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Multiple Positive Solutions Of A Singular Fractional Boundary Value Problem*

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Abstract

In this paper, we study the existence and multiplicity of positive solutions for the singular fractional boundary value problem

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u(t) = h(t) f(t, u(t)), 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

where $3 < \alpha \leq 4$ is a real number, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative, and $h \in C(0,1) \cap L(0,1)$ is nonnegative and may be singular at $t = 0$ and/or $t = 1$. We use fixed point index theory to establish our main results based on a priori estimates achieved by developing some spectral properties of associated linear integral operators. Our main results essentially extend and improve the corresponding ones in the literature.

1 Introduction

In this paper, we study the existence and multiplicity of positive solutions for the singular fractional boundary value problem

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u(t) = h(t) f(t, u(t)), t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{1}$$

where $\alpha \in (3, 4]$ is a real number, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f \in C([0,1] \times [0,\infty), [0,\infty))$, and $h \in C(0,1) \cap L(0,1)$ is nonnegative and may be singular at $t = 0$ and/or $t = 1$. We use fixed point index theory to establish our main results based on a priori estimates achieved by developing some spectral properties of associated linear integral operators.

Fractional differential equations can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetics, etc. This explains why many authors have studied existence and multiplicity questions of solutions (or positive solutions) of nonlinear fractional differential

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equation, see, for example, [1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 13] and references therein. It is of interest to note that the Riemann-Liouville fractional derivative is not suitable for nonzero boundary value conditions, see [11, 13].

By means of the Schauder fixed point theorem and fixed point index theory, Bai [1] discussed the existence of positive solutions for the fractional boundary value problem

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, t \in (0,1), \\ u(0) = 0, \beta u(\eta) = u(1), \end{cases} \tag{2}$$

where $\alpha \in (1,2]$, $\mathbf{D}_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $\eta \in (0,1)$, $\beta\eta^{\alpha-1} \in (0,1)$, and $f \in C([0,1] \times [0,\infty), [0,\infty))$ is sublinear. It should be remarked that our nonlinearity f here, unlike the f in [1], may be both sublinear and superlinear. Our first theorem involves the existence of at least one positive solution for (1) with f growing superlinearly, thereby complementing the results in [1]. We then establish two existence results of twin positive solutions for (1), two results that essentially improve and extend the corresponding ones in [12] (see REMARK 1).

2 Preliminaries

The Riemann-Liouville fractional derivative $\mathbf{D}_{0+}^{\alpha}$ is defined by

$$\mathbf{D}_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)ds}{(t-s)^{\alpha-n+1}},$$

where Γ is the gamma function and $n = [\alpha]+1$. For more details of fractional calculus, we refer the reader to the recent literature such as [1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 13]. Let

$$E = C[0,1], \quad \|u\| = \max_{t\in[0,1]} |u(t)|, \quad P = \{u \in E : u(t) \geq 0, \forall t \in [0,1]\}.$$

Then $(E, \|\cdot\|)$ is a real Banach space and P a cone on E. We denote $B_\rho = \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

LEMMA 1 ([12, Lemma 2.3]). Given $\phi \in C(0,1) \cap L(0,1)$ and $3 < \alpha \leq 4$, the unique solution of the fractional boundary value problem

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u(t) = \phi(t), t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

is represented by $u(t) = \int_0^1 G(t,s)\phi(s)ds$, where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-2}(1-s)^{\alpha-2}[(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3}$$

It is easy to verify that the Green's function $G \in C([0,1] \times [0,1], [0,\infty))$ satisfies the following relations(see [12, Lemma 2.4])

$$(\alpha-2)t^{\alpha-2}(1-t)^2 s^2 (1-s)^{\alpha-2} \leq \Gamma(\alpha) G(t,s) \leq m_0 s^2 (1-s)^{\alpha-2} \tag{4}$$

and

$$(\alpha-2)q(t)k(s) \le \Gamma(\alpha)G(t,s) \le m_0 k(s) \tag{5}$$

for all $0 \le t, s \le 1$, where $q(t) = t^{\alpha-2}(1-t)^2$, $k(s) = s^2(1-s)^{\alpha-2}$ and

$$m_0 = \max\{\alpha-1, (\alpha-2)^2\}. \tag{6}$$

LEMMA 2 ([5, p.314]). Suppose $A : P \to P$ is a completely continuous operator and has no fixed points on $\partial B_\rho \cap P$. Then the following are true:

1. If $\|Au\| \le \|u\|$ for all $u \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 1$, where i is the fixed point index on P .
2. If $\|Au\| \ge \|u\|$ for all $u \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 0$.

