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A Survey On D-Semiclassical Orthogonal Polynomials*

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Abstract

In this paper, a survey of the most basic results on characterizations, methods of classification and processes of construction of D-semiclassical orthogonal polynomials is presented. In particular, the standard perturbation, the symmetrization process are revisited and some examples are carefully analyzed. Symmetrization after perturbation is studied and some examples are given.

1 Introduction and Preliminaries

By D-classical monic orthogonal polynomials sequences: (MOPS), we refer to Hermite, Laguerre, Bessel and Jacobi polynomials where D is the derivative operator. The orthogonality considered here is related to a form (regular linear form) [6,30] not only to an inner product. Since 1939, a natural generalization of the D-classical character is the D-semiclassical one introduced by J. A. Shohat in [41]. From 1985, this theory has been developed, from an algebraic aspect and a distributional one, by P. Maroni and extensively studied by P. Maroni and coworkers in the last decade [1,12,16,30,34,36]. A form u is called D-semiclassical when it is regular and satisfies the Pearson equation

$$D(\Phi u) + \Psi u = 0 \tag{1}$$

where (Φ, Ψ) are two polynomials , Φ monic with $\deg \Phi \geq 0$ and $\deg \Psi \geq 1$. The corresponding (MOPS) $\{B_n\}_{n\geq 0}$ is called D-semiclassical. Moreover, if u is D-semiclassical, the class of u, denoted s is defined by

$$s := \min\Big(\max(\deg \Phi - 2, \deg \Psi - 1)\Big) \tag{2}$$

where the minimum is taken over all pairs (Φ, Ψ) satisfying (1). In particular, the class s is greater to 0 and when $s = 0$ the D-classical case is recovered [34].
In 1985, M. Bachene [3, page 87] gave the system fulfilled by the coefficients of the three-term recurrence relation of a D-semiclassical (MOPS) of class 1 using the structure

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§The second author is pleased to dedicate this work to Professor Pascal Maroni founder of the research group in orthogonal polynomials at Gabès, Tunisia.

relation (see (26) below). In 1992, S. Belmehdi [4,page 272] gave the same system (in a more simple way) using the Pearson equation (1). This system is not linear and it is very difficult to solve. As a consequence, D-semiclassical forms of class 1 are classified by S. Belmehdi in [4] through a distributional study by taking into account (1) and by giving an integral representation for any canonical case except the case of Bessel kind.

In 1996, J. Alaya and P. Maroni have established the linear system fulfilled by the coefficients of the three-term recurrence relation of a symmetric D-Laguerre-Hahn (MOPS) of class 1 [1] that is to say the (MOPS) associated with a regular form u satisfying the functional equation

$$D(\Phi u) + \Psi u + B\big(x^{-1}u^2\big) = 0$$

where Φ, Ψ, B are three polynomials with Φ monic and

$$\max\Bigg(\deg\Psi - 1, \max(\deg\Phi, \deg B) - 2\Bigg) = 1.$$

Consequently, the authors give the classification of symmetric D-semiclassical forms of class 1 as a particular case ($B \equiv 0$). There are three canonical situations:

- The generalized Hermite form $\mathcal{H}(\mu)$ ($\mu \neq 0$, $\mu \neq -n-\frac{1}{2}$, $n \geq 0$) and its (MOPS) satisfying

$$\begin{cases} \beta_n = 0 \ , \ \gamma_{n+1} = \frac{1}{2}(n+1+\mu(1+(-1)^n)) \ , \ n \geq 0, \\ D(x\mathcal{H}(\mu)) + \{2x^2 - (2\mu+1)\}\mathcal{H}(\mu) = 0. \end{cases} \tag{3}$$

- The generalized Gegenbauer $\mathcal{G}(\alpha,\beta)$ ($\alpha \neq -n-1$, $\beta \neq -n-1$, $\beta \neq -\frac{1}{2}$, $\alpha+\beta \neq -n-1$, $n \geq 0$) and its (MOPS) satisfying

$$\begin{cases} \beta_n = 0 \ , \ n \geq 0, \\ \gamma_{2n+1} = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \ , \ n \geq 0, \\ \gamma_{2n+2} = \frac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)} \ , \ n \geq 0, \\ D\Big(x(x^2-1)\mathcal{G}(\alpha,\beta)\Big) + \Big\{-2(\alpha+\beta+2)x^2 + 2(\beta+1)\Big\}\mathcal{G}(\alpha,\beta) = 0. \end{cases} \tag{4}$$

  For further properties of the generalized Hermite polynomials, the generalized Gegenbauer polynomials and their orthogonality relations see [1,4,6,14,40].

- The form $\mathcal{B}[\nu]$ of Bessel kind ($\nu \neq -n-1$, $n \geq 0$) and its (MOPS) satisfying

$$\begin{cases} \beta_n = 0 \ , \ \gamma_1 = -\frac{1}{4(\nu+1)}, \\ \gamma_{2n+2} = \frac{1}{4}\frac{n+1}{(2n+\nu+1)(2n+\nu+2)} \ , \ n \geq 0, \\ \gamma_{2n+3} = -\frac{1}{4}\frac{n+1+\nu}{(2n+\nu+2)(2n+\nu+3)} \ , \ n \geq 0, \\ D\Big(x^3\mathcal{B}[\nu]\Big) - \Big\{2(\nu+1)x^2 + \frac{1}{2}\Big\}\mathcal{B}[\nu] = 0. \end{cases} \tag{5}$$

For an integral representation of $\mathcal{B}[\nu]$ and some additional features of the associated (MOPS) see [13,39].

Other families of D-semiclassical orthogonal polynomials of class greater to 1 were revealed by solving functional equations of the type

$$P(x)\widetilde{u} = Q(x)u \tag{6}$$

where P, Q are two polynomials cunningly chosen and $u, \widetilde{u}$ two forms. In fact, the product of a form by a polynomial is one of the old construction process of forms; in 1858, Christoffel proved that the product of a positive definite form by a positive polynomial is still a positive definite form [7]. This result has been generalized by J. Dini and P. Maroni in [9] since 1989 where it was proved that, under certain regularity conditions, the product of a regular form $\widetilde{u}$ by a polynomial P is a regular form. In that work, the D-semiclassical character is studied. It is also interesting to consider the inverse problem, which consists of the determination of all regular form $\widetilde{u}$ satisfying (6) where $Q(x) = -\lambda \in \mathbb{C} - \{0\}$ and u is a given regular form. P. Maroni has considered the case $P(x) = x - \tau, \quad \tau \in \mathbb{C}$ in 1990 [31], the case $P(x) = x^2$ in 1996 [35] and the case $P(x) = x^3$ in 2003 [36].

In 1992, F. Marcellan and P. Maroni have considered the case $P(x) = Q(x) = x - \tau$, where u is a given regular D-semiclassical form [23]. For other published papers concerning the problem (6) see [12,15,17,21,22,43].

Symmetric D-semiclassical forms of class 2 satisfying (1) with $\Phi(0) = 0$ are well described in [39] by M. Sghaier and J. Alaya (in 2006) through there an original characterization by taking into account (6). In 2007, symmetric D-semiclassical forms of class 2 are also classified by a distributional study likewise in [4], by A. M. Delgado and F. Marcellan [8] but it seems that integral representations are given only in the positive definite cases (see Remark 4.2. below).
For other relevant research work on the subject from other points of view and with perhaps other operators see [2,5,20,38].

The aim of this survey is threefold. First, to present an overview about the characterizations and processes of construction of D-semiclassical orthogonal polynomials (see section 1 and section 2). Then, to revise the standard perturbation (see (6) for $\deg P = 1$ and $\deg Q = 0$) by taking into account the framework [31] and the limiting case ($q \to 1$) in [12] and to analyze the symmetrization process according to [2,33] by adding some complementary results (see section 3 and section 4). Finally, to highlight some examples of D-semiclassical orthogonal polynomials of class greater to 1 by using [37] (see section 2), by applying the standard perturbation (see section 3) and by combining the processes of symmetrization and perturbation (see section 4).

Now, we are going to introduce the material concerning orthogonal polynomials and regular forms that find their origin in the book of T. S. Chihara (1978) [6] and developed by P. Maroni since 1981 [24,25,29,30].

