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Strongly Summable Sequences Defined Over Real n-Normed Spaces*

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Abstract

The main aim of this article is to study subsets of real linear n-normed spaces consisting of strongly Cesàro summable and strongly lacunary summable sequences. Some standard facts as linearity, existence of norms and completeness with respect to these norms are investigated. Also some facts on equivalence of various norms on such constructed Banach spaces are presented, and we show that their topology can be fully described by using derived norm (norm). Further we investigate the relationship between the spaces and provide some examples and possible applications.

1 Introduction

Different types of complex sequences of the form $x = \{x_k\}_{k=1}^{\infty}$ or in short (x_k) , under various norms have been studied to great extent. In particular, the linear space w of all complex sequences (x_k) endowed with the usual operations and the supremum norm $\|x\|_\infty = \sup_k |x_k|$, as well as its subspaces ℓ_∞, c and c_0, consisting respectively of all, bounded, convergent and null sequences, are well studied.

The standard concept of a norm has, however, been extended. Therefore, the space w under these new norms may be of interests in various applications. In this paper, we intend to study the properties of several subsets of the linear space w under the so called n-norms.

Let us first recall the concept of an n-norm. Let $n \in N$ and X be a real linear space of dimension $d \geq n \geq 2$. A real valued function $\|., \ldots, .\| : X^n \to R$ satisfying the following four properties:

(N$_1$) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors,

(N$_2$) $\|x_1, x_2, \ldots, x_n\| = \|x_{j_1}, x_{j_2}, \ldots, x_{j_n}\|$ for every permutation $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$,

(N$_3$) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in R$,

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(N_4) $\|x+x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$ for all $x, x', x_2, \ldots, x_n \in X$,

is called an n-norm on X and the pair $(X, \|., \ldots, .\|)$ is called a linear n-normed space.

The concept of a 2-normed space was developed by Gähler [3] in the mid of 1960's, while that of an n-normed space can be found in Misiak [12]. Since then, many others have studied this concept and obtained various results; see for instance Gunawan [6, 7] and Gunawan and Mashadi [8, 9].

A trivial example of an n-normed space is $X = R^n$ equipped with the following Euclidean n-norm:

$$\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})|$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in R^n$ for each $i = 1, 2, \ldots, n$.

If $(X, \|., \ldots, .\|)$ is an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ a linearly independent set in X, then the following function $\|., \ldots, .\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \ldots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| : i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$ and this is known as a derived $(n-1)$-norm on X.

The standard n-norm on X, where X is a real inner product space of dimension $d \geq n$, is defined as

$$\|x_1, x_2, \ldots, x_n\|_S = \begin{vmatrix} \langle x_1, x_1\rangle & \ldots & \langle x_1, x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \ldots & \langle x_n, x_n\rangle \end{vmatrix}^{\frac{1}{2}},$$

where $\langle ., .\rangle$ denotes the inner product on X. If $X = R^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|., \ldots, .\|_E$ mentioned earlier. For $n = 1$, this n-norm reduces to the usual norm $\|x_1\| = \langle x_1, x_1\rangle^{\frac{1}{2}}$.

A sequence (x_k) in an n-normed space $(X, \|., \ldots, .\|)$ is said to *converge* to some $L \in X$ in the n-norm if $\lim_{k\to\infty} \|x_k - L, w_2, w_3 \ldots, w_n\| = 0$ for every $w_2, w_3 \ldots, w_n \in X$. A sequence (x_k) in an n-normed space $(X, \|., \ldots, .\|)$ is said to be *Cauchy* with respect to the n-norm if $\lim_{k,l\to\infty} \|x_k - x_l, w_2, w_3 \ldots, w_n\| = 0$ for every $w_2, w_3 \ldots, w_n \in X$. If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Now we state the following three useful results as Lemmas which can be found in [9].

LEMMA 1. Every n-normed space is an $(n-r)$-normed space for all $r = 1, 2, \ldots, n-1$. In particular, every n-normed space is a normed space.

LEMMA 2. A standard n-normed space is complete if and only if it is complete with respect to the usual norm $\|.\|_S = \langle ., .\rangle^{1/2}$.