LEMMA 3 ([5, p.144]). Suppose $\Omega \subset E$ is a bounded open set and $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that

$$u - Au \ne \lambda u_0, \forall \lambda \ge 0, u \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$.

LEMMA 4. ([5, p.164]) Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator and has no fixed points on $\partial\Omega \cap P$. If $u \ne \lambda Au, \forall u \in \partial\Omega \cap P, 0 \le \lambda \le 1$, then $i(A, \Omega \cap P, P) = 1$.

We assume the following conditions throughout this paper.

(H1) $f \in C([0,1] \times [0,\infty), [0,\infty))$.

(H2) $h \in L(0,1) \cap C(0,1)$ is nonnegative and does not vanish identically on any subinterval of $(0,1)$.

Define two operators A and T by

$$(Au)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds, u \in P$$

and

$$(Tu)(t) = \int_0^1 G(t,s)h(s)u(s)ds, u \in P.$$

Note that (H1) and (H2) imply $A : P \to P$ is a completely continuous operator and $T : P \to P$ is a completely continuous, linear, positive operator. A consequence of Lemma 1 is that $u \in P$ is a positive solution of (1) if and only if $u \in P \setminus \{0\}$ is a fixed point of A. Moreover, it is easy to prove that the spectral radius of T, denoted by $r(T)$, is positive. Now the well-known Krein-Rutman theorem [9] asserts that there exist two functions $\varphi \in P \setminus \{0\}$ and $\psi \in L(0,1)\setminus\{0\}$ with $\psi(x) \ge 0$ for which

$$\int_0^1 G(t,s)h(s)\varphi(s)ds = r(T)\varphi(t), \int_0^1 G(t,s)h(s)\psi(t)dt = r(T)\psi(s), \int_0^1 \psi(t)dt = 1. \tag{7}$$

Put

$$P_0 = \{u \in P : \int_0^1 \psi(t)u(t)dt \ge \omega\|u\|\},$$

where $\psi(t)$ is determined by (7) and $\omega = \frac{\alpha-2}{m_0}\int_0^1 q(t)\psi(t)dt > 0$. Clearly, P_0 is also a cone on E. The following is a result obtained by observing relations (5).

LEMMA 5. $A(P) \subset P_0$.

In addition to (H1) and (H2), we need the following hypotheses on f.

(H3) $\liminf_{u\to 0^+} \frac{f(t,u)}{u} > \lambda_1$ uniformly with respect to $t \in [0,1]$, where $\lambda_1 = 1/r(T) > 0$.

(H4) $\limsup_{u\to+\infty} \frac{f(t,u)}{u} < \lambda_1$ uniformly with respect to $t \in [0,1]$.

(H5) $\liminf_{u\to+\infty} \frac{f(t,u)}{u} > \lambda_1$ uniformly with respect to $t \in [0,1]$.

(H6) $\limsup_{u\to 0^+} \frac{f(t,u)}{u} < \lambda_1$ uniformly with respect to $t \in [0,1]$.

(H7) There is $\rho > 0$ such that the inequality $f(t,u) < \frac{\eta\Gamma(\alpha)\rho}{m_0}$ holds whenever $u \in [0,\rho]$ and $t \in [0,1]$, $m_0 > 0$ being defined in (6).

(H8) There are $\rho > 0$ and $\sigma \in (0, \frac{\alpha-2}{\alpha})$ such that $f(t,u) > \frac{\Gamma(\alpha)\rho}{\eta(\alpha-2)q(\frac{\alpha-2}{\alpha})}$ holds whenever $u \in [\theta\rho,\rho]$ and $t \in [\sigma, 1-\sigma]$, θ and η being defined by $\theta = \frac{\alpha-2}{m_0}\min\{q(\sigma), q(1-\sigma)\} > 0$ and $\eta = \left(\int_0^1 k(s)h(s)ds\right)^{-1} > 0$.

REMARK 1. Some simple computations show the following estimates:

$$\frac{\Gamma(\alpha)}{m_0}\left(\int_0^1 k(s)ds\right)^{-1} \leq \lambda_1 = (r(T))^{-1} \leq \frac{m_0}{\alpha-2}\left(\max_{t\in[0,1]}\int_0^1 G(t,s)q(s)ds\right)^{-1},$$

from which we obtain $M \leq \lambda_1 \leq \overline{N} \leq \widetilde{N}$, where $M, \overline{N}, \widetilde{N}$ are defined in [12, Section 3]. Notice that (H3) and (H5) considerably weaken (A1), and that (H4) and (H6) considerably (A2), where (A1) and (A2) are formulated in [12]. This means our main results, even in the case of h being nonsingular, essentially extend and improve the corresponding ones in the literature.