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. Moreover, a form u is called symmetric if $(u)_{2n+1} = 0$, $n \geq 0$. For any form u, any polynomial g , let gu , be the form defined by duality

$$\langle gu, f\rangle := \langle u, gf\rangle \ , f \in \mathcal{P}. \tag{7}$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial

$$(uf)(x) := \left\langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \right\rangle . \tag{8}$$

The derivative $u' = Du$ of the form u is defined by

$$\langle u', f \rangle := -\langle u, f' \rangle \ , f \in \mathcal{P}. \tag{9}$$

We have [30]

$$(fu)' = f'u + fu' \ , u \in \mathcal{P}' \ , f \in \mathcal{P} \ . \tag{10}$$

The formal Stieltjes function of $u \in \mathcal{P}'$ is defined by

$$S(u)(z) := -\sum_{n \geq 0} \frac{(u)_n}{z^{n+1}} \ . \tag{11}$$

Similarly, with the definitions

$$\langle h_a u, f \rangle := \langle u, h_a f \rangle = \langle u, f(ax) \rangle \ , u \in \mathcal{P}' \ , f \in \mathcal{P} \ , a \in \mathbb{C} - \{0\} , \tag{12}$$

$$\langle \tau_b u, f \rangle := \langle u, \tau_{-b} f \rangle = \langle u, f(x+b) \rangle \ , u \in \mathcal{P}' \ , f \in \mathcal{P} \ , b \in \mathbb{C} \tag{13}$$

and

$$\langle \delta_c, f \rangle := f(c) \ , f \in \mathcal{P} \ , c \in \mathbb{C}. \tag{14}$$

The form u is called *regular* if we can associate with it a polynomial sequence $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, such that

$$\langle u, P_m P_n \rangle = r_n \delta_{n,m} \ , \ n, m \geq 0 \ ; \ r_n \neq 0 \ , \ n \geq 0. \tag{15}$$

The polynomial sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. Necessarily, $\{P_n\}_{n\geq 0}$ is an (OPS) where every polynomial can be supposed monic and it fulfils the three-term recurrence relation

$$\begin{cases} P_0(x) = 1 \ , \ P_1(x) = x - \beta_0 \ , \\ \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x) \ , \ n \geq 0 \end{cases} \tag{16}$$

with $\beta_n = \frac{\langle u, xP_n^2 \rangle}{\langle u, P_n^2 \rangle}$ and $\gamma_{n+1} = \frac{\langle u, P_{n+1}^2 \rangle}{\langle u, P_n^2 \rangle} \neq 0, \quad n \geq 0$. Moreover, the regular form u, associated to the (MOPS) $\{P_n\}_{n\geq 0}$ satisfying (16), is said to be positive definite if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$ for all $n \geq 0$.

The form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any form will be normalized.

From the linear application $p \mapsto (\theta_c p)(x) = \frac{p(x) - p(c)}{x - c}$, $p \in \mathcal{P}$, $c \in \mathbb{C}$, we define $(x - c)^{-1} u$ by

$$\left\langle (x - c)^{-1} u, p \right\rangle := \langle u, \theta_c p \rangle \tag{17}$$

and we have [30,33]

$$(x - c)\big((x - c)^{-1} u\big) = u \quad ; \quad (x - c)^{-1}\big((x - c)u\big) = u - (u)_0 \delta_c, \ u \in \mathcal{P}', \ c \in \mathbb{C}. \tag{18}$$

Finally, we introduce the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ defined by $(\sigma f)(x) := f(x^2)$ for all $f \in \mathcal{P}$. Consequently, we define σu by duality

$$\langle \sigma u, f\rangle := \langle u, \sigma f\rangle \ , f \in \mathcal{P} \ , u \in \mathcal{P}' \tag{19}$$

and we have for $f \in \mathcal{P}$, $u \in \mathcal{P}'$ [33]

$$f(x)\sigma u = \sigma(f(x^2)u) \quad ; \quad \sigma u' = 2(\sigma(xu))'. \tag{20}$$

2 Overview about D-Semiclassical Forms

Now, we will expose the D-semiclassical character according to P. Maroni [26,27,28,30].

Let Φ monic and Ψ be two polynomials , $\deg \Phi = t$, $\deg \Psi = p \geq 1$. We suppose that the pair (Φ, Ψ) is *admissible* , i.e. when $p = t-1$, writing $\Psi(x) = a_p x^p + ...$, then $a_p \neq n+1$, $n \in \mathbb{N}$.

DEFINITION 2.1. A form u is called D-semiclassical when it is regular and satisfies the functional equation

$$(\Phi u)' + \Psi u = 0 \tag{21}$$

where the pair (Φ, Ψ) is admissible. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called D-semiclassical.

REMARKS 2.1.

1. The D-semiclassical character is kept by shifting(see [30]). In fact, let

$$\left\{a^{-n}\left(h_a \circ \tau_{-b} P_n\right)\right\}_{n\geq 0} \ , \ a \neq 0 \ , \ b \in \mathbb{C};$$

when u satisfies (21), then $h_{a^{-1}} \circ \tau_{-b} u$ fulfils the equation

$$\left(a^{-t}\Phi\left(ax+b\right)\left(h_{a^{-1}} \circ \tau_{-b} u\right)\right)'^{1-t} \Psi\left(ax+b\right)\left(h_{a^{-1}} \circ \tau_{-b} u\right) = 0. \tag{22}$$

2. The D-semiclassical form u is said to be of class $s = \max(p-1, t-2) \geq 0$ if and only if

$$\prod_{c\in\mathcal{Z}_\Phi}\left\{\left|\Psi(c) + \Phi'(c)\right| + \left|\left\langle u, \theta_c\Psi + \theta_c^2\Phi\right\rangle\right|\right\} > 0, \tag{23}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ will be known as of class s [30].

3. When $s = 0$, the form u is usually called *D-classical* (Hermite, Laguerre, Bessel, and Jacobi)[34].

We can state characterizations of D-semiclassical orthogonal sequences. $\{P_n\}_{n\geq 0}$ is D-semiclassical of class s, if and only if one of the following statements holds (see [31] and $q = 1$ in section 2. of [12])

(1). The formal Stieltjes function of u satisfies a non homogeneous first order linear differential equation

$$\Phi(z)S'\left(u\right)\left(z\right) = C_0\left(z\right)S\left(u\right)\left(z\right) + D_0\left(z\right) \ , \tag{24}$$

where

$$\begin{cases} C_0(z) = -\Psi(z) - \Phi'(z), \\ D_0(z) = -(u\theta_0\Phi)'(z) - (u\theta_0\Psi)(z). \end{cases} \tag{25}$$

Φ and Ψ are the same polynomials as in (21).

(2). $\{P_n\}_{n\geq 0}$ satisfies the following structure relation

$$\Phi\left(x\right)P'_{n+1}\left(x\right) = \frac{1}{2}\left(C_{n+1}\left(x\right) - C_0\left(x\right)\right)P_{n+1}\left(x\right) - \gamma_{n+1}D_{n+1}\left(x\right)P_n\left(x\right), n\geq 0, \tag{26}$$

where

$$C_{n+1}\left(x\right) = -C_n\left(x\right) + 2\left(x-\beta_n\right)D_n\left(x\right)\ , n\geq 0 \tag{27}$$

$$\begin{aligned} \gamma_{n+1}D_{n+1}\left(x\right) &= -\Phi\left(x\right) + \gamma_n D_{n-1}\left(x\right) \\ &\quad + \left(x-\beta_n\right)^2 D_n\left(x\right) - \left(x-\beta_n\right)C_n\left(x\right), \ n\geq 0, \end{aligned} \tag{28}$$

Φ, Ψ, C_0, D_0 are the same parameters introduced in **(1)**; β_n, γ_n are the coefficients of the three term recurrence relation (16). Notice that $D_{-1}\left(x\right) := 0$, $\deg C_n \leq s+1$ and $\deg D_n \leq s,\ n\geq 0$.

(3). Each polynomial $P_{n+1},\ n\geq 0$ satisfies a second order linear differential equation

$$J\left(x,n\right)P''_{n+1}\left(x\right) + K\left(x,n\right)P'_{n+1}\left(x\right) + L\left(x,n\right)P_{n+1}\left(x\right) = 0\ ,\ n\geq 0, \tag{29}$$

with

$$\begin{cases} J\left(x,n\right) = \Phi\left(x\right)D_{n+1}\left(x\right)\ , \\ K\left(x,n\right) = D_{n+1}\left(x\right)\Big(\Phi'\left(x\right) + C_0\left(x\right)\Big) - D'_{n+1}\left(x\right)\Phi\left(x\right)\ , \\ L\left(x,n\right) = \frac{1}{2}\Big(C_{n+1}\left(x\right) - C_0\left(x\right)\Big)D'_{n+1}\left(x\right) - \\ -\frac{1}{2}\Big(C'_{n+1} - C'_0\Big)\left(x\right)D_{n+1}\left(x\right) - D_{n+1}\left(x\right)\Sigma_n\left(x\right)\ ,\ n\geq 0. \end{cases} \tag{30}$$

and

$$\Sigma_n\left(x\right) := \sum_{k=0}^{n} D_k(x)\ ,\ n\geq 0. \tag{31}$$

Φ, C_n, D_n are the same in the previous characterization. Notice that $\deg J(.,n) \leq 2s+2$, $\deg K(.,n) \leq 2s+1$ and $\deg L(.,n) \leq 2s$. In particular, when $s=0$ that is to say the D-classical case, the coefficients of the structure relation (26) become

$$\begin{cases} \frac{C_{n+1}(x)-C_0(x)}{2} = \frac{1}{2}\Phi''(0)((n+1)x - S_n) + \\ \qquad +(\Psi'(0) - \Phi''(0)(n+1))\beta_{n+1} + (\Psi(0) - \Phi'(0)(n+1)), \\ D_{n+1}(x) = \frac{1}{2}\Phi''(0)(2n+1) - \Psi'(0),\ n\geq 0, \end{cases} \tag{32}$$

with $S_n = \sum_{k=0}^{n}\beta_k, n\geq 0$. Also we get for (30)

$$\begin{cases} J(x,n) = \Phi(x)\ , \\ K(x,n) = -\Psi(x)\ , \\ L(x,n) = (n+1)(\Psi'(0) - \frac{1}{2}\Phi''(0)n)\ ,\ n\geq 0. \end{cases} \tag{33}$$

In 1990, P. Maroni gives the following result about order of zero of a D-semiclassical polynomial P_{n+1} of class s [31].

