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A Study Of Boundary Value Problem For An Elliptic Equation In Hölder Spaces*

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Abstract

We give in this work some new results about the existence, uniqueness and optimal regularity for the strict solution of an abstract second-order differential equation set in an unbounded interval. We use similar techniques with those of Labbas [9], when the right-hand term is Hölder continuous function.

1 Introduction

The aim of this paper is to study the following second order abstract differential equation

$$u''(t) + Au(t) = f(t), \quad t \in (0, +\infty), \tag{1}$$

under the non-homogeneous boundary conditions

$$u(0) = \varphi, \quad u(+\infty) = 0, \tag{2}$$

where A is a closed linear operator with dense domain $D(A)$ in a complex Banach space E and φ is a given element of $D(A)$. The vector-valued function f is continuous on $[0, +\infty[$ into E and verifies

$$\lim_{t \to +\infty} \|f(t)\|_E = 0. \tag{3}$$

Throughout this work we assume that there exists $K > 0$ such that for all $\lambda \geqslant 0$,

$$\|(A - \lambda I)^{-1}\|_{L(E)} \leqslant \frac{K}{1 + \lambda}. \tag{4}$$

We recall that for $m \in \mathbb{N}$, $BUC^m([0, +\infty[; E)$ denotes the space of vector-valued functions with uniformly continuous and bounded derivatives up to the order m in $[0, +\infty[$.

For $\sigma \in]0, 1[$, the Banach space $C^\sigma([0, +\infty[; E)$ denotes the space of the bounded and σ-Hölder continuous functions $f : [0, +\infty[\longrightarrow E$, such that

$$\begin{cases} \sup_{t \in [0,+\infty[} \|f(t)\|_E < \infty, \\ \exists C > 0 : \forall t, \tau \in [0, +\infty[, \quad \|f(t) - f(\tau)\|_E \leqslant C|t - \tau|^\sigma, \end{cases}$$

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endowed with the norm

$$\|f\|_{C^\sigma([0,+\infty[;E)} = \sup_{t\in[0,+\infty[} \|f(t)\|_E + \sup_{t\neq\tau} \frac{\|f(t)-f(\tau)\|_E}{|t-\tau|^\sigma} = \|f\|_\infty + [f]_{C^\sigma([0,+\infty[;E)}.$$

For simplicity, we shall write $C^\sigma(E)$ instead of $C^\sigma([0,+\infty[;E)$.

We say that $u \in BUC([0,+\infty[;E)$ is a strict solution of (1)-(2) if

$$u \in BUC^2([0,+\infty[;E) \cap BUC([0,+\infty[;D(A)),$$

and u satisfies (1) and (2).

Observe that equation (1) has been studied by many authors, in different situations, but on a bounded domain. See, for example Krein [7], Sobolevskii [13], Kuyazyuk [8], Da Prato-Grisvard [3], Labbas [9], Favini-Labbas-Lemrabet-Sadallah [4], Favini-Labbas-Tanabe-Yagi [5].

In the present study, the principal goal is to give an alternative approach to that used in Berroug-Labbas-Sadallah [2]. The techniques we use are essentially based on the theory of fractional powers of linear operators in Banach spaces and on the semigroups estimates generated by them as in Krein [7] and in Sinestrari [12]. We make use of the real Banach interpolation space $D_A(\theta,+\infty)$, between $D(A)$ and E. It is characterized in [6], by

$$D_A(\theta,+\infty) = \{\xi \in E : \sup_{t>0} \|t^\theta A(A-tI)^{-1}\xi\|_E < \infty\}.$$

We will prove the following main results.

THEOREM 1 (Existence and uniqueness). Let $0<\theta<1/2$, $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$ with assumption (3). Then problem (1)-(2) admits a unique strict solution.

THEOREM 2 (Regularity). Let $0<\theta<1/2$, $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$ with assumption (3). If $f(0) - A\varphi \in D_A(\theta,+\infty)$, then the unique strict solution of (1)-(2) satisfies the property of maximal regularity $Au(.)$, $u''(.) \in C^{2\theta}(E)$.

2 Proof of Theorem 1

We start by some recall of the theory of fractional powers of linear operators as developed in Balakrishnan [1], Krein [7] and Pazy [11]. It is well known that assumption (4) implies that $(-(-A)^{1/2})$ is the infinitesimal generator of an analytic semigroup $\{V(t)\}$, $t \geqslant 0$ (for details, see [1]). Moreover we have the practical well known results

PROPOSITION 1.
(1) $\exists M, \delta > 0$ such that $\|V(t)\|_{L(E)} \leqslant Me^{-\delta t}$,
(2) there exists $C>0$ such that for $t>0$, $\|(-A)^{1/2}V(t)\|_{L(E)} \leqslant Ct^{-1}e^{-\delta t}$.