LEMMA 3. On a standard n-normed space X, the derived $(n-1)$-norm $\|., \ldots, .\|_\infty$, defined with respect to orthonormal set $\{e_1, e_2, \ldots, e_n\}$, is equivalent to the standard

$(n-1)$-norm $\|.,\dots,.\|_S$. Precisely, we have

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty \le \|x_1, x_2, \dots, x_{n-1}\|_S \le \sqrt{n}\|x_1, x_2, \dots, x_{n-1}\|_\infty$$

for all $x_1, x_2, \dots, x_{n-1}$, where

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, e_i\|_S : i = 1, 2, \dots, n\}.$$

Next we recall two subsets of the space w. The first is the space $|\sigma_1|$ of strongly Cesàro summable sequences (see e.g. Borwein [1], Freedman, Sember and Raphael [2] and Maddox [11]). It is defined as

$$|\sigma_1| = \left\{ x = (x_k) : \text{there exists } L \text{ such that } \lim_p \frac{1}{p} \sum_{k=1}^{p} |x_k - L| = 0 \right\}$$

and it is a Banach space normed by

$$\|x\| = \sup_p \left(\frac{1}{p} \sum_{k=1}^{p} |x_k| \right).$$

Next, by a lacunary sequence $\theta = (k_p)$; we mean an increasing sequence of non-negative integers with $h_p = (k_p - k_{p-1}) \to \infty$, where $k_0 = 0$, as $p \to \infty$. We denote $I_p = (k_{p-1}, k_p]$ and $\eta_p = \frac{k_p}{k_{p-1}}$ for $p = 1, 2, 3, \dots$. The space of strongly lacunary summable sequence N_θ was defined by Freedman, Sember and Raphael [2] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{p \to \infty} \frac{1}{h_p} \sum_{k \in I_p} |x_k - L| = 0, \text{ for some } L \right\}.$$

The space N_θ is a Banach space with the norm

$$\|x\|_\theta = \sup_p \left(\frac{1}{h_p} \sum_{k \in I_p} |x_k| \right).$$

Throughout the article $(X, \|.,\dots,.\|_X)$ will be an n-normed space and $w(X)$ will denote X-valued sequence space. The n-norm $\|.,\dots,.\|_X$ on X is either a standard n-norm or non-standard n-norm. In general, we write $\|.,\dots,.\|_X$ and for standard case we write $\|.,\dots,.\|_S$. Again for derived norms we use $\|.,\dots,.\|_\infty$.

2 Main Results

In this section we extend the notion of strongly Cesàro summable sequences and strongly lacunary summable sequences to n-normed space valued sequences.

We denote by $|\sigma_1|(X)$ the set of all X-valued strongly Cesàro summable sequences defined as the set of all $x \in w(X)$ such that

$$\lim_{p \to \infty} \frac{1}{p} \sum_{k=1}^{p} \|x_k - L, z_1, \dots, z_{n-1}\|_X = 0 \text{ for every } z_1, \dots, z_{n-1} \in X \text{ and for some } L.$$

For $L = 0$, we write this corresponding space as $|\sigma_1|^0(X)$.

Let θ be a lacunary sequence. Then we denote by $N_\theta(X)$ the set of all X-valued strongly lacunary summable sequences $x \in w(X)$ such that

$$\lim_{p\to\infty} \frac{1}{h_p} \sum_{k\in I_p} \|x_k - L, z_1, ..., z_{n-1}\|_X = 0 \text{ for every } z_1, ..., z_{n-1} \in X \text{ and for some } L.$$

For $L = 0$, we write this space as $N_\theta^0(X)$.

In the special case where $\theta = (2^p)$, we have $N_\theta(X) = |\sigma_1|(X)$.

THEOREM 1. The following are true:

(i) If X is an n-Banach space then $|\sigma_1|(X)$ is a Banach space normed by

$$\|x\| = \sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k, z_1, ..., z_{n-1}\|_X \right). \tag{1}$$

(ii) If X is an n-Banach space then $N_\theta(X)$ is a Banach space normed by

$$\|x\|_\theta = \sup_p \left(\frac{1}{h_p} \sum_{k\in I_p} \|x_k, z_1, ..., z_{n-1}\|_X \right). \tag{2}$$

PROOF. It is easy to see that $|\sigma_1|(X)$ is a normed linear space. To prove completeness, let (x^i) be a Cauchy sequence in $|\sigma_1|(X)$, where $x^i = (x_k^i) = (x_1^i, x_2^i, ...)$ for each $i \in N$. Then for a given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\|x^i - x^j\| = \sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k^j, z_1, ..., z_{n-1}\|_X \right) < \varepsilon, \text{ for all } i, j \geq n_0.$$