REMARK 2.2. Taking into account the structure relation (26), the polynomial sequence $\{D_{n+1}\}_{n\geq 0}$ gives us some information about zeros of the polynomial P_{n+1}. In fact, if c is a zero of order η of $P_{n+1}, n \geq 1$ with $\eta \geq 2$, then $\eta \leq s+1$ and c is a zero of order $\eta - 1$ of D_{n+1}.

In 1996, J. Alaya and P. Maroni give the following result to a symmetric D-Laguerre-Hahn form [1] which remains valid in the symmetric D-semiclassical case

PROPOSITION 2.1. Let u be a symmetric D-semiclassical form of class s satisfying (21). The following statements hold

i) When s is odd then the polynomial Φ is odd and Ψ is even.

ii) When s is even then the polynomial Φ is even and Ψ is odd.

Finally, about integral representation (P. Maroni [32] in 1995), let u be a D-semiclassical form satisfying (21). We are looking for an integral representation of u and consider

$$\langle u, f\rangle = \int_{-\infty}^{+\infty} U(x)f(x)dx \ , \ f \in \mathcal{P}, \tag{34}$$

where we suppose the function U to be absolutely continuous on $\mathbb{R}$, and is decaying as fast as its derivative U'. From (21) we get

$$\int_{-\infty}^{+\infty} \big((\Phi U)' + \Psi U\big) f(x)dx - \Phi(x)U(x)f(x)\big]_{-\infty}^{+\infty} = 0 \ , \ f \in \mathcal{P}.$$

Hence, from the assumptions on U, the following conditions hold

$$\Phi(x)U(x)f(x)\big]_{-\infty}^{+\infty} = 0 \ , \ f \in \mathcal{P}, \tag{35}$$

$$\int_{-\infty}^{+\infty} \big((\Phi U)' + \Psi U\big) f(x)dx = 0 \ , \ f \in \mathcal{P}. \tag{36}$$

Condition (36) implies

$$(\Phi U)' + \Psi U = \omega g, \tag{37}$$

where $\omega \neq 0$ arbitrary and g is a locally integrable function with rapid decay representing the null-form

$$\int_{-\infty}^{+\infty} x^n g(x)dx = 0 \ , \ n \geq 0. \tag{38}$$

Conversely, if U is a solution of (37) verifying the hypothesis above and the condition

$$\int_{-\infty}^{+\infty} U(x)dx \neq 0, \tag{39}$$

then (35)-(36) are fulfilled and (34) defines a form u which is a solution of (21).

In particular, from $\big((\Phi u)' + \Psi u\big)_n = 0,\ n \geq 0$, writing

$$\Phi(x) = \sum_{k=0}^{t} c_k x^k; \quad \Psi(x) = \sum_{k=0}^{p} a_k x^k$$

and by taking into account (9), we have for the moments of u

$$(u)_0 = 1\ ;\ \sum_{k=0}^{p} a_k (u)_k = 0; \quad \sum_{k=1}^{p+1} a_{k-1} (u)_{n+k} - (n+1) \sum_{k=0}^{t} c_k (u)_{n+k} = 0,\ n \geq 0. \tag{40}$$

In [37], P. Maroni and M. Ihsen Tounsi (2004) have described all (MOPS) which are identical to their second associated sequence that is to say those satisfying (16) with

$$\beta_{n+2} = \beta_n; \quad \gamma_{n+3} = \gamma_{n+1}\ , \quad n \geq 0.$$

The resulting polynomials are D-semiclassical of class $s \leq 3$. In fact there are five canonical situations itemized (a)-...-(e) (see Proposition 3.5 in [37]). The characteristic elements of the structure relation (26) and the second-order differential equation (29) are given explicitly by using the quadratic decomposition [33]. Integral representations of the corresponding forms are also given by an other process which consists of representing the corresponding Stieljes function (11). So, our interest in the following example is to describe the canonical situation (d) in that work but with processes exposed in this section.

EXAMPLE 2.1. Let's consider the D-semiclassical form u of class 1 and its (MOPS) $\{P_n\}_{n\geq 0}$. We have [37]

$$\begin{cases} \beta_n = (-1)^n, \quad \gamma_{n+1} = -\frac{1}{4}\ ,\ n \geq 0\ \ , \\ \Big(x(x^2-1))u\Big)'^2 + x + 2)u = 0. \end{cases} \tag{41}$$

For the moments of u, from (40)-(41) we get

$$\begin{cases} (u)_0 = 1, \qquad (u)_1 = 1\ , \\ -(n+4)(u)_{n+2} + (u)_{n+1} + (n+2)(u)_n = 0\ ,\ n \geq 0. \end{cases} \tag{42}$$

Consequently, it is easy to prove by induction that

$$(u)_{2n+1} = (u)_{2n}\ ,\ n \geq 0. \tag{43}$$

Now, taking $n \leftarrow 2n$ in the second equality in (42) and by virtue of (43) we get

$$(u)_{2n+2} = \frac{2n+3}{2n+4}(u)_{2n}\ ,\ n \geq 0$$

from which we derive the following

$$(u)_{2n+1} = (u)_{2n} = \frac{1}{n+1}\ \frac{(2n+1)!}{2^{2n}(n!)^2}\ ,\ n \geq 0. \tag{44}$$

For an integral representation of u of type (34), equation (37) with $\omega = 0$ and hypothesis becomes

$$U'(x) + \left\{-\frac{1}{x} + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right\}U(x) = 0. \tag{45}$$

A possible solution of (45) is the function

$$U(x) = \begin{cases} 0 & , \ x \leq -1, \\ \rho\ x\ \sqrt{\frac{1+x}{1-x}} & , \ -1 < x < 1, \\ 0 & , \ x \geq 1. \end{cases} \tag{46}$$

First, condition (35) is fulfilled because

$$x(x^2-1)x^n \rho\ x\ \sqrt{\frac{1+x}{1-x}}\ \Bigg]_{-1}^{+1} = 0\ ,\ n \geq 0.$$

Second, with the change of variable $t = \sqrt{\frac{1+x}{1-x}}$ the normalization constant ρ satisfies

$$\rho^{-1} = \int_{-1}^{1} x\ \sqrt{\frac{1+x}{1-x}}\ dx = \int_{0}^{+\infty} \frac{4t^2(t^2-1)}{(t^2+1)^3}\ dt = 4J_1 - 12J_2 + 8J_3$$

where

$$J_k = \int_0^{+\infty} \frac{dt}{(t^2+1)^k}\ ,\ k \geq 1.$$

But, upon integration by parts we get

$$J_{k+1} = \frac{2k-1}{2k}\ J_k,\ k \geq 1.$$

In particular, $J_1 = \frac{\pi}{2}$, $J_2 = \frac{\pi}{4}$ and $J_3 = \frac{3\pi}{16}$. Thus, $\rho^{-1} = \frac{\pi}{2}$ and u has the following integral representation

$$\langle u, f\rangle = \frac{2}{\pi}\ \int_{-1}^{1} x\ \sqrt{\frac{1+x}{1-x}}\ f(x)\ dx\ ,\ f \in \mathcal{P}. \tag{47}$$

For the structure relation (26), we may write it as follows

$$x(x^2-1)P'_{n+1}(x) = \big((n+1)x^2 + b_n x + c_n\big)P_{n+1}(x) + \frac{1}{4}\ \big(d_n x + e_n\big)P_n(x)\ ,\ n \geq 0. \tag{48}$$

The problem is then to determine the coefficients b_n, c_n, d_n, e_n, $n \geq 0$. Taking $x = 0$, $x = 1$, $x = -1$ respectively in (48) we get

$$\begin{cases} c_n P_{n+1}(0) + \frac{1}{4} e_n P_n(0) = 0 \\ \big((n+1) + b_n + c_n\big)P_{n+1}(1) + \frac{1}{4}(d_n + e_n)P_n(1) = 0 \\ \big((n+1) - b_n + c_n\big)P_{n+1}(-1) + \frac{1}{4}(-d_n + e_n)P_n(-1) = 0 \end{cases} ,n \geq 0. \tag{49}$$

Moreover, from (41) and (48), the formulas in (25)-(26) become

$$\begin{cases} C_n(x) = (2n+1)x^2 + (2b_{n-1}-1)x + 2c_{n-1} - 1 \\ D_n(x) = d_{n-1}x + e_{n-1} \end{cases} , n \geq 0 \tag{50}$$

where $b_{-1} := 0,\ c_{-1} := 0,\ d_{-1} := 2,\ e_{-1} := 2$. Consequently, replacing in (27)-(28) we get after identification

$$\begin{cases} d_n = 2(n+2) \\ 2b_n + 2b_{n-1} - 2e_{n-1} = 2\big(1 - 2(-1)^n(n+1)\big) \\ 2c_n + 2c_{n-1} + 2(-1)^n e_{n-1} = 2 \\ 2b_{n-1} + e_{n-1} = 1 + (-1)^n(-2n+1) \\ -2(-1)^n b_{n-1} + 2c_{n-1} - 2(-1)^n e_{n-1} = 2n + \frac{1}{2} - (-1)^n \\ -2(-1)^n c_{n-1} + \frac{1}{4}e_n + \frac{3}{4}e_{n-1} + = (-1)^{n+1} \end{cases} , n \geq 0. \tag{51}$$

On the other hand, regarding (49) we are going to compute $P_n(0)$, $P_n(1)$, $P_n(-1)$ for any $n \geq 0$.