PROPOSITION 2. For all $x \in E$ we have
(1) $\int_0^t V(s)x\,ds = (-A)^{-1/2}(x - V(t)x)$,
(2) $\int_t^{+\infty} V(s)x\,ds = (-A)^{-1/2}V(t)x$.

Let $0 < \theta < 1/2$ and $f \in X = C^{2\theta}(E)$ with assumption (3). First, we seek for a particular solution $v(.)$ to equation (1). Let us set for $t \in [0, +\infty[$

$$v(t) = -\frac{1}{2}\int_0^t V(t-s)(-A)^{-1/2} f(s)ds - \frac{1}{2}\int_t^{+\infty} V(s-t)(-A)^{-1/2} f(s)ds.$$

Notice that the second integral is convergent. Indeed, Proposition 1 implies

$$\left\| \int_t^{+\infty} V(s-t)(-A)^{-1/2} f(s)ds \right\|_E \leqslant M' \int_t^{+\infty} e^{-\delta(s-t)} ds \|f\|_X.$$

We can see that the derivative $v'(t)$ exists and

$$v'(t) = \frac{1}{2}\int_0^t V(t-s)f(s)ds - \frac{1}{2}\int_t^{+\infty} V(s-t)f(s)ds.$$

To show that $v(t) \in D(A)$ and $Av(.)$ is continuous we write (thanks to Proposition 2)

$$\begin{aligned} Av(t) &= -\frac{1}{2}A(-A)^{-1/2}\int_0^t V(t-s)(f(s)-f(t))ds \\ &\quad -\frac{1}{2}\left(A(-A)^{-1/2}\int_0^t V(t-s)ds\right) f(t) \\ &\quad +\frac{1}{2}A(-A)^{-1/2}\int_t^{+\infty} V(s-t)(f(t)-f(s))ds \\ &\quad -\frac{1}{2}\left(A(-A)^{-1/2}\int_t^{+\infty} V(s-t)ds\right) f(t) \\ &= \frac{1}{2}(U(t)+S(t)) + f(t) - \frac{1}{2}V(t)f(t). \end{aligned}$$

where

$$U(t) = \int_0^t \frac{\partial V}{\partial s}(t-s)(f(s)-f(t))ds = \int_0^t (-A)^{1/2} V(t-s)(f(s)-f(t))ds,$$

$$S(t) = \int_t^{+\infty} \frac{\partial V}{\partial s}(s-t)(f(t)-f(s))ds = \int_t^{+\infty} -(-A)^{1/2} V(s-t)(f(t)-f(s))ds.$$

Furthermore, as f is Hölder-continuous, $v'(.)$ is differentiable with

$$\begin{aligned} v''(t) &= f(t) - \frac{1}{2}(-A)^{1/2}\int_0^t e^{-(t-s)(-A)^{1/2}} f(s)ds \\ &\quad -\frac{1}{2}(-A)^{1/2}\int_t^{+\infty} e^{(t-s)(-A)^{1/2}} f(s)ds \\ &= f(t) - \frac{1}{2}(-A)^{1/2}\int_0^t e^{-(t-s)(-A)^{1/2}} (f(s)-f(t))ds \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\left((-A)^{1/2}\int_0^t e^{-(t-s)(-A)^{1/2}}ds\right)f(t) \\
& -\frac{1}{2}(-A)^{1/2}\int_t^{+\infty} e^{(t-s)(-A)^{1/2}}(f(s)-f(t))ds \\
& -\frac{1}{2}\left((-A)^{1/2}\int_t^{+\infty} e^{(t-s)(-A)^{1/2}}ds\right)f(t) \\
= \; & f(t)-\frac{1}{2}U(t)-\frac{1}{2}(I-V(t))f(t)-\frac{1}{2}S(t)-\frac{1}{2}f(t) \\
= \; & -\frac{1}{2}U(t)-\frac{1}{2}S(t)+\frac{1}{2}V(t)f(t),
\end{aligned}
$$