It follows that

$$\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k^j, z_1, ..., z_{n-1}\|_X < \varepsilon, \text{ for all } i, j \geq n_0 \text{ and for all } p \geq 1.$$

Hence (x_k^i) is a Cauchy sequence in X for all $k \in N$. Since X is an n-Banach space, (x_k^i) is convergent in X for all $k \in N$. For simplicity, let $\lim_{i\to\infty} x_k^i = x_k$ (say), exists for each $k \in N$. Now we can find that

$$\lim_{j\to\infty} \frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k^j, z_1, ..., z_{n-1}\|_X < \varepsilon, \text{ for all } i \geq n_0 \text{ and for all } p \geq 1.$$

Thus,

$$\sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k, z_1, ..., z_{n-1}\|_X \right) < \varepsilon, \text{ for all } i \geq n_0.$$

It follows that $(x^i - x) \in |\sigma_1|(X)$. Since $(x^i) \in |\sigma_1|(X)$ and $|\sigma_1|(X)$ is a linear space, so we have $x = x^i - (x^i - x) \in |\sigma_1|(X)$. This completes the proof of part (i).

The proof of (ii) similar and skipped. The proof is complete.

COROLLARY 2. Let X be equipped with the standard n-norm. Then
(i) if X is a Banach space then $|\sigma_1|(X)$ is a Banach space normed by

$$\|x\| = \sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k, z_1, ..., z_{n-1}\|_X \right),$$

(ii) if X is a Banach space then $N_\theta(X)$ is a Banach space normed by

$$\|x\|_\theta = \sup_r \left(\frac{1}{h_r} \sum_{k \in I_r} \|x_k, z_1, ..., z_{n-1}\|_X \right)$$

Indeed, the proof follows by combining Lemma 2 and Theorem 1, and is skipped.

We now use the notion derived norms to define some other norms on the spaces and investigate the relationship among these norms.

Let $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in X. Then

$$\|x_k, z_1, ..., z_{n-r-1}\|_\infty = \max\{\|x_k, z_1, ..., z_{n-r-1}, a_{i_1}, a_{i_2}, ..., a_{i_r}\|_X\},\ \{i_1, ..., i_r\} \subseteq \{1, ..., n\}$$

is an derived $(n-r)$-norm on X for each $r = 1, 2, ..., n-1$ and for each $k \geq 1$. Hence we have the following Theorem.

THEOREM 3. Let $\{a_1, a_2, \cdots, a_n\}$ be a linearly independent set in X. Then
(i) $|\sigma_1|(X)$ is a normed linear space, with norm $\|.\|^r$ defined by

$$\|x\|^r = \sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k, z_1, ..., z_{n-r-1}\|_\infty \right) \text{ for each } r = 1, 2, ..., n-1, \tag{3}$$

(ii) $N_\theta(X)$ is a normed linear space, with norm $\|.\|_\theta^r$ defined by

$$\|x\|_\theta^r = \sup_r \left(\frac{1}{h_r} \sum_{k \in I_r} \|x_k, z_1, ..., z_{n-r-1}\|_\infty \right) \text{ for each } r = 1, 2, ..., n-1. \tag{4}$$

We call the above norms as the derived $(n-r)$-norm for each $r = 1, 2, ..., n-1$.

Proof is a routine verification and so omitted.

THEOREM 4. If X is an $(n-r)$-Banach spaces for each $r = 1, 2, ..., n-1$, then $|\sigma_1|(X)$ is a Banach with norm $\|.\|^r$ defined by (3) and $N_\theta(X)$ is a Banach space with norm $\|.\|_\theta^r$ defined by (4).

Proof is same with the proof of Theorem 1 and is omitted.

THEOREM 5. If (x^i) converges to an x in $|\sigma_1|(X)$ in the norm $\|.\|$ defined by (1), then (x^i) also converges to x in the norm $\|.\|^r$ defined by (3) for $r = 1$.

