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Existence Of Ψ-Bounded Solutions For Nonhomogeneous Linear Difference Equations*

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Abstract

In this paper, we give a necessary and sufficient condition for the existence of Ψ-bounded solutions for the nonhomogeneous linear difference equation $x(n+1) = A(n)x(n) + f(n)$ and a result in connection with the asymptotic behavior of the solutions of this equation.

1 Introduction

The aim of this paper is to give a necessary and sufficient condition so that the nonhomogeneous linear difference equation

$$x(n+1) = A(n)x(n) + f(n) \tag{1}$$

have at least one Ψ-bounded solution for every Ψ-bounded sequence f.

Here, Ψ is a matrix function. The introduction of the matrix function Ψ allows us to obtain a mixed asymptotic behavior of the components of the solutions.

The problem of boundedness of the solutions for the system of ordinary differential equations $x' = A(t)x + f(t)$ was studied by Coppel in [2]. In [3], [4] and [5], the author proposes a novel concept, Ψ-boundedness of solutions (Ψ being a matrix function), which is interesting and useful in some practical cases and presents the existence condition for such solutions. Also, in [1], the author associates this problem with the concept of Ψ-dichotomy on R of the system $x' = A(t)x$.

In [6], the authors extend the concept of Ψ-boundedness to the solutions of difference equation (via Ψ-bounded sequence) and establish a necessary and sufficient condition for existence of Ψ-bounded solutions for the nonhomogeneous linear difference equation (1) in case f is a Ψ-summable sequence on N.

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2 Preliminaries

Let R^d be the Euclidean space. For $x = (x_1, ..., x_d)^T \in R^d$, let $\|x\| = \max\{|x_1|, ..., |x_d|\}$ be the norm of x. For a $d \times d$ real matrix $A = (a_{ij})$, the norm $|A|$ is defined by $|A| = \sup_{\|x\| \leq 1} \|Ax\|$. Let $N = \{1, 2, ...\}$ and $\Psi_i : N \to (0, \infty)$, $i = 1, 2, ..., d$, and let the matrix function $\Psi = diag[\Psi_1, \Psi_2, ..., \Psi_d]$. Then, $\Psi(n)$ is invertible for each $n \in N$.

DEFINITION 2.1. A sequence $\varphi : N \to R^d$ is said to be Ψ-bounded if the sequence $\Psi\varphi$ is bounded (i.e. there exists $M > 0$ such that $\|\Psi(n)\varphi(n)\| \leq M$ for all $n \in N$).

Consider the nonautonomous difference linear equation

$$y(n+1) = A(n)y(n) \tag{2}$$

where the $d \times d$ real matrix $A(n)$ is invertible at $n \in N$. Let Y be the fundamental matrix of (2) with $Y(1) = I_d$ (identity $d \times d$ matrix). It is well-known that $Y(n) = A(n-1)A(n-2)\cdots A(2)A(1)$ for $n \geq 2$, $Y(n+1) = A(n)Y(n)$ for all $n \in N$ and the solution of (2) with the initial condition $y(1) = y_0$ is $y(n) = Y(n)y_0$, $n \in N$.

Let X_1 denote the subspace of R^d consisting of all vectors which are values for $n = 1$ of Ψ-bounded solutions of (2) and let X_2 be an arbitrary fixed subspace of R^d, supplementary to X_1. Let P_1, P_2 denote the corresponding projections of R^d onto X_1, X_2 respectively.

3 Main Result

The main result of this note is the following.

THEOREM 3.1. The equation (1) has at least one Ψ-bounded solution on N for every Ψ-bounded sequence f on N if and only if there is a positive constant K such that, for all $n \in N$,

$$\sum_{k=1}^{n-1} \left|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\right| + \sum_{k=n}^{\infty} \left|\Psi(n)Y(n)P_2Y^{-1}(k+1)\Psi^{-1}(k)\right| \leq K, \tag{3}$$

where we have adopted the convention that empty sums are 0.

PROOF. First, we prove the "only if" part. We define the sets:

$$\begin{aligned} B &= \{x : N \longrightarrow R^d \mid x \text{ is } \Psi\text{-bounded}\} \\ D &= \{x : N \longrightarrow R^d \mid x \in B, x(1) \in X_2, (x(n+1) - A(n)x(n)) \in B\}. \end{aligned}$$

Obviously, B and D are vector spaces over R and the functionals

$$\begin{aligned} x &\longmapsto \|x\|_B = \sup_{n \in N} \|\Psi(n)x(n)\|, \\ x &\longmapsto \|x\|_D = \|x\|_B + \|x(n+1) - A(n)x(n)\|_B \end{aligned}$$

are norms on B and D respectively.

