b-y))\mathrm{d}R(y) + \overline{Q}(b) = 1.$$

This illustrates that ruin is certain when there is a horizontal barrier b.

4 Exponential Premium

In this section, we pay attention to the situation in which the premium sizes are exponentially distributed. Let $p(x) = \alpha e^{-\alpha x}$, $x \geq 0$, $\alpha \geq 0$, then (3) simplifies to

$$\begin{aligned}(\lambda+\mu+\delta)m_b(u) &= \lambda\alpha e^{\alpha u}\int_u^b m_b(y)e^{-\alpha y}\mathrm{d}y + \lambda m_b(b)e^{-\alpha(b-u)} \\ &\quad +\mu\int_0^u m_b(x)q(u-x)\mathrm{d}x + \mu\omega(u), \quad 0 \leq u \leq b.\end{aligned} \tag{8}$$

Differentiating the above equation with respect to u, we obtain for $0 \leq u \leq b$,

$$\begin{aligned}(\lambda+\mu+\delta)m_b'(u) &= \alpha(\mu+\delta)m_b(u) \\ &\quad + \mu\left(\frac{\mathrm{d}}{\mathrm{d}u} - \alpha\right)\left(\int_0^u m_b(x)q(u-x)\mathrm{d}x\right) + h(u),\end{aligned} \tag{9}$$

where $h(u) = \mu\omega'(u) - \alpha\mu\omega(u)$. Replacing u by x in (9) and then integrating both sides of the equation from 0 to u with respect to x, we obtain for $0 \leq u \leq b$,

$$\begin{aligned}&(\lambda+\mu+\delta)(m_b(u) - m_b(0)) \\ &= \alpha(\mu+\delta)\int_0^u m_b(x)\mathrm{d}x - \mu\int_0^u m_b(x)(\alpha Q(u-x) - q(u-x))\mathrm{d}x + \int_0^u h(x)\mathrm{d}x.\end{aligned}$$

Rearranging this equation, we have the following theorem.

THEOREM 2. If the premium size distribution P is an exponential distribution with mean $1/\alpha$, $\alpha > 0$. Then the integral equation (3) can be represented as the Volterra integral equation of the second kind

$$m_b(u) = \int_0^u k(u,x) m_b(x) \mathrm{d}x + \ell(u), \quad 0 \le x \le u \le b,$$

where

$$\begin{aligned} k(u,x) &= \frac{\alpha(\mu+\delta) - \alpha\mu Q(u-x) + \mu q(u-x)}{\lambda+\mu+\delta}, \\ \ell(u) &= m_b(0) + \frac{\int_0^u h(x)\mathrm{d}x}{\lambda+\mu+\delta}. \end{aligned} \tag{10}$$

REMARK. If $m_b(0)$ is available, then the solution for $m_b(u)$ is available. Therefore, we have to determine $m_b(0)$. It is easy to verify that $\ell(u)$ is continuous in $0 \le u \le b$ since $\omega(x,y)$ is bounded and $Q(x)$ is continuous. Obviously, $k(u,x)$ is continuous in $0 \le x \le u$ in that both $Q(x)$ and $q(x)$ are continuous functions. Then, according to Cai and Dickson (2002), the unique solution for $m_b(u)$ has the following representation, for $0 \le u \le b$

$$m_b(u) = \ell(u) + \sum_{m=1}^{\infty} \int_0^u k_m(u,x)\ell(x)\mathrm{d}x, \tag{11}$$

where $k_m(u,x) = \int_x^u k(u,t)k_{m-1}(t,x)\mathrm{d}t, \quad m = 2,3,..., 0 \le x \le u$, with $k_1(u,x) = k(u,x)$. Setting $u = b$ in (11) and combining with (10), we see that

$$m_b(0) = \frac{(\lambda+\mu+\delta)m_b(b) - \int_0^b h(x)\mathrm{d}x - \int_0^b K(b,x)\int_0^x h(y)\mathrm{d}y\mathrm{d}x}{(\lambda+\mu+\delta)\left(1 - \int_0^b K(b,x)\mathrm{d}x\right)}, \tag{12}$$

where $K(b,x) = \sum_{m=1}^{\infty} k_m(b,x), \quad 0 \le x \le b$ and $m_b(b)$ is given in (5).