3 Main results

First we have the following.

THEOREM 1. Suppose that (H1), (H2), (H5) and (H6) are satisfied, then (1) has at least one positive solution.

PROOF. By (H5), there exist $\varepsilon > 0$ and $b > 0$ such that $f(t,u) \geq (\lambda_1+\varepsilon)u - b$ for all $u \geq 0$ and $t \in [0,1]$. This implies

$$(Au)(t) \geq (\lambda_1+\varepsilon)\int_0^1 G(t,s)h(s)u(s)ds - b\int_0^1 G(t,s)h(s)ds \tag{8}$$

for all $u \in P$. Let $M_1 = \{u \in P : u = Au + \lambda\varphi, \lambda \geq 0\}$, where $\varphi \in P$ is determined by (7). We shall prove that M_1 is bounded. Indeed, if $u \in M_1$, then we have $u \geq Au$ by definition. This together with (8) leads to

$$u(t) \geq (\lambda_1+\varepsilon)\int_0^1 G(t,s)h(s)u(s)ds - b\int_0^1 G(t,s)h(s)ds.$$

Multiply by $\psi(t)$ on both sides of the above and integrate over $[0, 1]$ and use (7) to obtain

$$\int_0^1 \psi(t)u(t)dt \geq (\lambda_1 + \varepsilon)\lambda_1^{-1} \int_0^1 \psi(t)u(t)dt - b\lambda_1^{-1},$$

so that $\int_0^1 \psi(t)u(t)dt \leq \frac{b}{\varepsilon}$ for all $u \in M_1$. Note we have $M_1 \subset P_0$ by Lemma 5. This together with the preceding inequality implies $\|u\| \leq (\varepsilon\omega)^{-1}b$ for all $u \in M_1$, which establishes the boundedness of M_1, as required. Taking $R > (\varepsilon\omega)^{-1}b$, we obtain

$$u \neq Au + \lambda\varphi, \quad \forall u \in \partial B_R \cap P, \quad \lambda \geq 0.$$

Now Lemma 3 yields

$$i(A, B_R \cap P, P) = 0. \tag{9}$$

By (H6), there exist $r \in (0, R)$ and $\varepsilon \in (0, \lambda_1)$ such that $f(t, u) \leq (\lambda_1 - \varepsilon)u$ for all $u \in [0, r]$ and $t \in [0, 1]$. This implies

$$(Au)(t) \leq (\lambda_1 - \varepsilon) \int_0^1 G(t, s)h(s)u(s)ds \tag{10}$$

for all $u \in \overline{B}_r \cap P$. Now we claim

$$u \neq \mu Au, \quad \forall u \in \partial B_r \cap P, \quad 0 \leq \mu \leq 1. \tag{11}$$

Indeed, if there exist $u_0 \in \partial B_r \cap P$ and $\mu_0 \in [0, 1]$ for which $u_0 = \mu_0 Au_0$, then this together with (10) leads to $u_0(t) \leq (\lambda_1 - \varepsilon) \int_0^1 G(t, s)h(s)u_0(s)ds$. Multiply by $\psi(t)$ on both sides of the preceding inequality and integrate over $[0, 1]$ and use (7) to obtain

$$\int_0^1 \psi(t)u_0(t)dt \leq \frac{\lambda_1 - \varepsilon}{\lambda_1} \int_0^1 \psi(t)u_0(t)dt,$$

so that $\int_0^1 \psi(t)u_0(t)dt = 0$, whence $u_0(t) \equiv 0$, contradicting $u_0 \in \partial B_r \cap P$. As a result, (11) is true and we have by Lemma 4

$$i(A, B_r \cap P, P) = 1. \tag{12}$$

Now (9) and (12) combined imply

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_r \cap P, P) = -1.$$

Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. Therefore (1) has at least one positive solution, which completes the proof.

THEOREM 2. Suppose that (H1)-(H3), (H5) and (H7) are satisfied, then (1) has at least two positive solutions.