PROOF. Let (x^i) converges to x in $|\sigma_1|(X)$ in the norm $\|.\|$. Then $\|x^i - x\| \longrightarrow 0$ as $i \longrightarrow \infty$. Using the definition of norm (1), we get

$$\sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k, z_1, ..., z_{n-1}\|_X \right) \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

Let $\{a_1, a_2, \cdots, a_n\}$ be any linearly independent set in X. Then

$$\sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k, z_1, ..., z_{n-2}, a_j\|_X \right) \longrightarrow 0 \text{ as } i \longrightarrow \infty \text{ for each } j = 1, 2, ..., n.$$

Hence

$$\sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k, z_1, ..., z_{n-2}\|_\infty \right) \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

Thus $\|x^i - x\|^1 \longrightarrow 0$ as $i \longrightarrow \infty$. Hence (x^i) converges to x in the norm $\|.\|^1$.

If X is equipped with the standard n-norm and derived norm on X are with respect to an orthonormal set then the converse of the above Theorem is also true. Consequently we have the following Theorem.

THEOREM 6. Let X be a standard n-normed space and the derived $(n-1)$-norm on X is with respect to an orthonormal set. Then (x^i) is convergent in $|\sigma_1|(X)$ in the norm $\|.\|$ defined by (1), if and only if (x^i) is convergent in $|\sigma_1|(X)$ in the norm $\|.\|^r$ defined by (3) for $r = 1$.

PROOF. In view of the above Theorem it is enough to prove that (x^i) is convergent in the norm $\|.\|^1$ implies (x^i) is convergent in the norm $\|.\|$. Let (x^i) is converges to x in $|\sigma_1|(X)$ in the norm $\|.\|_1$. Then $\|x^i - x\|^1 \longrightarrow 0$ as $i \longrightarrow \infty$. Using (3) with $r = 1$, we get

$$\sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k, z_1, ..., z_{n-2}\|_\infty \right) \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

Now one can observe that $\|x_k^i - x_k, z_1, \cdots, z_{n-1}\|_S \leq \|x_k^i - x, z_1, ..., z_{n-2}\|_S \|z_{n-1}\|_S$, where $\|., \ldots, .\|_S$ and $\|.\|_S$ on the right hand side denote the standard $(n-1)$-norm and the usual norm on X respectively. Since the derived $(n-1)$-norm on X is with respect to an orthonormal set, using Lemma 3, we have

$$\|x_k^i - x_k, z_1, ..., z_{n-1}\|_S \leq \sqrt{n} \|x_k^i - x, z_1, ..., z_{n-2}\|_\infty \|z_{n-1}\|_S,$$

and in this case $\|., \ldots, .\|_\infty$ on the right hand side is the derived $(n-1)$-norm which we used to define the norm $\|.\|^1$. Therefore

$$\sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k, z_1, ..., z_{n-1}\|_S \right) \leq \sup_p \frac{1}{p} \sum_{k=1}^{p} \left(\sqrt{n} \left\|x_k^i - x_k, z_1, ..., z_{n-2}\right\|_\infty \|z_{n-1}\|_S \right).$$

Hence

$$\sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k^i - x_k, z_1, ..., z_{n-1}\|_S \right) \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

Thus $\|x^i - x\| \longrightarrow 0$ as $i \longrightarrow \infty$. That is, (x^i) is converges to x in $|\sigma_1|(X)$ in the norm $\|.\|$. The proof is complete.

COROLLARY 7. Let X be a standard n-normed space and the derived $(n-r)$-norms on X are with respect to an orthonormal set. Then a sequence in $|\sigma_1|(X)$ is convergent in the norm $\|.\|$ defined by (1) if and only if it is convergent in the norm $\|.\|^1$ and, by induction, in the norm $\|.\|^r$ defined by (3) for all $r = 1, 2, ..., n-1$. In particular, a sequence in $|\sigma_1|(X)$ is convergent in the norm $\|.\|$ if and only if it is convergent in the norm $\|.\|^{n-1}$ defined by

$$\|x\|^{n-1} = \sup_p \left(\frac{1}{p} \sum_{k=1}^{p} \|x_k\|_\infty \right). \tag{5}$$

THEOREM 8. Let X be a standard n-normed space and the derived $(n-r)$-norms on X for all $r = 1, 2, ..., n-1$ are with respect to an orthonormal set. Then $|\sigma_1|(X)$ is complete with respect to the norm $\|.\|$ defined by (1) if and only if it is complete with respect to the norm $\|.\|^1$ defined by (3). By induction, $|\sigma_1|(X)$ is complete with respect to the norm $\|.\|$ if and only if it is complete with respect to the norm $\|.\|^{n-1}$ defined by (5).