THEOREM 3. When $\delta = 0$, $m_b(u)$ satisfies the following defective renewal equation, for $0 \le u \le b$,

$$m_b(u) = \int_0^u q_1(z) m_b(u-z)\mathrm{d}z + \ell_1(u), \tag{13}$$

where

$$\begin{aligned} q_1(x) &= \frac{\alpha\mu\overline{Q}(x)}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}q(x), \\ \ell_1(u) &= m_b(0) + \int_0^u h(x)\mathrm{d}x/(\lambda+\mu), \\ m_b(0) &= \frac{m_b(b) - \frac{1}{\lambda+\mu}\int_0^b\int_0^{b-y} h(x)\mathrm{d}x\mathrm{d}H(y)}{H(b)}, \end{aligned} \tag{14}$$

and $m_b(b)$ is given in (7).

PROOF. Setting $\delta = 0$ in (8), we obtain (13). Since the positive loading condition, then

$$\int_0^\infty q_1(x)\mathrm{d}x = \frac{\alpha\mu m_Y}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} < \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} = 1.$$

Since $\ell_1(u)$ is a bounded function in $0 \le u \le b$, then the unique solution to $m_b(u)$ in (13) can be expressed as

$$m_b(u) = \ell_1 * H(u), \quad \text{for } \ 0 \le u \le b,$$

where $H(x) = \sum_{n=0}^\infty \overline{Q_1^{*n}}(x)$ with $Q_1(x) = \int_0^x q_1(y)\mathrm{d}y$. Setting $u = b$ in the above equation and rearranging lead to (14).

4.1 Explicit Results for Exponential Claim Size

In this subsection, we assume that Q is an exponential distribution function with mean $1/\beta$, $\beta > 0$.

THEOREM 4. Let $P(x)$ be an exponential distribution with mean $1/\alpha$, $\alpha > 0$ and $Q(y)$ an exponential distribution with mean $1/\beta$, $\beta > 0$. Then for $0 \le u \le b$,

$$\begin{aligned}(\lambda+\mu+\delta)m_b''(u) \quad &- \quad [\alpha(\mu+\delta) - \beta(\lambda+\delta)]m_b'(u) \\ &- \quad \alpha\beta\delta m_b(u) - \beta h(u) - h'(u) = 0.\end{aligned} \tag{15}$$

Indeed, this equation can be obtained directly from (8).

EXAMPLE 1. (The distribution function of deficit at ruin) If $\delta = 0$, $\omega(x,y) = I(y \le z)$, then $m_b(u)$ reduces to the distribution function of the deficit at ruin, denoted by $F_b(z|u)$ for $z > 0$. Note that $\omega(u) = \int_0^z \beta e^{-\beta(u+y)}\mathrm{d}y = e^{-\beta u}(1 - e^{-\beta z})$ and $\beta h(u) + h'(u) = 0$. Therefore,

$$F_b(z|u) = C_1 + C_2 e^{\frac{\alpha\mu - \lambda\beta}{\lambda+\mu}u}, \qquad 0 \le u \le b, \tag{16}$$

where

$$\begin{aligned} C_1 &= (1 - e^{-\beta z})\left(1 - e^{-\beta b}\left(1 - \frac{\mu(\alpha+\beta)}{\beta(\lambda+\mu)}\exp\left(\frac{\alpha\mu - \lambda\beta}{\lambda+\mu}b\right)\right)^{-1}\right), \\ C_2 &= \mu e^{-\beta b}(1 - e^{-\beta z})\left(\frac{\beta(\lambda+\mu)}{\alpha+\beta} - \mu\exp\left(\frac{\alpha\mu - \lambda\beta}{\lambda+\mu}b\right)\right). \end{aligned}$$

Then the distribution function of the deficit at ruin is given by

$$F_b(z|u) = (1 - e^{-\beta z})(1 - e^{-\beta b}\hbar(u)/\hbar(b)),\ z > 0,\ 0 \le u \le b,$$

where $\hbar(u) = \mu\exp(\frac{\alpha\mu - \lambda\beta}{\lambda+\mu}u) - \frac{\beta(\lambda+\mu)}{\alpha+\beta}$.