Step 1. It is a simple exercise that $(B, \|\cdot\|_B)$ is a Banach space.

Step 2. $(D, \|\cdot\|_D)$ is a Banach space. Indeed, let $(x_p)_{p\in N}$ be a fundamental sequence in D. Then, $(x_p)_{p\in N}$ is a fundamental sequence in B. Therefore, there exists a Ψ-bounded sequence x in B such that $\|x_p - x\|_B \longrightarrow 0$ as $p \longrightarrow \infty$. From $\|x_p(1) - x(1)\| \le \left|\Psi^{-1}(1)\right| \|\Psi(1)(x_p(1) - x(1))\| \le \left|\Psi^{-1}(1)\right| \|x_p - x\|_B$, it follows that $\lim_{p\to\infty} x_p(1) = x(1)$. Thus, $x(1) \in X_2$. On the other hand, the sequence $((x_p(n+1) - A(n)x_p(n)))_{p\in N}$ is a fundamental sequence in B. Thus, there exists a function $f \in B$ such that

$$\sup_{n\ge 1} \|\Psi(n)(x_p(n+1) - A(n)x_p(n)) - \Psi(n)f(n)\| \longrightarrow 0 \text{ as } p \longrightarrow \infty.$$

It follows that $\lim_{p\to\infty}(x_p(n+1) - A(n)x_p(n)) = f(n)$, for $n \in N$. Because $\lim_{p\to\infty} x_p(n) = x(n)$ for all $n \in N$, we have that $x(n+1) - A(n)x(n) = f(n)$, for all $n \in N$. Thus,

$$\sup_{n\ge 1} \|\Psi(n)(x_p(n+1) - A(n)x_p(n)) - \Psi(n)(x(n+1) - A(n)x(n))\| \longrightarrow 0$$

and then

$$\|x_p - x\|_D = \|x_p - x\|_B + \|(x_p - x)(n+1) - A(n)(x_p - x)(n)\|_B \longrightarrow 0.$$

Thus, $(D, \|\cdot\|_D)$ is a Banach space.

Step 3. There exists a positive constant K_0 such that, for every $f \in B$ and for corresponding solution $x \in D$ of (1), we have

$$\sup_{n\ge 1} \|\Psi(n)x(n)\| \le K_0 \cdot \sup_{n\ge 1} \|\Psi(n)f(n)\| . \tag{4}$$

Indeed, we define the operator $T : D \longrightarrow B$ by

$$(Tx)(n) = x(n+1) - A(n)x(n),\ n \in N.$$

Clearly, T is linear and bounded, with $\|T\| \le 1$. Let $Tx = 0$. Then, $x(n+1) = A(n)x(n)$, and $x \in D$. This shows that x is a Ψ-bounded solution of (2) with $x(1) \in X_2$. From the definition of X_1, we have $x(1) \in X_1$. Thus, $x(1) \in X_1 \cap X_2 = \{0\}$. It follows that $x = 0$. This means that the operator T is one-to-one. Now, for $f \in B$, let x be the Ψ-bounded solution of the equation (1). Let z be the solution of the Cauchy problem $z(n+1) = A(n)z(n) + f(n), z(1) = P_2x(1)$. Then, the sequence $(x(n) - z(n))$ is a solution of the equation (2) with $P_2(x(1) - z(1)) = 0$, i.e. $x(1) - z(1) \in X_1$. It follows that $(x(n)-z(n))$ is Ψ-bounded on N. Thus, $(z(n))$ is Ψ-bounded on N. It follows that $(z(n)) \in D$ and $Tz = f$. Consequently, the operator T is onto. From a fundamental result of Banach (If T is a bounded one-to-one linear operator from a Banach space onto another, then the inverse operator T^{-1} is also bounded), we conclude that our claim is true (K_0 being $\left\|T^{-1}\right\| - 1$).