PROOF. By replacing the phrases '(x^i) converges to x' with '(x^i) is Cauchy' and '$x^i - x$' with '$x^i - x^j$', we see that the analogues of Theorem 5, Theorem 6 and Corollary 7 hold for Cauchy sequences. This completes the proof.

REMARK 1. It we replace the space $|\sigma_1|(X)$ by $N_\theta(X)$, analogues of Theorem 5, Theorem 6, Corollary 7 and Theorem 8 hold for $N_\theta(X)$.

EXAMPLE 4. Let us take $X = R^3$ and consider a 3-norm $\|., ., .\|_X$ defined as:

$$\|x_1, x_2, x_3\|_X = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, x_{i3}) \in R^3$ for each $i = 1, 2, 3$. Consider the divergent sequence $x = \{\overline{0}, \overline{1}, \overline{0}, \overline{1}, \overline{0}, \cdots\} \in w(X)$, where $\overline{k} = (k, k, k)$ for each $k = 0, 1$. Let us consider a basis $\{e_1, e_2, e_3\}$ of $X = R^3$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Now $\|x_k\|_\infty = \max\{\|x_k, e_{i_1}, e_{i_2}\|_X\}$, $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ is an derived norm on X. Then x belong to $N_\theta(X)$ and $|\sigma_1|(X)$, for $\theta = (2^p)$. Here actually strongly Cesàro summability method transform the sequence x into the sequence y, where $y = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{2}{5}, \cdots\right\}$, which converges to $\frac{1}{2}$. In other words we can say that x has the generalized limit $\frac{1}{2}$. Hence x is a 3-nls valued strongly Cesàro summable sequence.

REMARK 2. Associated to the derived norm $\|.\|^{n-1}$, we can define balls (open) $S(x, \varepsilon)$ centered at x and radius ε as follows:

$$S(x, \varepsilon) = \{y : \|x - y\|^{n-1} < \varepsilon\}.$$

Using these balls, Theorem 8 becomes:

THEOREM 9. A sequence (x_k) is convergent to x in $|\sigma_1|(X)$ if and only if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $x_k \in S(x, \varepsilon)$ for all $k \geq n_0$.

Hence we have the following important result.

THEOREM 10. A space $|\sigma_1|(X)$ is a normed space and its topology agrees with that generated by the derived norm $\|.\|^{n-1}$.

Our next aim is to investigate the relationship among the spaces $|\sigma_1|(X)$ and $N_\theta(X)$.

PROPOSITION 11. Let $\theta = (k_p)$ be a lacunary sequence with $\liminf_p \eta_p > 1$, then $|\sigma_1|(X) \subseteq N_\theta(X)$.

PROOF. Let $\liminf \eta_p > 1$. Then there exists a $\nu > 0$ such that $1+\nu \leq \eta_p$ for all $p \geq 1$. Let $x \in |\sigma_1|(X)$. Then there exists some $L \in X$ such that

$$\lim_{t\to\infty} \frac{1}{t}\sum_{k=1}^{t} \|x_k - L, z_1, ..., z_{n-1}\|_X = 0, \text{ for every } z_1, ..., z_{n-1} \in X.$$

Now we write

$$\begin{aligned}
& \frac{1}{h_p}\sum_{k\in I_p} \|x_k - L, z_1, ..., z_{n-1}\|_X \\
= \; & \frac{1}{h_p}\sum_{1\leq i\leq k_p} \|x_i - L, z_1, ..., z_{n-1}\|_X - \frac{1}{h_p}\sum_{1\leq i\leq k_{p-1}} \|x_i - L, z_1, ..., z_{n-1}\|_X \\
= \; & \frac{k_p}{h_p}\left(\frac{1}{k_p}\sum_{1\leq i\leq k_p} \|x_i - L, z_1, ..., z_{n-1}\|_X\right) \\
& -\frac{k_{p-1}}{h_p}\left(\frac{1}{k_{p-1}}\sum_{1\leq i\leq k_{p-1}} \|x_i - L, z_1, ..., z_{n-1}\|_X\right). \qquad (6)
\end{aligned}$$

Now we have $\frac{k_p}{h_p} \leq \frac{1+\nu}{\nu}$ and $\frac{k_{p-1}}{h_p} \leq \frac{1}{\nu}$, since $h_p = k_p - k_{p-1}$. Hence using (6), we have $x \in N_\theta(X)$.