For $P_n(0)$, taking $x = 0$ in (16) and on account of (41) we obtain

$$P_{n+2}(0) = -(-1)^{n+1}P_{n+1}(0) + \frac{1}{4}P_n(0),\ n \geq 0\ ;\ P_0(0) = 1\ ;\ P_1(0) = -1.$$

Thus,

$$\begin{cases} \left(P_{n+2}(0) + \frac{(-1)^{n+1}}{2}P_{n+1}(0)\right) = \frac{(-1)^n}{2}\left(P_{n+1}(0) + \frac{(-1)^n}{2}P_n(0)\right),\ n \geq 0, \\ P_1(0) + \frac{1}{2}P_0(0) = -\frac{1}{2}. \end{cases}$$

Therefore,

$$P_{n+1}(0) = -\frac{(-1)^n}{2}P_n(0) - \frac{(-1)^{\frac{(n-1)n}{2}}}{2^{n+1}},\ n \geq 0$$

from which we deduce that

$$P_n(0) = \frac{n+1}{2^n}\ (-1)^{\frac{n(n+1)}{2}},\ n \geq 0. \tag{52}$$

For $P_n(1)$, taking $x = 1$ in (16) and on account of (41) we obtain

$$P_{n+2}(1) = (1 - (-1)^{n+1})P_{n+1}(1) + \frac{1}{4}P_n(1),\ n \geq 0\ ;\ P_0(1) = 1\ ;\ P_1(1) = 0. \tag{53}$$

With $n \leftarrow 2n$ and $n \leftarrow 2n-1$ successively in (53) we deduce

$$\begin{cases} P_{2n+2}(1) = 2P_{2n+1}(1) + \frac{1}{4}P_{2n}(1),\ n \geq 0 \\ P_{2n+1}(1) = \frac{1}{4}P_{2n-1}(1),\ n \geq 1 \end{cases}$$

which gives

$$P_{2n+1}(1) = 0,\ n \geq 0 \quad ; \quad P_{2n}(1) = \frac{1}{2^{2n}},\ n \geq 0. \tag{54}$$

For $P_n(-1)$, taking $x=-1$ in (16) and on account of (41) we obtain

$$\begin{cases} P_{n+2}(-1) = -(1+(-1)^{n+1})P_{n+1}(-1) + \frac{1}{4}P_n(-1) \ , \ n \geq 0, \\ P_0(-1) = 1 \ ; \ P_1(-1) = -2. \end{cases} \tag{55}$$

With $n \leftarrow 2n$ and $n \leftarrow 2n-1$ successively in (55) we deduce

$$\begin{cases} P_{2n+2}(-1) = \frac{1}{4}P_{2n}(-1), \ n \geq 0 \\ P_{2n+1}(-1) = \frac{1}{4}P_{2n-1}(-1) - \frac{1}{2^{2n-1}}, \ n \geq 1 \end{cases}$$

which gives

$$P_{2n}(-1) = \frac{1}{2^{2n}}, \ n \geq 0 \quad ; \quad P_{2n+1}(-1) = -\frac{n+1}{2^{2n-1}} \ n \geq 0. \tag{56}$$

Now, replacing the numbers in (52), (54), (56) in the system (49) and by taking into account the second system (51) we obtain

$$\begin{cases} b_{2n} = 1 \quad ; \quad b_{2n+1} = 0 \\ c_{2n} = -(2n+1) \quad ; \quad c_{2n+1} = -2(n+1) \\ d_n = 2(n+2) \\ e_{2n} = -4(n+1) \quad ; \quad e_{2n+1} = 2(2n+3) \end{cases} , \ n \geq 0. \tag{57}$$

So, it is quite straightforward to get the expressions in (26) and (30) for the structure relation and the second order linear differential equation.

3 The Standard Perturbation Revisited

3.1 On the standard perturbation

Let u be a regular form. Denoting by $\{P_n\}_{n\geq 0}$ its (MOPS) satisfying (16). Let $\widetilde{u} \in \mathcal{P}'$ satisfying

$$(x-\tau)\widetilde{u} = \lambda u \ , \ \tau \in \mathbb{C}, \ \lambda \in \mathbb{C} - \{0\}. \tag{58}$$

According to (18) relation (58) is equivalent to

$$\widetilde{u} = \delta_\tau + \lambda(x-\tau)^{-1}u \tag{59}$$

with constraints $(\widetilde{u})_0 = 1$, $(u)_0 = 1$ and $\lambda + \tau = (\widetilde{u})_1$.

Suppose $\widetilde{u}$ regular and let $\{\widetilde{P}_n\}_{n\geq 0}$ be its (MOPS)

$$\begin{cases} \widetilde{P}_0(x) = 1 \ , \ \widetilde{P}_1(x) = x - \widetilde{\beta}_0 \ , \\ \widetilde{P}_{n+2}(x) = (x-\widetilde{\beta}_{n+1})\widetilde{P}_{n+1}(x) - \widetilde{\gamma}_{n+1}\widetilde{P}_n(x) \ , \ n \geq 0. \end{cases} \tag{60}$$

The connection between $\widetilde{P}_n$ and P_n is (see [31])

$$\widetilde{P}_0(x) = 1 \ , \ \widetilde{P}_{n+1}(x) = P_{n+1}(x) + a_n P_n(x) \ , \ n \geq 0, \tag{61}$$

with

$$a_n = -\frac{P_{n+1}(\tau) + \lambda P_n^{(1)}(\tau)}{P_n(\tau) + \lambda P_{n-1}^{(1)}(\tau)} \neq 0\ ,\ n \geq 0, \tag{62}$$

where

$$P_n^{(1)}(x) := \left\langle u, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} \right\rangle\ ,\ n \geq -1.$$

We have [30]

$$P_{n+1}^{(1)}(x) P_{n+1}(x) - P_{n+2}(x) P_n^{(1)}(x) = \prod_{k=0}^{n} \gamma_{k+1}\ ,\ n \geq 0. \tag{63}$$

Denoting

$$\lambda_n = -\frac{P_n(\tau)}{P_{n-1}^{(1)}(\tau)}\ ,\ n \geq 1\ ,\ \lambda_0 = 0. \tag{64}$$

Let us recall the fundamental result owing to P. Maroni (1990) [31]

PROPOSITION 3.1. Let u be a regular form. The following statements are equivalent

i) The form $\widetilde{u} = \delta_\tau + \lambda (x - \tau)^{-1} u$ is regular.

ii) $\lambda \neq \lambda_n,\ n \geq 0$.

We may write

$$\frac{\gamma_{n+1}}{a_n} + a_{n+1} - \beta_{n+1} = -\tau\ ,\ n \geq 0, \tag{65}$$

$$\widetilde{\beta}_0 = \beta_0 - a_0 = \tau + \lambda\ ,\ \widetilde{\beta}_{n+1} = \beta_{n+1} + a_n - a_{n+1}\ ,\ \widetilde{\gamma}_{n+1} = -a_n(a_n - \beta_n + \tau)\ ,\ n \geq 0, \tag{66}$$

$$\begin{cases} (x-\tau) P_n(x) = \widetilde{P}_{n+1}(x) + (\beta_n - a_n - \tau) \widetilde{P}_n(x)\ ,\ n \geq 0\ , \\ \\ (x-\tau) P_{n+1}(x) = (x - a_n - \tau) \widetilde{P}_{n+1}(x) + a_n (a_n - \beta_n + \tau) \widetilde{P}_n(x)\ ,\ n \geq 0. \end{cases} \tag{67}$$

In particular, if the regular form u is symmetric then the form $\widetilde{u} = \delta_0 + \lambda x^{-1} u$ is regular for any $\lambda \neq 0$ and we have ($\gamma_0 := 1\ ;\ \gamma_{-1} := 1$) **

$$\begin{cases} a_{2n} = -\lambda \prod_{k=0}^{n} \frac{\gamma_{2k}}{\gamma_{2k-1}}\ ,\ n \geq 0, \\ \\ a_{2n+1} = \frac{1}{\lambda} \prod_{k=0}^{n} \frac{\gamma_{2k+1}}{\gamma_{2k}}\ ,\ n \geq 0. \end{cases} \tag{68}$$

3.2 The D-semiclassical case

Suppose u be a D-semiclassical form of class s satisfying (21). Multiplying (21) by λ and on account of (58) we get

$$\left(\widetilde{\Phi} \widetilde{u}\right)' + \widetilde{\Psi} \widetilde{u} = 0, \tag{69}$$

with

$$\widetilde{\Phi}(x) = (x-\tau)\Phi(x) \ , \ \widetilde{\Psi}(x) = (x-\tau)\Psi(x)\,. \tag{70}$$

Now, taking $q=1$ in section 3.1 of [12], we recover the following result concerning the class of $\widetilde{u}$ [31]

THEOREM 3.1. If the D-semiclassical form u is of class s then the form $\widetilde{u}$ is D-semiclassical of class $\widetilde{s}=s+1$ for

$$\Phi(\tau)\neq 0\ ,\ \lambda\neq\lambda_n,\ n\geq 0 \ \text{ or } \ \Phi(\tau)=0\ ,\ \lambda\neq\lambda_n,\ n\geq -1, \tag{71}$$

where

$$\lambda_{-1} = -\frac{\Psi(\tau)+\Phi'(\tau)}{\langle u, \theta_\tau\Psi + \theta_\tau^2\Phi\rangle}. \tag{72}$$

REMARK 3.1. Also, taking $q=1$ in section 3.2 of [12], we recover again the results in [31] concerning the structure relation of $\left\{\widetilde{P}_n\right\}_{n\geq 0}$ and the second order linear differential equation satisfied by $\widetilde{P}_{n+1},\ n\geq 0$.