REMARK. When $b \to \infty$, $F_b(z|u) \to 1 - e^{-\beta z}$, $z > 0$. This is clear in that the claim sizes are exponentially distributed.

EXAMPLE 2. (The Laplace transform of the time to ruin) When $\delta > 0$, $\omega(x, y) = 1$, $m_b(u)$ reduces to the Laplace transform of ruin probability $\phi_b(u)$ with $\omega(u) = e^{-\beta u}$, $\beta h(u) + h'(u) = 0$. Thus (15) simplifies to

$$(\lambda + \mu + \delta)\phi_b''(u) - [\alpha(\mu + \delta) - \beta(\lambda + \delta)]\phi_\delta'(u) - \alpha\beta\delta\phi_\delta(u) = 0,$$

which leads us to

$$\phi_b(u) = c_1 e^{-Ru} + c_2 e^{\rho u}, \qquad 0 \leq u \leq b, \tag{17}$$

where $-R < 0$ and $\rho > 0$ are solutions of

$$(\lambda + \mu + \delta)s^2 - [\alpha(\mu + \delta) - \beta(\lambda + \delta)]s - \alpha\beta\delta = 0.$$

Substituting (17) into (8) and comparing the coefficients of $e^{\alpha u}$ and $e^{-\beta u}$ respectively, yields

$$\begin{cases} -\frac{c_1 R}{R+\alpha} e^{-(R+\alpha)b} + \frac{c_2 \rho}{\alpha - \rho} e^{-(\alpha-\rho)b} = 0, \\ \frac{c_1 \beta}{\beta - R} + \frac{c_2 \beta}{\rho + \beta} = 1. \end{cases}$$

Solving the system of equations above gives

$$\begin{aligned} c_1 &= \frac{\rho}{\alpha - \rho}\left(\frac{\beta\rho}{(\beta - R)(\alpha - \rho)} + \frac{\beta R}{(\beta + \rho)(R + \alpha)} e^{-(\rho+R)b}\right)^{-1}, \\ c_2 &= \frac{R}{R + \alpha}\left(\frac{\beta\rho}{(\beta - R)(\alpha - \rho)} e^{(\rho+R)b} + \frac{\beta R}{(\beta + \rho)(R + \alpha)}\right)^{-1}. \end{aligned}$$

REMARKS: 1. When $b \to \infty$, then $c_2 = 0$, $c_1 = \frac{\beta - R}{\beta}$, and

$$\phi_\delta(u) = \frac{\beta - R}{\beta} e^{-Ru} = \frac{\mu(\alpha + R)}{(\lambda + \mu + \delta)(\alpha + R) - \alpha\lambda} e^{-Ru}, \tag{18}$$

since $-R$ satisfies the Lundberg equation

$$\lambda + \mu + \delta = \frac{\alpha\lambda}{\alpha - s} + \frac{\beta\mu}{s + \beta}.$$

(18) is identical to Theorem 2.2 of Yao et al. [12] with $\lambda = \lambda_1$, $\alpha = a$, $R = -\beta_1$, $\mu = \lambda_2$. This also illustrates that Theorem 2.2 of Yao et al. [12] can be rewritten in a simpler style:

$$\phi_\delta(u) = \left(1 + \frac{\beta_1}{b}\right) e^{\beta_1 u},$$

which can also be obtained by substituting $\psi_\delta(u) = -c_1 e^{\beta_1 u}$ into (2.1) of Yao et al. [12].

2. When $\delta = 0$, $\rho = 0$, then $c_1 = 0$, $c_2 = 1$, $\psi_\delta(u) = 1$, $0 \leq u \leq b$. This illustrates that the ruin is certain when there is a horizontal barrier $b < \infty$.

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