so

$$v''(t)+Av(t)=f(t).$$

Hence v is a strict solution to (1) satisfying the boundary conditions

$$v(0)=-\frac{1}{2}\int_0^{+\infty}V(s)(-A)^{-1/2}f(s)ds, \quad v(+\infty)=0,$$

for the last condition we use the estimate

$$
\begin{aligned}
\left\|\int_{t/2}^t V(t-s)(-A)^{-1/2}f(s)ds\right\|_E & \leqslant M \max_{r\in[\frac{t}{2},t]}\|f(r)\|_E\left(\int_{t/2}^t e^{-\delta(t-s)}ds\right) \\
& \leqslant \frac{M}{\delta}\max_{r\in[\frac{t}{2},t]}\|f(r)\|_E\left(1-e^{-\frac{\delta}{2}t}\right).
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
(-A)v(0) & = -\frac{1}{2}(-A)^{1/2}\int_0^{+\infty}V(s)(f(s)-f(0))ds-\frac{1}{2}(-A)^{1/2}\int_0^{+\infty}V(s)f(0)ds \\
& = \frac{1}{2}\int_0^{+\infty}\frac{\partial V}{\partial s}(s)(f(s)-f(0))ds-\frac{1}{2}f(0),
\end{aligned}
$$

from Proposition 1, we conclude since f is Hölder-continuous, that $v(0)\in D(A)$.

We will also use the following lemma

LEMMA 1. Assume (4) and let $\xi\in D(A)$. Then the homogeneous Problem

$$
\begin{cases} u''(t)+Au(t)=0, \quad t\in[0,+\infty[, \\ u(0)=\xi, \quad u(+\infty)=0, \end{cases} \tag{5}
$$

admits a unique strict solution $u(.)$.

PROOF. Let us set $u(t)=V(t)\xi$. Since $\xi\in D(A)$ we can easily see that

$$u'(t)=-V(t)(-A)^{1/2}\xi,$$

and

$$u''(t)=V(t)(-A)\xi,$$

then

$$\begin{cases} u''(t) = (-A)u(t), \\ u(0) = \xi, \quad u(+\infty) = 0. \end{cases}$$

Let us return to the proof of Theorem 1. The Problem

$$\begin{cases} u''(t) + Au(t) = 0, \quad t \in [0, +\infty[, \\ u(0) = x_0, \\ u(+\infty) = 0, \end{cases}$$

with

$$x_0 = \varphi - v(0),$$

admits a unique strict solution $\overline{u}$. Indeed, we know that $v(0) \in D(A)$ and $\varphi \in D(A)$, thus Lemma 1 applies. Therefore,

$$u(.) = v(.) + \overline{u}(.),$$

is the unique strict solution to problem (1)-(2).

3 Proof of Theorem 2

Let $0 < \theta < 1/2$, $\varphi \in D(A)$ and $f \in X = C^{2\theta}(E)$ with assumption (3). Let us suppose that $f(0) - A\varphi \in D_A(\theta, +\infty)$. It is enough to do it for $Au(.)$, for this purpose we write

$$\begin{aligned} Au(t) &= Av(t) + V(t)(A\varphi - Av(0)) \\ &= \frac{1}{2}(U(t) + S(t)) + f(t) - \frac{1}{2}V(t)f(t) \\ &\quad + V(t)\left(A\varphi + \frac{1}{2}\int_0^{+\infty} \frac{\partial V}{\partial s}(s)(f(s) - f(0))ds - \frac{1}{2}f(0)\right) \\ &= \frac{1}{2}(U(t) + S(t)) + f(t) + K(t), \end{aligned}$$

where

$$\begin{cases} U(t) = \int_0^t (-A)^{1/2} e^{-(-A)^{1/2}(t-s)}(f(s) - f(t))ds \\ S(t) = \int_t^{+\infty} -(-A)^{1/2} e^{-(-A)^{1/2}(s-t)}(f(t) - f(s))ds \\ K(t) = V(t)\left(A\varphi + \frac{1}{2}\int_0^{+\infty} \frac{\partial V}{\partial s}(s)(f(s) - f(0))ds - \frac{1}{2}f(0)\right) - \frac{1}{2}V(t)f(t). \end{cases}$$

Let us show the holderianity of $U(.)$, $S(.)$ and $K(.)$. For $0 \leqslant r < t$, we get

$$\begin{aligned} U(t) - U(r) &= \int_r^t (-A)^{1/2} e^{-(-A)^{1/2}(t-s)}(f(s) - f(t))ds \\ &\quad + \int_0^r (-A)^{1/2} e^{-(-A)^{1/2}(t-s)}(f(s) - f(t))ds \\ &\quad - \int_0^r (-A)^{1/2} e^{-(-A)^{1/2}(r-s)}(f(s) - f(r))ds \end{aligned}$$