Finally, if we suppose that the form u has the following integral representation:

$$\langle u, f\rangle = \int_{-\infty}^{+\infty} U(x)\,f(x)\,dx\ ,\ f\in\mathcal{P}\ ;\ \ \int_{-\infty}^{+\infty} U(x)\,dx = 1$$

where U is a locally integrable function with rapid decay and continuous at the point $x=\tau$ then in view of (59) we may write

$$\langle \widetilde{u}, f\rangle = \left\{1-\lambda P\int_{-\infty}^{+\infty}\frac{U(x)}{x-\tau}dx\right\} f(0) + \lambda P\int_{-\infty}^{+\infty}\frac{U(x)}{x-\tau}f(x)\,dx\ , f\in\mathcal{P}\ , \tag{73}$$

where

$$P\int_{-\infty}^{+\infty}\frac{U(x)}{x-\tau}dx := \lim_{\varepsilon\rightsquigarrow 0^+}\left(\int_{-\infty}^{\tau-\varepsilon}\frac{U(x)}{x-\tau}dx + \int_{\tau+\varepsilon}^{+\infty}\frac{U(x)}{x-\tau}dx\right). \tag{74}$$

In particular, taking $q=1$ in (5) of the document [12], the moments of $\widetilde{u}$ are given by

$$(\widetilde{u})_0 = 1 \quad ; \quad (\widetilde{u})_n = \tau^n + \lambda\sum_{k=1}^{n}\tau^{n-k}(u)_{k-1},\ n\geq 1. \tag{75}$$

EXAMPLE 3.1. Let us consider u the D-semiclassical form of class 1 studied in Example 2.1. and $\widetilde{u} = \delta_{-1} + \lambda(x+1)^{-1}u$ be its standard perturbed ($\tau=-1$). Taking $x=-1$ in (63), by virtue of (41) and the fact that $P_n(-1)\neq 0,\ n\geq 0$ (see (56)) we get

$$P_{n+1}^{(1)}(-1) = \frac{P_{n+2}(-1)}{P_{n+1}(-1)}\ P_n^{(1)}(-1) + \frac{(-1)^{n+1}}{2^{2n+2}P_{n+1}(-1)}\ ,\ n\geq 0$$

which gives

$$P_{n+1}^{(1)}(-1) = \prod_{k=0}^{n}\left(\frac{P_{k+2}(-1)}{P_{k+1}(-1)}\right)\left\{1+\sum_{k=0}^{n}\frac{(-1)^{k+1}}{2^{2k+2}P_{k+1}(-1)}\prod_{l=0}^{k}\left(\frac{P_{l+1}(-1)}{P_{l+2}(-1)}\right)\right\},\ n\geq 0.$$

Consequently, in accordance with (56) it follows

$$P^{(1)}_{2n+1}(-1) = 0 \quad ; \quad P^{(1)}_{2n}(-1) = \frac{1}{2^{2n}}\,,\ n \geq 0. \tag{76}$$

Thus, (62) and (64) can be written as

$$a_{2n} = 2n+2-\lambda \quad ; \quad a_{2n+1} = \frac{1}{4(2n+2-\lambda)}\,,\ n \geq 0 \tag{77}$$

$$\lambda_{2n+1} = 2(n+1)\,,\ n \geq 0. \tag{78}$$

By virtue of Proposition 3.1., The form $\widetilde{u} = \delta_{-1} + \lambda(x+1)^{-1}u$ is regular if and only if $\lambda \neq 2(n+1),\ n \geq -1$. In particular, from (41) and (77), the relations in (66) become

$$\begin{cases} \widetilde{\beta}_0 = \lambda - 1 \\ \widetilde{\beta}_{2n+1} = 2n+1-\lambda - \frac{1}{4(2n+2-\lambda)} \\ \widetilde{\beta}_{2n+2} = \frac{1}{4(2n+2-\lambda)} - 2n - 3 + \lambda \\ \widetilde{\gamma}_{2n+1} = -(2n+2-\lambda)(2n-\lambda)\ ;\ \widetilde{\gamma}_{2n+2} = -\frac{1}{16(2n+2-\lambda)^2} \end{cases} ,\ n \geq 0. \tag{79}$$

In accordance with Theorem 3.1. $\widetilde{u}$ is D-semiclassical of class 2 for any $\lambda \neq 2(n+1),\ n \geq -1$ satisfying the functional equation

$$\Big(x(x-1)(x+1)^2)\widetilde{u}\Big)'^2 + x + 2)\widetilde{u} = 0. \tag{80}$$

For the moments of $\widetilde{u}$, with (44) and (75) we get

$$(\widetilde{u})_{2n} = 1 \quad ; \quad (\widetilde{u})_{2n+1} = \lambda - 1\,,\ n \geq 0. \tag{81}$$

For an integral representation of $\widetilde{u}$, in view of (47) and (73) and by the fact that $\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = 0$ we may write

$$\langle \widetilde{u}, f \rangle = f(-1) + \lambda \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} f(x)\,dx\,,\ f \in \mathcal{P}. \tag{82}$$

3.3 The study of $\delta_0 + \lambda x^{-1}\mathcal{B}(\alpha)$

Consider $u = \mathcal{B}(\alpha),\ \alpha \neq -\frac{n}{2}, n \geq 0$ the D-classical Bessel form. we have [32]

$$\begin{cases} \beta_0 = -\frac{1}{\alpha},\ \ \beta_{n+1} = \frac{1-\alpha}{(n+\alpha)(n+\alpha+1)}\ \ ,\ \ n \geq 0, \\ \gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}, n \geq 0, \\ \Big(x^2\mathcal{B}(\alpha)\Big)' - 2(\alpha x + 1)\mathcal{B}(\alpha) = 0, \end{cases} \tag{83}$$

and for $\alpha \geq 6\big(\frac{2}{\pi}\big)^4, f \in \mathcal{P}$

$$\langle \mathcal{B}(\alpha), f \rangle = T_\alpha^{-1} \int_0^{+\infty} \frac{1}{x^2} \int_x^{+\infty} \Big(\frac{x}{t}\Big)^{2\alpha} \exp\Big(\frac{2}{t} - \frac{2}{x}\Big) s(t)dtf(x)dx, \tag{84}$$

with

$$T_\alpha = \int_0^{+\infty} \frac{1}{x^2} \int_x^{+\infty} \left(\frac{x}{t}\right)^{2\alpha} \exp\left(\frac{2}{t} - \frac{2}{x}\right) s(t) dt dx, \tag{85}$$

$$s(x) = \begin{cases} 0, & x \leq 0, \\ \\ \exp(-x^{\frac{1}{4}}) \ \sin x^{\frac{1}{4}}, & x > 0. \end{cases} \tag{86}$$

where s represents the null-form and given in 1895 by T. J. Stieljes [42]. The integral representation (84) of the D-classical Bessel form is given by P. Maroni (1995) in [32]. In fact, the history of the Bessel form is more tortured than that other D-classical forms and the reason is certainly that the Bessel form is not positive definite for any value of the parameter α. For others representations through a function, a distribution or through an ultra-distribution see A. J. Duran (1993) [10], A. M. Krall (1981)[19] and W. D. Evans et al. (1992)[11], S. S. Kim et al. (1991) [18] respectively.

Taking into account the functional equation in (83), it is easy to see that the moments of $\mathcal{B}(\alpha)$ are

$$(\mathcal{B}(\alpha))_n = (-1)^n 2^n \frac{\Gamma(2\alpha)}{\Gamma(n+2\alpha)} \ , \ n \geq 0, \tag{87}$$

where Γ is the Gamma function.