Step 4. Let $n_0 \in N$, $n_0 > 1$, a fixed but arbitrary number. Let f be a function which vanishes for $n > n_0$. Then, the sequence $(x(n))_{n\in N}$ with

$$x(n) = \begin{cases} -\sum_{k=1}^{n_0} P_2Y^{-1}(k+1)f(k), & n = 1 \\ \sum_{k=1}^{n-1} Y(n)P_1Y^{-1}(k+1)f(k) - \sum_{k=n}^{\infty} Y(n)P_2Y^{-1}(k+1)f(k), & n > 1 \end{cases}$$

is the solution in D of the equation (1). In fact, since

$$\begin{aligned}
x(2) &= Y(2)P_1Y^{-1}(2)f(1) - \sum_{k=2}^{n_0} Y(2)P_2Y^{-1}(k+1)f(k) \\
&= Y(2)P_1Y^{-1}(2)f(1) - \sum_{k=1}^{n_0} A(1)Y(1)P_2Y^{-1}(k+1)f(k) + Y(2)P_2Y^{-1}(2)f(1) \\
&= A(1)x(1) + Y(2)(P_1+P_2)Y^{-1}(2)f(1) = A(1)x(1) + f(1)
\end{aligned}$$

and, for $n > 1$,

$$\begin{aligned}
x(n+1) &= \sum_{k=1}^{n} Y(n+1)P_1Y^{-1}(k+1)f(k) - \sum_{k=n+1}^{\infty} Y(n+1)P_2Y^{-1}(k+1)f(k) \\
&= A(n)[\sum_{k=1}^{n} Y(n)P_1Y^{-1}(k+1)f(k) - \sum_{k=n+1}^{\infty} Y(n)P_2Y^{-1}(k+1)f(k)] \\
&= A(n)[\sum_{k=1}^{n-1} Y(n)P_1Y^{-1}(k+1)f(k) - \sum_{k=n}^{\infty} Y(n)P_2Y^{-1}(k+1)f(k)] \\
&\quad + A(n)Y(n)(P_1+P_2)Y^{-1}(n+1)f(n) \\
&= A(n)x(n) + f(n),
\end{aligned}$$

we deduce that x is a solution of the equation (1). From $f \in B$, it follows that the sequence $(x(n+1) - A(n)x(n)) \in B$. In addition, $x(1) = -\sum_{k=1}^{n_0} P_2Y^{-1}(k+1)f(k) \in X_2$. Finally, we have $x(n) = \sum_{k=1}^{n-1} Y(n)P_1Y^{-1}(k+1)f(k) = Y(n)P_1u$ for $n > n_0$, where $u = \sum_{k=1}^{n_0} Y^{-1}(k+1)f(k)$. By the definition of X_1, the solution $y(n) = Y(n)P_1u$ of (2) is Ψ-bounded on N. Because $x(n) = y(n)$ for $n > n_0$, it follows that x is Ψ-bounded on N. Thus, x is the solution in D of the equation (1).

Putting

$$G(n,k) = \begin{cases} Y(n)P_1Y^{-1}(k), & \text{for } 1 \le k \le n \\ -Y(n)P_2Y^{-1}(k), & \text{for } 1 \le n < k \end{cases},$$

it is easy to see that $x(n) = \sum_{k=1}^{n_0} G(n,k+1)f(k)$, for all $n \in N$. Thus, the inequality (4) becomes

$$\sup_{n\ge 1} \left\| \sum_{k=1}^{n_0} \Psi(n)G(n,k+1)\Psi^{-1}(k)(\Psi(k)f(k)) \right\| \le K_0 \max_{1\le n\le n_0} \|\Psi(n)f(n)\| .$$

Putting $\Psi(n)G(n,k+1)\Psi^{-1}(k) = (G_{ij}(n,k))$, the above inequality becomes

$$\left| \sum_{k=1}^{n_0} \sum_{j=1}^{d} G_{ij}(n,k)\Psi_j(k)f_j(k) \right| \le K_0 \max_{1\le n\le n_0} \max_{1\le i\le d} |\Psi_i(n)f_i(n)| ,$$

for $i = 1, ...d, n \in N$ and for every $f = (f_1, ...f_d) : N \longrightarrow R^d$ which vanishes for $n > n_0$.