PROOF. By (H7), we have

$$\|Au\| = \max_{t\in[0,1]} \int_0^1 G(t, s)h(s)f(s, u(s))ds < \int_0^1 \frac{m_0}{\Gamma(\alpha)} k(s)h(s)\frac{\eta\Gamma(\alpha)\rho}{M_0} ds = \|u\|$$

for all $u \in \partial B_\rho \cap P$. Now Lemma 2 yields

$$i(A, B_\rho \cap P, P) = 1. \tag{13}$$

On the other hand, in view of (H5), we may take $R > \rho$ so that (9) holds (see the proof of Theorem 1). By (H3), there exist $r \in (0, \rho)$ and $\varepsilon > 0$ such that $f(t, u) \geq (\lambda_1 + \varepsilon)u$, for all $u \in [0, r]$ and $t \in [0, 1]$. This implies

$$(Au)(t) \geq (\lambda_1 + \varepsilon) \int_0^1 G(t, s)h(s)u(s)ds \tag{14}$$

for all $u \in \overline{B}_r \cap P$. Now we claim

$$u - Au \neq \mu\varphi, \forall u \in \partial B_r \cap P, \mu \geq 0, \tag{15}$$

where φ is determined by (7). Indeed, if the claim is false, then there exist $u_1 \in \partial B_r \cap P$ and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1\varphi$ and thus $u_1 \geq Au_1$. Combining the last inequality with (14) (with u replaced by u_1), we obtain

$$u_1(t) \geq (\lambda_1 + \varepsilon) \int_0^1 G(t, s)h(s)u_1(s)ds.$$

Multiply by $\psi(t)$ on both sides of the above and integrate over [0,1] and use (7) to obtain

$$\int_0^1 u_1(t)\psi(t)dt \geq (\lambda_1 + \varepsilon)\lambda_1^{-1} \int_0^1 u_1(t)\psi(t)dt,$$

so that $\int_0^1 u_1(t)\psi(t)dt = 0$, whence $u_1(t) \equiv 0$, contradicting $u_1 \in \partial B_r \cap P$. As a result, (15) is true, as claimed. Now Lemma 3 yields

$$i(A, B_r \cap P, P) = 0. \tag{16}$$

Combining (9), (13) and (16), we arrive at

$$i(A, (B_R \backslash \overline{B}_\rho) \cap P, P) = 0 - 1 = -1$$

and

$$i(A, (B_\rho \backslash \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Consequently, A has at least two fixed points, with one on $(B_R \backslash \overline{B}_\rho) \cap P$ and the other on $(B_\rho \backslash \overline{B}_r) \cap P$. Therefore (1) has at least two positive solutions, which completes the proof.

To prove Theorem 3 below, we need an extra cone P_1, which is defined by

$$P_1 = \{u \in P : u(t) \geq \theta\|u\|, \forall t \in [\sigma, 1 - \sigma]\},$$

where $\theta = \frac{\alpha - 2}{m_0} \min\{q(\sigma), q(1 - \sigma)\}$. We have the following

Claim 1. $A(P) \subset P_1$.

PROOF. On the one hand, $u \in P$ implies $\|Au\| \le \frac{1}{\Gamma(\alpha)} \int_0^1 m_0 k(s)h(s)f(s,u(s))ds$. On the other hand, $u \in P$ and $t \in [\sigma, 1-\sigma]$ imply

$$(Au)(t) \ge \frac{(\alpha-2)q(t)}{m_0\Gamma(\alpha)} \int_0^1 m_0 k(s)h(s)f(s,u(s))ds \ge \frac{\theta}{\Gamma(\alpha)} \int_0^1 m_0 k(s)h(s)f(s,u(s))ds$$

and thus $(Au)(t) \ge \theta\|Au\|$ for all $u \in P$ and $t \in [\sigma, 1-\sigma]$. This implies $A(P) \subset P_1$, as claimed.

THEOREM 3. Suppose that (H1), (H2), (H4), (H6) and (H8) are satisfied, then (1) has at least two positive solutions.

PROOF. Recall $A(P) \subset P_1$. By (H8), we have

$$\begin{aligned}
\|Au\| &= \max_{0\le t\le 1}(Au)(t) \ge \max_{t\in[\sigma,1-\sigma]}(Au)(t) \\
&= \max_{t\in[\sigma,1-\sigma]} \int_0^1 G(t,s)h(s)f(s,u(s))ds \\
&> \max_{t\in[\sigma,1-\sigma]} \int_0^1 \frac{(\alpha-2)q(t)k(s)}{\Gamma(\alpha)} h(s)f(s,u(s))ds \\
&= \frac{(\alpha-2)q(\frac{\alpha-2}{\alpha})}{\Gamma(\alpha)} \int_0^1 k(s)h(s)\frac{\Gamma(\alpha)\rho}{\eta(\alpha-2)q(\frac{\alpha-2}{\alpha})}ds \\
&= \|u\|,
\end{aligned}$$