Putting $x = 0$ in (26) and by virtue of (83) and (32) we get

$$P_n(0) = 2^n \frac{\Gamma(n+2\alpha-1)}{\Gamma(2n+2\alpha-1)}, \ n \geq 0. \tag{88}$$

Also, taking $x = 0$ in (63) and in accordance with(88) and (83) we obtain the recurrence relation

$$P_{n+1}^{(1)}(0) = \frac{n+2\alpha}{(2n+2\alpha+1)(n+\alpha+1)} P_n^{(1)}(0) + (-1)^{n+1} 2^{n+1} \frac{\Gamma(2\alpha)}{\Gamma(2n+2\alpha+2)}, \ n \geq 0.$$

Therefore, an easy computation yields the equality

$$P_n^{(1)}(0) = 2^{n+1} \frac{\Gamma(2\alpha)\Gamma(n+2\alpha)}{\Gamma(2n+2\alpha+1)} \sum_{k=0}^{n} \frac{(-1)^k k!(k+\alpha)}{\Gamma(k+2\alpha)}, \ n \geq 0.$$

By induction, we can easily prove that

$$\sum_{k=0}^{n} \frac{(-1)^k k!(k+\alpha)}{\Gamma(k+2\alpha)} = \frac{1}{2} \frac{2\alpha-1}{\Gamma(2\alpha)} + \frac{1}{2}(-1)^n \frac{(n+1)!}{\Gamma(n+2\alpha)}, \ n \geq 0,$$

which gives (compare with [31])

$$P_n^{(1)}(0) = 2^n \frac{(2\alpha-1)\Gamma(n+2\alpha) + (-1)^n (n+1)! \Gamma(2\alpha)}{\Gamma(2n+2\alpha+1)}, \ n \geq 0. \tag{89}$$

Thus, for (62) and (64) we get for $n \geq 0$

$$a_n = -2 \frac{\Gamma(2n+2\alpha-1)}{\Gamma(2n+2\alpha+1)}$$

$$\times \frac{2\Gamma(n+2\alpha)+\lambda\big\{(2\alpha-1)\Gamma(n+2\alpha)+(-1)^n(n+1)!\Gamma(2\alpha)\big\}}{2\Gamma(n+2\alpha-1)+\lambda\big\{(2\alpha-1)\Gamma(n+2\alpha-1)+(-1)^{n-1}n!\Gamma(2\alpha)\big\}}, \tag{90}$$

and

$$\lambda_n = -2\frac{\Gamma(n+2\alpha-1)}{(2\alpha-1)\Gamma(n+2\alpha-1)+(-1)^{n-1}n!\Gamma(2\alpha)} \ , \ \ n \geq 1. \tag{91}$$

Consequently, in accordance with theorem 3.1. and (69)-(72) the form $\widetilde{u} = \delta_0 + \lambda x^{-1}\mathcal{B}(\alpha)$ is D-semiclassical of class $\widetilde{s}=1$ for any $\lambda \neq \lambda_n$, $n \geq -1$ with $\lambda_{-1} = \frac{2}{1-2\alpha}$ and fulfils the functional equation (69) with

$$\widetilde{\Phi}(x) = x^3, \qquad \widetilde{\Psi}(x) = -2x(\alpha x+1). \tag{92}$$

Also, the recurrence coefficients of $\{\widetilde{P}_n\}_{n\geq 0}$ are given by (66) with the above results (88) and (83).

From (75) and (87), the moments of $\widetilde{u}$ are

$$(\widetilde{u})_n = \lambda(-1)^{n-1}2^{n-1}\frac{\Gamma(2\alpha)}{\Gamma(n+2\alpha-1)} \ , \ n \geq 1 \ , \ (\widetilde{u})_0 = 1. \tag{93}$$

When $\alpha \geq 6(\frac{2}{\pi})^4$ and on account of (73)-(74), (84)-(86) let us consider the function

$$\widetilde{U}(x) = \frac{1}{x^3}\int_x^{+\infty}\Big(\frac{x}{\xi}\Big)^{2\alpha}\exp\Big(\frac{2}{\xi}-\frac{2}{x}\Big)s(\xi)d\xi \ , \ x>0. \tag{94}$$

We have for $n \geq 0$,

$$\begin{aligned}
\big|x^n\widetilde{U}(x)\big| &\leq x^{n+2\alpha-3}\exp(-\frac{2}{x})\int_x^{+\infty}\xi^{-2\alpha}\exp\bigg(\frac{2}{\xi}\bigg)\exp(-\xi^{\frac{1}{4}})d\xi \\
&\leq x^{n+2\alpha-3}\exp(-\frac{2}{x})\exp(-\frac{1}{2}x^{\frac{1}{4}})\int_x^{+\infty}\xi^{-2\alpha}\exp\bigg(\frac{2}{\xi}\bigg)\exp\bigg(-\frac{1}{2}\ \xi^{\frac{1}{4}}\bigg)d\xi \\
&= o\bigg(x^{n+2\alpha-3}\exp\big(-\frac{1}{2}x^{\frac{1}{4}}\big)\bigg).
\end{aligned}$$

For $0 < x \leq 1$ and $n \geq 0$

$$x^n\widetilde{U}(x) = \Theta_n(x) + O\bigg(x^{n+2\alpha-3}\exp\Big(-\frac{2}{x}\Big)\bigg)$$

with

$$\Theta_n(x) = x^{n+2\alpha-3}\exp\bigg(-\frac{2}{x}\bigg)\int_x^1\xi^{-2\alpha}\exp\bigg(\frac{2}{\xi}\bigg)s(\xi)d\xi.$$

Hence,

$$\int_0^1|\Theta_n(x)|dx \leq \int_0^1\xi^{-2\alpha}\exp\Big(\frac{2}{\xi}\Big)|s(\xi)|\bigg(\int_0^\xi x^{n+2\alpha-3}\exp\Big(-\frac{2}{x}\Big)dx\bigg)d\xi.$$

Moreover, upon integration by parts we have

$$\begin{aligned}
&\int_0^\xi x^{n+2\alpha-3}\exp\bigl(-\frac{2}{x}\bigr)dx \\
= \;& \frac{1}{2}\xi^{n+2\alpha-1}\exp\bigl(-\frac{2}{\xi}\bigr) - \frac{1}{2}(n+2\alpha-1)\int_0^\xi x^{n+2\alpha-2}\exp\bigl(-\frac{2}{x}\bigr)dx \\
\leq \;& \frac{1}{2}\xi^{n+2\alpha-1}\exp\bigl(-\frac{2}{\xi}\bigr) + \frac{1}{2}(n+2\alpha-1)\xi\int_0^\xi x^{n+2\alpha-3}\exp\bigl(-\frac{2}{x}\bigr)dx,
\end{aligned}$$

and hence

$$\int_0^\xi x^{n+2\alpha-3}\exp\bigl(-\frac{2}{x}\bigr)dx \leq \frac{1}{2}\;\frac{\xi^{n+2\alpha-1}\exp\bigl(-\frac{2}{\xi}\bigr)}{1-\frac{1}{2}(n+2\alpha-1)\xi}$$

for all $0 \leq \xi \leq \frac{2}{n+2\alpha-1}$, $n \geq 0$.

It results in

$$\int_0^1 |\Theta_n(x)|dx < +\infty.$$

Consequently, we get the natural integral representation of $\widetilde{u}$: for $\alpha \geq 6(\frac{2}{\pi})^4$ and $f \in \mathcal{P}$

$$\begin{aligned}
\langle \widetilde{u}, f\rangle &= (1-\lambda T_\alpha^{-1}\widetilde{T}_\alpha)f(0) \\
&+\lambda T_\alpha^{-1}\int_0^{+\infty}\frac{1}{x^3}\int_x^{+\infty}\left(\frac{x}{t}\right)^{2\alpha}\exp\left(\frac{2}{t}-\frac{2}{x}\right)s(t)dtf(x)dx, \qquad (95)
\end{aligned}$$

with

$$\widetilde{T}_\alpha = \int_0^{+\infty}\frac{1}{x^3}\int_x^{+\infty}\left(\frac{x}{t}\right)^{2\alpha}\exp\left(\frac{2}{t}-\frac{2}{x}\right)s(t)dtdx. \qquad (96)$$

4 The Symmetrization Process Revisited

4.1 On the symmetrization process

Let $\widehat{u}$ be a symmetric regular form and $\{\widehat{P}_n\}_{n\geq 0}$ be its (MOPS). They satisfy a three term recurrence relation

$$\begin{cases} \widehat{P}_0\,(x) = 1, \;\; \widehat{P}_1\,(x) = x\;, \\ \widehat{P}_{n+2}\,(x) = x\widehat{P}_{n+1}\,(x) - \widehat{\gamma}_{n+1}\widehat{P}_n\,(x)\;\;,\;\; n\geq 0. \end{cases} \qquad (97)$$

It is very well known (see T.S. Chihara [6] and P. Maroni [33]) that

$$\widehat{P}_{2n}(x) = \widetilde{P}_n(x^2), \qquad \widehat{P}_{2n+1}(x) = xP_n(x^2)\;,\; n\geq 0, \qquad (98)$$

where $\{\widetilde{P}_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ are the two (MOPS) related to regular forms $\widetilde{u}$ and $x\widetilde{u}$, respectively, with

$$\widetilde{u} = \sigma\widehat{u}. \qquad (99)$$

The form $\widehat{u}$ is said to be the symmetrized of the form $\widetilde{u}$.