For a fixed i and n, we consider the functions f_j, $j = 1, 2, ...d$, such that

$$f_j(k) = \begin{cases} \Psi_j^{-1}(k) sgn G_{ij}(n,k), & \text{for } 1 \le k \le n_0 \\ 0, & \text{for } k > n_0 \end{cases}.$$

The above inequality becomes $\sum_{k=1}^{n_0} \sum_{j=1}^{d} |G_{ij}(n,k)| \le K_0$, for $i = 1, 2, ...d$ and $n \in N$. Thus,

$$\begin{aligned} \sum_{k=1}^{n_0} \left|\Psi(n) G(n,k+1) \Psi^{-1}(k)\right| &= \sum_{k=1}^{n_0} \max_{1\le i\le d} \sum_{j=1}^{d} |G_{ij}(n,k)| \le \sum_{k=1}^{n_0} \sum_{i=1}^{d} \sum_{j=1}^{d} |G_{ij}(n,k)| \\ &= \sum_{i=1}^{d} \sum_{k=1}^{n_0} \sum_{j=1}^{d} |G_{ij}(n,k)| \le K_0 d = K. \end{aligned}$$

It follows that

$$\sum_{k=1}^{n-1} \left|\Psi(n) Y(n) P_1 Y^{-1}(k+1) \Psi^{-1}(k)\right| + \sum_{k=n}^{n_0} \left|\Psi(n) Y(n) P_2 Y^{-1}(k+1) \Psi^{-1}(k)\right| \le K,$$

for all $n_0 \in N$ and $n \in N$.

Thereafter, the inequality (3) holds for all $n \in N$.

Now, we prove the "if" part. For a Ψ-bounded sequence f on N, we consider the sequence $(x(n))_{n\in N}$ with

$$x(n) = \begin{cases} -\sum_{k=1}^{\infty} P_2 Y^{-1}(k+1) f(k), & \text{for } n = 1 \\ \sum_{k=1}^{n-1} Y(n) P_1 Y^{-1}(k+1) f(k) - \sum_{k=n}^{\infty} Y(n) P_2 Y^{-1}(k+1) f(k), & \text{for } n > 1 \end{cases}.$$

For $m \ge n \ge 1$, we have

$$\begin{aligned} & \sum_{k=n}^{m} \left\|Y(n) P_2 Y^{-1}(k+1) f(k)\right\| \\ = & \sum_{k=n}^{m} \left\|\Psi^{-1}(n) (\Psi(n) Y(n) P_2 Y^{-1}(k+1) \Psi^{-1}(k)) (\Psi(k) f(k))\right\| \\ \le & \left|\Psi^{-1}(n)\right| \sum_{k=n}^{m} \left|\Psi(n) Y(n) P_2 Y^{-1}(k+1) \Psi^{-1}(k)\right| \|\Psi(k) f(k)\| \\ \le & \left|\Psi^{-1}(n)\right| (\sup_{k\ge1} \|\Psi(k) f(k)\|) \sum_{k=n}^{m} \left|\Psi(n) Y(n) P_2 Y^{-1}(k+1) \Psi^{-1}(k)\right|. \end{aligned}$$

It follows that $\sum_{k=n}^{\infty} Y(n) P_2 Y^{-1}(k+1) f(k)$ is an absolutely convergent series. Thus, the sequence $(x(n))_{n\in N}$ is well-defined.

As in the Step 4, we can show that the sequence $(x(n))_{n\in N}$ is a solution of the equation (1). On the other hand,

$$\begin{aligned}
& \|\Psi(n)x(n)\| \\
= & \left\| \sum_{k=1}^{n-1} \Psi(n)Y(n)P_1Y^{-1}(k+1)f(k) - \sum_{k=n}^{\infty} \Psi(n)Y(n)P_2Y^{-1}(k+1)f(k) \right\| \\
\leq & \left(\sum_{k=1}^{n-1} \left|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\right| + \sum_{k=n}^{\infty} \left|\Psi(n)Y(n)P_2Y^{-1}(k+1)\Psi^{-1}(k)\right| \right) \\
& \cdot \left(\sup_{k\geq 1} \|\Psi(k)f(k)\| \right) \\
\leq & K \cdot \sup_{k\geq 1} \|\Psi(k)f(k)\| .
\end{aligned}$$

Thus, the sequence $(x(n))_{n\in N}$ is Ψ-bounded on N.

Therefore, the sequence $(x(n))_{n\in N}$ is a Ψ-bounded solution on N of the equation (1).

The proof is now complete.

Finally, we give a result in which we will see that the asymptotic behavior of solutions of (1) is determined completely by the asymptotic behavior of f.