$$= \quad a+b-c.$$

We have

$$\|a\|_E \leqslant C\int_r^t \frac{(t-s)^{2\theta}}{(t-s)}ds\|f\|_X \leqslant C(t-r)^{2\theta}\|f\|_X,$$

on the other hand we can see that

$$\begin{aligned}
b-c &= \int_0^r (-A)^{1/2}\left(e^{-(-A)^{1/2}(t-s)} - e^{-(-A)^{1/2}(r-s)}\right)(f(s)-f(r))ds \\
&\quad + \int_0^r (-A)^{1/2}e^{-(-A)^{1/2}(t-s)}(f(r)-f(t))ds \\
&= \int_0^r\int_{r-s}^{t-s} -((-A)^{1/2})^2 e^{-(-A)^{1/2}\sigma}(f(s)-f(r))d\sigma ds \\
&\quad + \int_0^r (-A)^{1/2}e^{-(-A)^{1/2}(t-s)}(f(r)-f(t))ds \\
&= \int_0^r\int_{r-s}^{t-s} -((-A)^{1/2})^2 e^{-(-A)^{1/2}\sigma}(f(s)-f(r))d\sigma ds \\
&\quad + \left[e^{-(-A)^{1/2}t} - e^{-(-A)^{1/2}(t-r)}\right](f(t)-f(r)) \\
&= b_1 + c_1,
\end{aligned}$$

and

$$\begin{aligned}
\|b_1\|_E &\leqslant \int_0^r\int_{r-s}^{t-s} \left\|-((-A)^{1/2})^2 e^{-(-A)^{1/2}\sigma}(f(s)-f(r))\right\|_E d\sigma ds \\
&\leqslant C\int_0^r (r-s)^{2\theta}\int_{r-s}^{t-s}\frac{1}{\sigma^2}d\sigma ds\|f\|_X \\
&\leqslant C\int_0^r \frac{(r-s)^{2\theta-1}(t-r)}{(t-r+r-s)}ds\|f\|_X.
\end{aligned}$$

Now, by making the change of variable $(r-s)=(t-r)\xi$, it follows

$$\int_0^r \frac{(r-s)^{2\theta-1}(t-r)}{(t-r+r-s)}ds \leqslant (t-r)^{2\theta}\int_0^{+\infty}\frac{\xi^{2\theta-1}}{1+\xi}d\xi \leqslant C(t-r)^{2\theta}.$$

Holderianity of c_1 is obvious.

For $S(.)$, one has

$$\begin{aligned}
S(r)-S(t) &= \int_t^{+\infty} -(-A)^{1/2}e^{-(-A)^{1/2}(s-t)}(f(s)-f(t))ds \\
&\quad + \int_r^t (-A)^{1/2}e^{-(-A)^{1/2}(s-r)}(f(s)-f(r))ds \\
&\quad + \int_t^{+\infty} (-A)^{1/2}e^{-(-A)^{1/2}(s-r)}(f(s)-f(r))ds
\end{aligned}$$

$$\begin{aligned}
&= \int_r^t (-A)^{1/2} e^{-(-A)^{1/2}(s-r)}(f(s)-f(r))ds \\
&\quad + \int_t^{+\infty} (-A)^{1/2} e^{-(-A)^{1/2}(s-r)}(f(t)-f(r))ds \\
&\quad + \int_t^{+\infty} (-A)^{1/2} \left(e^{-(-A)^{1/2}(s-r)} - e^{-(-A)^{1/2}(s-t)}\right)(f(s)-f(t))ds \\
&= \tilde{a} + \tilde{b} + \tilde{c},
\end{aligned}$$

thus

$$\|\tilde{a}\|_E \leqslant C \int_r^t \frac{(s-r)^{2\theta}}{(s-r)} ds \|f\|_X \leqslant C(t-r)^{2\theta}\|f\|_X.$$

It is easy to check the result for $\tilde{b}$. Finally

$$\begin{aligned}
\|\tilde{c}\|_E &= \left\| \int_t^{+\infty} \int_{s-t}^{s-r} -\left[(-A)^{1/2}\right]^2 e^{-(-A)^{1/2}\sigma}(f(s)-f(t))d\sigma ds \right\|_E \\
&\leqslant C \int_t^{+\infty} (s-t)^{2\theta} \int_{s-t}^{s-r} \frac{d\sigma}{\sigma^2} ds \|f\|_X \\
&\leqslant C \int_t^{+\infty} (s-t)^{2\theta-1} \frac{(t-r)}{(s-t+t-r)} ds \|f\|_X,
\end{aligned}$$

setting $(s-t) = \xi(t-r)$ in this last inequality we obtain

$$\|\tilde{c}\|_E \leqslant C \int_0^{+\infty} \frac{\xi^{2\theta-1}(t-r)^{2\theta}}{(1+\xi)} d\xi \|f\|_X \leqslant C(t-r)^{2\theta}\|f\|_X.$$