In fact [6,33]

$$\widehat{u} \text{ is regular } \Longleftrightarrow \widetilde{u} \text{ and } x\widetilde{u} \text{ are regular.}$$

$$\widehat{u} \text{ is positive definite } \Longleftrightarrow \widetilde{u} \text{ and } x\widetilde{u} \text{ are positive definite.}$$

Furthermore, taking into account the three term recurrence relations for $\{\widetilde{P}_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ which are (60) and (16) respectively, we get [6,33]

$$\begin{cases} \widetilde{\beta}_0 = \widehat{\gamma}_1, \\ \widetilde{\beta}_{n+1} = \widehat{\gamma}_{2n+2} + \widehat{\gamma}_{2n+3}\ ,\ \ n \geq 0, \\ \widetilde{\gamma}_{n+1} = \widehat{\gamma}_{2n+1}\widehat{\gamma}_{2n+2},\ \ n \geq 0. \end{cases} \tag{100}$$

and

$$\begin{cases} \beta_n = \widehat{\gamma}_{2n+1} + \widehat{\gamma}_{2n+2}\ \ ,\ \ n \geq 0, \\ \gamma_{n+1} = \widehat{\gamma}_{2n+2}\widehat{\gamma}_{2n+3}\ \ ,\ \ n \geq 0. \end{cases} \tag{101}$$

Consequently,

$$\begin{cases} \widehat{\gamma}_1 = \widetilde{\beta}_0, \qquad \widehat{\gamma}_2 = \frac{\widetilde{\gamma}_1}{\widetilde{\beta}_0}, \\ \widehat{\gamma}_{2n+1} = \widetilde{\beta}_0 \frac{\prod_{k=1}^{n} \gamma_k}{\prod_{k=1}^{n} \widetilde{\gamma}_k}, \qquad \widehat{\gamma}_{2n+2} = \frac{1}{\widetilde{\beta}_0} \frac{\prod_{k=1}^{n+1} \widetilde{\gamma}_k}{\prod_{k=1}^{n} \gamma_k},\ n \geq 1. \end{cases} \tag{102}$$

4.2 The D-semiclassical case

Regarding section two in [2], a study of the D-semiclassical character and the class of $\widehat{u}$ is done by J. Arvesú, M. J. Atia and F. Marcellán (in 2002) after determination of the non homogeneous first order linear differential equation satisfied by the formal Stieltjes function of $\widehat{u}$ which derived from that of $\widetilde{u}$. In the following theorem, we are going to resume these results in terms of the polynomials coefficients in the Pearson equation of $\widetilde{u}$.

THEOREM 4.1. Let $\widetilde{u}$ be a regular form and $\{\widetilde{P}_n\}_{n\geq 0}$ its (MOPS) such that $\widetilde{P}_{n+1}(0) \neq 0,\ n \geq 0$. Then its symmetrized form $\widehat{u}$ is regular.

If $\widetilde{u}$ is D-semiclassical of class $\widetilde{s}$ satisfying the functional equation

$$(\widetilde{\Phi}\widetilde{u})' + \widetilde{\Psi}\widetilde{u} = 0 \tag{103}$$

then $\widehat{u}$ is D-semiclassical of class $\widehat{s}$ satisfying the functional equation

$$(\widehat{\Phi}\widehat{u})' + \widehat{\Psi}\widehat{u} = 0 \tag{104}$$

with

i)

$$\widehat{\Phi}(x) = (\theta_0\widetilde{\Phi})(x^2) \quad , \quad \widehat{\Psi}(x) = x\big\{(\theta_0^2\widetilde{\Phi})(x^2) + 2(\theta_0\widetilde{\Psi})(x^2)\big\} \tag{105}$$

if

$$\widetilde{\Phi}(0) = 0 \quad , \quad \widetilde{\Phi}'(0) + 2\widetilde{\Psi}(0) = 0, \tag{106}$$

and $\widehat{s} = 2\widetilde{s}$ for (105).

ii)

$$\widehat{\Phi}(x) = x(\theta_0\widetilde{\Phi})(x^2) \quad , \quad \widehat{\Psi}(x) = 2\widetilde{\Psi}(x^2) \tag{107}$$

if

$$\widetilde{\Phi}(0) = 0 \quad , \quad \widetilde{\Phi}'(0) + 2\widetilde{\Psi}(0) \neq 0, \tag{108}$$

and $\widehat{s} = 2\widetilde{s} + 1$ for (107).

iii)

$$\widehat{\Phi}(x) = x\widetilde{\Phi}(x^2) \quad , \quad \widehat{\Psi}(x) = 2\big\{ -\widetilde{\Phi}(x^2) + x^2\widetilde{\Psi}(x^2)\big\} \tag{109}$$

if

$$\widetilde{\Phi}(0) \neq 0, \tag{110}$$

and $\widehat{s} = 2\widetilde{s} + 3$ for (109).

REMARK 4.1. Some calculation allows to give the structure relation of $\{\widehat{P}_n\}_{n\geq 0}$ by taking into account the results of the components $\{\widetilde{P}_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$ and Theorem 4.1.

Finally, let us suppose that the form $\widetilde{u}$ has the following integral representation:

$$\langle \widetilde{u}, f\rangle = \int_0^{+\infty} \widetilde{U}\,(x)\, f\,(x)\, dx\ ,\ f \in \mathcal{P};\ \int_0^{+\infty} \widetilde{U}\,(x)\, dx = 1 \tag{111}$$

where $\widetilde{U}$ is a locally integrable function with rapid decay. Consider now $f \in \mathcal{P}$ and let us split it up into its even and odd parts

$$f(x) = f^e(x^2) + x f^o(x^2). \tag{112}$$

From the symmetric character of the form $\widehat{u}$, (99) and (112) we get

$$\langle \widehat{u}, f\rangle = \langle \widehat{u}, \sigma f^e\rangle = \langle \sigma\widehat{u}, f^e\rangle = \langle \widetilde{u}, f^e\rangle. \tag{113}$$

In view of (111) and (113), with the fact that

$$f^e(x) = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2} \quad , \quad x \geq 0,$$

and after a change of variables, we obtain the following integral representation of $\widehat{u}$

$$\langle \widehat{u}, f\rangle = \int_{-\infty}^{+\infty} |x|\widetilde{U}(x^2) f(x) dx \quad , \quad f \in \mathcal{P} \tag{114}$$

since

$$\left| \int_0^{+\infty} x^{n+\frac{1}{2}}\, \widetilde{U}(x) dx \right| < +\infty \quad , \quad n \geq 0. \tag{115}$$

In particular, the moments of $\widehat{u}$ are

$$(\widehat{u})_0 = 1 \quad ; \quad (\widehat{u})_{2n+1} = 0 \ \ , \ \ (\widehat{u})_{2n+2} = (\widetilde{u})_{n+1} \quad , \quad n \geq 0. \tag{116}$$

4.3 The symmetrized of $\delta_0+\lambda x^{-1}\mathcal{B}(\alpha)$

Let $\widetilde{u}=\delta_0+\lambda x^{-1}\mathcal{B}(\alpha)$ and its associated (MOPS) $\{\widetilde{P}_n\}_{n\geq 0}$ the D-semiclassical of class $\widetilde{s}=1$ for any $\lambda\neq\lambda_n,\ n\geq -1$ with $\lambda_{-1}=\frac{2}{1-2\alpha}$ and $\lambda_n,\ n\geq 0$are given by (91), taking $x=0$ in the first equality of (67) we get

$$\widetilde{P}_{n+1}(0)=(a_n-\beta_n)\widetilde{P}_n(0),\ n\geq 0.$$

Consequently, $\widetilde{P}_{n+1}(0)\neq 0,\ n\geq 0$, which implies that the form $\widehat{u}$, symmetrized of $\widetilde{u}$, is regular.

On the other hand, from (92) and (106) we have

$$\widetilde{\Phi}(0)=0\ \ ,\ \ \widetilde{\Phi}'(0)+2\widetilde{\Psi}(0)=0.$$

Then with (106) the form $\widehat{u}$ is symmetric D-semiclassical of class 2 satisfying the functional equation (104) with

$$\widehat{\Phi}(x)=x^4\ \ ,\ \ \widehat{\Psi}(x)=-x\big\{(4\alpha-1)x^2+4\big\}. \tag{117}$$

Taking into account (102) and (65)-(66), the recurrence coefficients of $\{\widehat{P}_n\}_{n\geq 0}$ are

$$\widehat{\gamma}_1=\lambda\ ,\ \ \widehat{\gamma}_2=a_0\ ,\ \ \widehat{\gamma}_{2n+1}=\frac{a_0}{a_{n-1}}\gamma_n\ ,\ \ \widehat{\gamma}_{2n+2}=\frac{a_n}{a_0}\ ,\ \ n\geq 1, \tag{118}$$

where $\gamma_n, n\geq 1$ and $a_n, n\geq 0$ are given by (83) and (90) respectively.