PROPOSITION 12. Let $\theta = (k_p)$ be a lacunary sequence with $\limsup_p \eta_p < \infty$, then $N_\theta(X) \subseteq |\sigma_1|(X)$.

PROOF. Let $\limsup \eta_p < \infty$. Then there exists a $M > 0$ such that $\eta_p < M$ for all $p \geq 1$. Let $x \in N_\theta^0(X)$ and $\varepsilon > 0$. Then we can find $R > 0$ and $K > 0$ such that

$$\sup_{i\geq R} S_i = \sup_{i\geq R}\left(\frac{1}{h_i}\sum_{i=1}^{k_i} \|x_i, z_1, \cdots, z_{n-1}\|_X - \frac{1}{h_i}\sum_{i=1}^{k_{i-1}} \|x_i, z_1, \cdots, z_{n-1}\|_X\right) < \varepsilon$$

and $S_i < K$ for all $i = 1, 2, ...$. Then if t is any integer with $k_{p-1} < t \leq k_p$, where $p > R$, we can write

$$\begin{aligned}
& \frac{1}{t}\sum_{i=1}^{t} \|x_i, z_1, \cdots, z_{n-1}\|_X \\
\leq \; & \frac{1}{k_{p-1}}\sum_{i=1}^{k_p} \|x_i, z_1, \cdots, z_{n-1}\|_X \\
= \; & \frac{1}{k_{p-1}}\left(\sum_{I_1} \|x_i, z_1, \cdots, z_{n-1}\|_X + \cdots + \sum_{I_p} \|x_i, z_1, \cdots, z_{n-1}\|_X\right) \\
= \; & \frac{k_1}{k_{p-1}}S_1 + \frac{k_2 - k_1}{k_{p-1}}S_2 + \cdots + \frac{k_R - k_{R-1}}{k_{p-1}}S_R + \frac{k_{R+1} - k_R}{k_{p-1}}S_{R+1}
\end{aligned}$$

$$
\begin{aligned}
&+\cdots+\frac{k_p-k_{p-1}}{k_{p-1}}S_p\\
\leq\ & \left(\sup_{i\geq 1} S_i\right)\frac{k_R}{k_{p-1}}+\left(\sup_{i\geq R} S_i\right)\frac{k_p-k_R}{k_{p-1}}\\
\leq\ & K\frac{k_R}{k_{p-1}}+\varepsilon M.
\end{aligned}
$$

Since $k_{p-1}\longrightarrow\infty$ as $i\longrightarrow\infty$, it follows that $x\in|\sigma_1|^0(X)$. The general inclusion $N_\theta(X)\subseteq|\sigma_1|(X)$ follows by linearity.

PROPOSITION 13. Let $\theta=(k_p)$ be a lacunary sequence with $1<\liminf\eta_p\leq\limsup\eta_p<\infty$, then $|\sigma_1|(X)=N_\theta(X)$.

PROOF. Proof follows by combining Proposition 11 and Proposition 12.

3 Examples and Remarks

The concept of 2-normed spaces was introduced and studied by Siegfried Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of paper in German language published in Mathematische Nachrichten in the mid of 1960's. Later on it was further generalized, and the notion of n-norm was introduced by Misiak. Very often Gähler has raised the following questions: What is the real motivation for studying 2-norm structures? Is there a physical situation or an abstract concept where norm topology does not work but 2-norm topology does? After the investigations of this paper, we can comment that while studying n-normed structure or summability methods for sequences with a real n-normed linear space as base space the main issue should be the use of the n-norms. We also observe that if a term in the definition of n-norm represents the change of shape, and the n-norm stands for the associated area or center of gravity of the term, we can think of some plausible applicable of the notion of n-norm. As an example, we can think of use of the notion of n-norm for a process where for a particular output we need n-inputs but with one main input and other $(n-1)$-inputs as dummy inputs to complete the process. Keeping all these factors in mind we provide some further examples.