For $K(.)$, we note that

$$\begin{aligned}
K(t) - K(r) &= (V(t)-V(r)) \left(A\varphi - f(0) + \frac{1}{2}\int_0^{+\infty} \frac{\partial V}{\partial s}(s)(f(s)-f(0))ds\right) \\
&\quad -\frac{1}{2}\left(V(t)-V(r)\right)\left(f(r)-f(0)\right) - \frac{1}{2}V(t)(f(t)-f(r)) \\
&= k_1 + k_2 + k_3,
\end{aligned}$$

we then have the estimate

$$\begin{aligned}
\|k_2\|_E &\leqslant C \int_r^t \left\|(-A)^{1/2} e^{-(-A)^{1/2}s}(f(r)-f(0))\right\|_E ds \\
&\leqslant C \int_r^t s^{-1} r^{2\theta} ds \|f\|_X \\
&\leqslant C \int_r^t s^{2\theta-1} ds \|f\|_X \\
&\leqslant C(t-r)^{2\theta}\|f\|_X,
\end{aligned}$$

moreover

$$\|k_3\|_E \leqslant C\,(t-r)^{2\theta}\,\|f\|_X.$$

Now, for k_1 we use the following result proved in Sinestrari [12]

LEMMA 2. Setting for $x \in E$ and $t \geqslant 0$

$$v(t) = V(t)x = e^{-(-A)^{1/2}t}x,$$

if $x \in D_{(-A)^{1/2}}(2\theta, +\infty)$ then $v \in C^{2\theta}(E)$.

Thanks to the reiteration theorem in interpolation theory (see [10]) we have the equality

$$D_{(-A)^{1/2}}(2\theta, +\infty) = D_A(\theta, +\infty). \tag{6}$$

Therefore, it suffices to show that (see Sinestrari [12, p.24])

$$\sup_{r>0} \left\| r^{1-2\theta} \left(-(-A)^{1/2}\right) V(r) \int_0^{+\infty} \left(-(-A)^{1/2}\right) V(s)(f(s) - f(0))ds \right\|_E \leqslant K.$$

Let $r > 0$, we have

$$\begin{aligned}
& \left\| r^{1-2\theta} \left(-(-A)^{1/2}\right) V(r) \int_0^{+\infty} \left(-(-A)^{1/2}\right) V(s)(f(s) - f(0))ds \right\|_E \\
= & \left\| r^{1-2\theta} \int_0^{+\infty} \left(-(-A)^{1/2}\right)^2 V(s+r)(f(s) - f(0))ds \right\|_E \\
\leqslant & \; r^{1-2\theta} \int_0^{+\infty} \frac{s^{2\theta}}{(s+r)^2} ds \|f\|_X,
\end{aligned}$$

by making the change of variable $s = r\xi$, we obtain

$$r^{1-2\theta} \int_0^{+\infty} \frac{s^{2\theta}}{(s+r)^2} ds = \int_0^{+\infty} \frac{\xi^{2\theta}}{(1+\xi)^2} d\xi.$$

Consequently

$$V(.) \left(A\varphi - f(0) + \frac{1}{2} \int_0^{+\infty} \frac{\partial V}{\partial s}(s)(f(s) - f(0))ds \right) \in C^{2\theta}(E).$$

Hence $Au(.) \in C^{2\theta}(E)$. This ends the proof of Theorem 2.

EXAMPLE. We present now a simple example to illustrate equations (1)-(2). Consider, for instance, in $E = L^2(\mathbb{R})$ the operator A defined by

$$\begin{cases} D(A) = H^4(\mathbb{R}), \quad Au = au^{(4)} - bu, \\ \text{with } a < 0, \quad b > 0, \end{cases}$$

for more details concerning A, see [5]. All previous abstract results can be applied to the following problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a\dfrac{\partial^4 u}{\partial x^4} - bu = f(t,x), \quad (t,x) \in \Sigma, \\ u(0,x) = u_0(x), \quad u(+\infty, x) = 0, \quad x \in \mathbb{R}, \end{cases}$$

where $\Sigma = (0, +\infty) \times \mathbb{R}$, $u_0 \in H^4(\mathbb{R})$ and $f \in C^{2\theta}([0, +\infty[; L^2(\mathbb{R}))$.

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