The moments of $\widehat{u}$ are

$$(\widehat{u})_{2n+1}=0\ ,\ \ (\widehat{u})_{2n+2}=\lambda(-1)^n2^n\frac{\Gamma(2\alpha)}{\Gamma(n+2\alpha)}\ ,\ \ n\geq 0\ ,\ \ (\widehat{u})_0=1. \tag{119}$$

Regarding the natural integral representation (95) of $\widetilde{u}$, for $\alpha\geq 6\left(\frac{2}{\pi}\right)^4-\frac{1}{4}$ the condition (115) is fulfilled and from (114) we obtain the following representation of $\widehat{u}$, for $f\in\mathcal{P}$

$$\begin{aligned}&\langle\widehat{u},f\rangle=(1-\lambda T_\alpha^{-1}\widetilde{T}_\alpha)f(0)\\ &+\lambda T_\alpha^{-1}\int_{-\infty}^{+\infty}|x|^{4\alpha-5}\exp\big(-\frac{2}{x^2}\big)\int_{x^2}^{+\infty}t^{-2\alpha}\exp\big(\frac{2}{t}\big)s(t)dtf(x)dx.\end{aligned} \tag{120}$$

REMARK 4.2. The representation (120) doesn't exist in [8] which proves that the list of integral representations given in [8] is not complete. Another integral representation analogous to (120) is given in [39].

4.4 Symmetrization after Perturbation when u is a Symmetric D-Semiclassical Form

Let us suppose u to be a symmetric D-semiclassical form of class s satisfying the Pearson equation (21) and $\widetilde{u}=\delta_0+\lambda x^{-1}u$ or equivalently $x\widetilde{u}=\lambda u$. It has been shown in section 3 that $\widetilde{u}$ is D-semiclassical satisfying the equation (69) with (70) and $\widetilde{u}$ is of

class $\widetilde{s} = s + 1$ when (71)-(72) are valid.
If $\widehat{u}$ is the symmetrized of $(\widetilde{u})$ then $\widetilde{u} = \sigma\widehat{u}$ and from (58)-(59) we get

$$\begin{cases} \sigma\widehat{u} = \delta_0 + \lambda x^{-1}u \\ x\sigma\widehat{u} = \lambda u. \end{cases} \tag{121}$$

Thus, $\sigma\widehat{u}$ and $x\sigma\widehat{u}$ are well known.

Now, we are able to give $\widehat{\gamma}_{n+1}$, $n \geq 0$. From (102) and by virtue of (65)-(66) we obtain

$$\widehat{\gamma}_{4n+1} = -a_{2n}\ ,\ \widehat{\gamma}_{4n+2} = a_{2n}\ ,\ \widehat{\gamma}_{4n+3} = -a_{2n+1}\ ,\ \widehat{\gamma}_{4n+4} = a_{2n+1}\ ,\ n \geq 0. \tag{122}$$

From (11), (121) and the fact that u is symmetric, the moments of $\widehat{u}$ are

$$(\widehat{u})_0 = 1\ ;\ (\widehat{u})_{4n+1} = (\widehat{u})_{4n+3} = (\widehat{u})_{4n+4} = 0\ ,\ (\widehat{u})_{4n+2} = \lambda(u)_{2n}\ ,\ n \geq 0. \tag{123}$$

On the other hand, in accordance with (70) we have

$$\widetilde{\Phi}(x) = x\Phi(x) \quad , \quad \widetilde{\Psi}(x) = x\Psi(x).$$

Therefore,

$$\widetilde{\Phi}(0) = 0 \quad , \quad \widetilde{\Phi}'(0) + 2\widetilde{\Psi}(0) = \Phi(0).$$

Consequently, with Theorem 4.1., two cases arise

i)

$$\widehat{\Phi}(x) = \Phi(x^2), \quad \widehat{\Psi}(x) = x\big\{(\theta_0\Phi)(x^2) + 2\Psi(x^2)\big\} \tag{124}$$

if

$$\Phi(0) = 0, \tag{125}$$

and $\widehat{s} = 2\widetilde{s} = 2s + 2$ for (124).

Or

ii)

$$\widehat{\Phi}(x) = x\Phi(x^2), \quad \widehat{\Psi}(x) = 2x^2\Psi(x^2) \tag{126}$$

if

$$\Phi(0) \neq 0, \tag{127}$$

and $\widehat{s} = 2\widetilde{s} + 1 = 2s + 3$ for (126).

EXAMPLE 4.1. Let u be the symmetric D-semiclassical form of class 1 of generalized Hermite $\mathcal{H}(\mu)$ ($\mu \neq 0\ ,\ \mu \neq -n - \frac{1}{2}\ ,\ n \geq 0$), then the symmetrized $\widehat{u}$ of $\delta_0 + \lambda x^{-1}u$ is of class $\widehat{s} = 4$ satisfying

$$\big(x^2\widehat{u}\big)' + x\big\{4x^4 - (4\mu + 1)\big\}\widehat{u} = 0. \tag{128}$$

In this case (122) becomes

$$\begin{cases} \widehat{\gamma}_{4n+1} = \lambda\ \frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(n+\mu+\frac{1}{2})}n! \\ \widehat{\gamma}_{4n+2} = -\lambda\ \frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(n+\mu+\frac{1}{2})}n! \\ \widehat{\gamma}_{4n+3} = -\frac{1}{\lambda}\ \frac{\Gamma(n+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})}\frac{1}{n!} \\ \widehat{\gamma}_{4n+4} = \frac{1}{\lambda}\ \frac{\Gamma(n+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})}\frac{1}{n!} \end{cases} ,\ n \geq 0. \tag{129}$$

The moments are

$$\begin{cases} (\widehat{u})_0 = 1 \\ (\widehat{u})_{4n+1} = (\widehat{u})_{4n+3} = (\widehat{u})_{4n+4} = 0 \\ (\widehat{u})_{4n+2} = \frac{\lambda}{2^{2n}} \ \frac{\Gamma(\mu+1)\Gamma(2n+2\mu+1)}{\Gamma(2\mu+1)\Gamma(n+\mu+1)} \end{cases} , \ n \geq 0. \tag{130}$$

EXAMPLE 4.2. Let u be the symmetric D-semiclassical form of class 1 of generalized Gegenbauer $\mathcal{G}(\alpha,\beta)$ ($\alpha \neq -n-1$, $\beta \neq -n-1$, $\beta \neq -\frac{1}{2}$, $\alpha+\beta \neq -n-1$, $n \geq 0$), then the symmetrized $\widehat{u}$ of $\delta_0 + \lambda x^{-1}u$ is of class $\widehat{s} = 4$ satisfying

$$\big(x^2(x^4-1)\widehat{u}\big)' + x\big\{-(4\alpha+4\beta+7)x^4 + 4\beta + 3\big\}\widehat{u} = 0. \tag{131}$$

In this case (122) becomes

$$\begin{cases} \widehat{\gamma}_{4n+1} = \frac{\lambda n!}{2n+\alpha+\beta+1} \ \frac{\Gamma(n+\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\beta+1)} \\ \widehat{\gamma}_{4n+2} = -\frac{\lambda n!}{2n+\alpha+\beta+1} \ \frac{\Gamma(n+\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\beta+1)} \\ \widehat{\gamma}_{4n+3} = -\frac{1}{\lambda n!(2n+\alpha+\beta+2)} \ \frac{\Gamma(\alpha+1)\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(\beta+1)} \\ \widehat{\gamma}_{4n+4} = \frac{1}{\lambda n!(2n+\alpha+\beta+2)} \ \frac{\Gamma(\alpha+1)\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(\beta+1)} \end{cases} , \ n \geq 0. \tag{132}$$

The moments are

$$\begin{cases} (\widehat{u})_0 = 1 \\ (\widehat{u})_{4n+1} = (\widehat{u})_{4n+3} = (\widehat{u})_{4n+4} = 0 \\ (\widehat{u})_{4n+2} = \lambda\frac{\Gamma(\alpha+\beta+2)\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+\alpha+\beta+2)} \end{cases} , \ n \geq 0. \tag{133}$$

EXAMPLE 4.3. Let u be the symmetric D-semiclassical form of class 1 of Bessel kind $\mathcal{B}[\nu]$ ($\nu \neq -n-1$, $n \geq 0$), then the symmetrized $\widehat{u}$ of $\delta_0 + \lambda x^{-1}u$ is of class $\widehat{s} = 4$ with

$$\big(x^6\widehat{u}\big)' - x\big\{(4\nu+3)x^4 + 1\big\}\widehat{u} = 0. \tag{134}$$

In this case (122) becomes

$$\begin{cases} \widehat{\gamma}_{4n+1} = \lambda\frac{(-1)^n n!}{(2n+\nu)} \ \frac{\Gamma(\nu+1)}{\Gamma(n+\nu)} \\ \widehat{\gamma}_{4n+2} = -\lambda\frac{(-1)^n n!}{(2n+\nu)} \ \frac{\Gamma(\nu+1)}{\Gamma(n+\nu)} \\ \widehat{\gamma}_{4n+3} = \frac{1}{4\lambda}\frac{(-1)^n}{(2n+\nu+1)n!} \ \frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)} \\ \widehat{\gamma}_{4n+4} = -\frac{1}{4\lambda}\frac{(-1)^n}{(2n+\nu+1)n!} \ \frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)} \end{cases} , \ n \geq 0. \tag{135}$$

The moments are

$$\begin{cases} (\widehat{u})_0 = 1 \\ (\widehat{u})_{4n+1} = (\widehat{u})_{4n+3} = (\widehat{u})_{4n+4} = 0 \\ (\widehat{u})_{4n+2} = \lambda\frac{(-1)^n\Gamma(\nu+1)}{2^{2n}\Gamma(n+\nu+1)} \end{cases} , \ n \geq 0. \tag{136}$$

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