EXAMPLE 1. Consider the linear space P_m of real polynomials of degree $\leq m$ on the interval $[0,1]$. Let $\{x_i\}_{i=0}^{nm}$ be $nm+1$ arbitrary but distinct fixed points in $[0,1]$. For $f_1,f_2,\ldots,f_n$ in P_m, let us define

$$
\|f_1,f_2,\ldots,f_n\|=\begin{cases} 0 & \text{if}\quad f_1,\ldots,f_n \text{ are lin. independent,}\\ \sum\limits_{i=0}^{nm}|f_1(x_i)f_2(x_i)\ldots f_n(x_i)| & \text{if}\quad f_1,\ldots,f_n \text{ are lin. dependent.}\end{cases}
$$

Then $\|.,\ldots,.\|$ is an n-norm on P_m.

PROOF. We prove only the property $\|f_1,f_2,\ldots,f_n\|=0$ if and only if $f_1,f_2,\ldots,f_n$ are linearly dependent. Other properties of n-norm can be easily verified. If $f_1,f_2,\ldots,f_n$ are linearly dependent, then $\|f_1,f_2,\ldots,f_n\|=0$. Conversely assume

$$
\sum_{i=0}^{nm}|f_1(x_i)f_2(x_i)\ldots f_n(x_i)|=0.
$$

This implies that

$$f_1(x_i)f_2(x_i)\dots f_n(x_i) = 0 \text{ at } nm+1 \text{ distinct points.}$$

Since the degree of each $f_i \leq m$, we must have at least one $f_i = 0$. Thus

$$\|f_1, f_2, \dots, f_n\| = 0 \text{ if and only if } f_1, f_2, \dots, f_n \text{ are linearly dependent.}$$

EXAMPLE 2. Consider the space C_0 of real sequences with only finite number of non-zero terms. Let us define:

$$\|x_1, x_2, ..., x_n\| = \begin{cases} 0 & \text{if} \quad x_1,\ x_2, ..., x_n \text{ lniearly dependent,} \\ \sum_{k=1}^{\infty}(|x_k{}^1||x_k{}^2|\dots|x_k{}^n|) & \text{if} \quad x_1, x_2, ..., x_n \text{ linearly independent.} \end{cases}$$

Then $\|., \dots, .\|$ is an n-norm on C_0. But it is not an n-norm on c_0 and l_∞ consisting of real sequences.

In view of Lemma 1, Lemma 2 and definitions of convergence and Cauchy sequence in n-norm, the concept of derived norm has special role through the subject.

Associated to the derived norm $\|., ..., .\|_\infty$, we can define the balls (open) $B(x, \varepsilon)$ centered at x having radius ε by

$$B(x, \varepsilon) := \{y : \|x - y, z_2, ..., z_{n-1}\|_\infty < \varepsilon\},$$

where

$$\|x - y, z_2, ..., z_{n-1}\|_\infty := \max\{\|x - y, z_2, ..., z_{n-1}, u_j\| : j = 1, 2, ..., d\}.$$

We may want to view an n-norm on a real linear space M, say as a norm on the Cartesian product space M^n which is invariant under permutation. But this is not true. One may find it interesting to see the differences between these two concepts through the condition (N_1) in the definition of n-norm. We now give the following example which seems to be a 2-norm but not true.

EXAMPLE 3. Let Y be the space of all bounded real-valued functions on R. For $f, g \in Y$, let us define

$$\|f, g\| = \begin{cases} 0 & \text{if} \quad f, g \text{ are linearly dependent,} \\ \sup_{t \in R}|f(t)g(t)| & \text{if} \quad f, g \text{ are linearly independent.} \end{cases}$$

Then $\|., .\|$ is not a 2-norm. To see this,

$$f(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ \sin t & \text{if} \quad 0 < t < \pi, \\ 0 & \text{if} \quad t \geq \pi, \end{cases}$$

and

$$g(t) = \begin{cases} 0 & \text{if} \quad t \leq -\pi, \\ \sin t & \text{if} \quad -\pi < t < 0, \\ 0 & \text{if} \quad t \geq 0. \end{cases}$$

Then f and g are linearly independent. But $\|f, g\| = 0$.

EXAMPLE 4. Let X be a 2-normed space of all bounded real-valued functions on R and $\|.\|_\infty$ be a derived norm on X. Let $T : X \longrightarrow X$ be defined by

$$h(t) = Tf(t) = f(t - \Delta),$$

where $\Delta > 0$ is a constant. This is a model of a delay line, which is an electric device whose output h is a delayed version of the input f, the time delay be Δ. Then T is linear and bounded with respect to the derived norm.

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