for all $u \in \partial B_\rho \cap P$, and by Lemma 2

$$i(A, B_\rho \cap P, P) = 0. \tag{17}$$

On the other hand, in view of (H6) , we may take $r \in (0,\rho)$ so that (12) holds (see the proof of Theorem 1). In addition, by (H4), there exist $\varepsilon \in (0,\lambda_1)$ and $m > 0$ such that $f(t,u) \le (\lambda_1-\varepsilon)u + m$ for all $u \ge 0$ and $t \in [0,1]$. Let

$$M_2 = \{u \in P: \ u = \mu Au, \ 0 \le \mu \le 1\}. \tag{18}$$

We shall prove that M_2 is bounded. Indeed, if $u \in M_2$, then, by definition, we have for some $\mu \in [0,1]$

$$\begin{aligned}
u(t) &= \mu(Au)(t) \le \textstyle\int_0^1 G(t,s)h(s)f(s,u(s))ds \\
&\le \textstyle\int_0^1 G(t,s)h(s)((\lambda_1-\varepsilon)u(s)+m)ds \\
&= (\lambda_1-\varepsilon)(Tu)(t) + u_0(t),
\end{aligned} \tag{19}$$

$u_0 \in P \setminus \{0\}$ being defined by $u_0(t) = m\int_0^1 G(t,s)h(s)ds$. Notice $r((\lambda_1-\varepsilon)T) < 1$. This implies the inverse operator of $I - (\lambda_1-\varepsilon)T$ exists and equals

$$(I-(\lambda_1-\varepsilon)T)^{-1} = I + (\lambda_1-\varepsilon)T + (\lambda_1-\varepsilon)^2T^2 + \ldots + (\lambda_1-\varepsilon)^nT^n + \ldots,$$

from which we obtain $(I-(\lambda_1-\varepsilon)T)^{-1}(P) \subset P$. Applying this to (19) gives $u \le (I-(\lambda_1-\varepsilon)T)^{-1}u_0$ for all $u \in M_2$. This proves the boundedness of M_2, as required.

Choosing $R > \sup\{\|u\| : u \in M_2\}$ and $R > \rho$, we have $u \neq \lambda Au$ for all $u \in \partial B_R \cap P$ and $\lambda \in [0,1]$. Now Lemma 4 yields

$$i(A, B_R \cap P, P) = 1. \tag{20}$$

Combining (12), (17) and (20), we arrive at

$$i(A, (B_R \backslash \overline{B}_\rho) \cap P, P) = 1 - 0 = 1, i(A, (B_\rho \backslash \overline{B}_r) \cap P, P) = 0 - 1 = -1.$$

Hence A has at least two fixed points, with one on $(B_R \backslash \overline{B}_\rho) \cap P$ and the other on $(B_\rho \backslash \overline{B}_r) \cap P$. Therefore (1) has at least two positive solutions, which completes the proof.

References

[1] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, Nonlinear Analysis, 72(2010), 916–924.

[2] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl., 311(2005), 495–505.

[3] A. Babakhani and V. D. Gejji, Existence of positive solutions of nonlinear fractional differential equations, J. Math. Anal. Appl., 278(2003), 434–442.

[4] D. Delbosco and L. Rodino, Existence and uniqueness for a fractional differential equation, J. Math. Anal. Appl., 204(1996), 609–625.

[5] D. Guo, Nonlinear Functional Analysis, Jinan: Science and Technology Press of Shandong, 1985 (in Chinese).

[6] D. Jiang and C. Yuan, The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application, Nonlinear Analysis, 72(2010), 710–719.

[7] A. A. Kilbas and J. J. Trujillo, Differential equations of fractional order: methods, results and problems-I, Appl. Anal., 78(2001), 153–192.

[8] A. A. Kilbas and J. J. Trujillo, Differential equations of fractional order: methods, results and problems-II, Appl. Anal. 81 (2002), 435–493.

[9] M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space, Transl. Amer. Math. Soc., 10(1962), 199–325.

[10] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, New York, London, Toronto, 1999.

[11] X. Su and S. Zhang, Solutions to boundary-value problems for nonlinear differential equations of fractional order, Electronic Journal of Differential Equations, 2009(2009), No. 26, pp. 1-15.

[12] X. Xu, D. Jiang and C. Yuan, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation, Nonlinear Analysis, 71(2009), 4676–4688.

[13] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, Electronic Journal of Differential Equations, Vol. 2006(2006), No. 36, pp. 1-12.