THEOREM 3.2. Suppose that

$1°$. The fundamental matrix Y of (2) satisfies the inequality (3) for all $n \geq 1$, where K is a positive constant;

$2°$. The matrix Ψ satisfies the condition $\left|\Psi(n)\Psi^{-1}(n+1)\right| \leq T$ for all $n \in N$, where T is a positive constant;

$3°$. The (Ψ-bounded) function $f : N \longrightarrow R^d$ is such that $\lim_{n\to\infty} \|\Psi(n)f(n)\| = 0$. Then, every Ψ-bounded solution x of (1) is such that $\lim_{n\to\infty} \|\Psi(n)x(n)\| = 0$.

PROOF. Let x be a Ψ-bounded solution of (1). We consider the sequence $(y(n))_{n\in N}$, where $y(n)$ is equal to

$$P_2x(1) + \sum_{k=1}^{\infty} P_2Y^{-1}(k+1)f(k),$$

for $n = 1$, and to

$$x(n) - Y(n)P_1x(1) - \sum_{k=1}^{n-1} Y(n)P_1Y^{-1}(k+1)f(k) + \sum_{k=n}^{\infty} Y(n)P_2Y^{-1}(k+1)f(k),$$

for $n > 1$. As in the proof of the above theorem, the sequence $(y(n))_{n\in N}$ is well-defined and is a solution of the equation (2).

On the other hand,

$$\begin{aligned}\|\Psi(n)y(n)\| &\leq \|\Psi(n)x(n)\| + |\Psi(n)Y(n)P_1| \, \|x(1)\| \\ &\quad + \sum_{k=1}^{n-1} \left|\Psi(n)Y(n)P_1 Y^{-1}(k+1)\Psi^{-1}(k)\right| \|\Psi(k)f(k)\| \\ &\quad + \sum_{k=n}^{\infty} \left|\Psi(n)Y(n)P_2 Y^{-1}(k+1)\Psi^{-1}(k)\right| \|\Psi(k)f(k)\| \\ &\leq \sup_{n\geq 1}\|\Psi(n)x(n)\| + |\Psi(n)Y(n)P_1| \, \|x(1)\| + K \cdot \sup_{n\geq 1}\|\Psi(n)f(n)\| .\end{aligned}$$

From the hypotheses, we have that

$$\sum_{k=1}^{n-1} \left|\Psi(n)Y(n)P_1 Y^{-1}(k+1)\Psi^{-1}(k)\right| \leq K, \ n \geq 2.$$

Let $a(n) = |\Psi(n)Y(n)P_1|^{-1}$ for $n \geq 1$. From the identity

$$\begin{aligned}&\left[\sum_{k=1}^{n-1} a(k+1)\right] \Psi(n)Y(n)P_1 \\ =& \sum_{k=1}^{n-1} (\Psi(n)Y(n)P_1 Y^{-1}(k+1)\Psi^{-1}(k))(\Psi(k)\Psi^{-1}(k+1)) \\ &\cdot(\Psi(k+1)Y(k+1)P_1)a(k+1),\end{aligned}$$

it follows that, for $n \geq 2$,

$$\begin{aligned}&|\Psi(n)Y(n)P_1| \left[\sum_{k=1}^{n-1} a(k+1)\right] \\ \leq& \sum_{k=1}^{n-1} \left|\Psi(n)Y(n)P_1 Y^{-1}(k+1)\Psi^{-1}(k)\right| \left|\Psi(k)\Psi^{-1}(k+1)\right| \\ &\cdot |\Psi(k+1)Y(k+1)P_1| \, a(k+1) \\ \leq& \ TK.\end{aligned}$$

Thus,

$$\frac{1}{a(n)} = |\Psi(n)Y(n)P_1| \leq \frac{TK}{\sum_{k=1}^{n-1} a(k+1)} \leq \frac{TK}{a(2)}, \text{ or } a(n) \geq \frac{a(2)}{TK}.$$

Therefore, $\sum_{k=1}^{\infty} a(k) = +\infty$ and then, $\lim_{n\to\infty} |\Psi(n)Y(n)P_1| = 0$.

Thus, we come to the conclusion that the sequence $(y(n))_{n\in N}$ is a Ψ-bounded solution of (2).

Now, by the definition of $X_1, y(1) \in X_1$. Since $y(1) = P_2 x(1) + \sum_{k=1}^{\infty} P_2 Y^{-1}(k+1)f(k) \in X_2$, we have $y(1) \in X_1 \cap X_2 = \{0\}$. Thus, $y = 0$. It follows that

$$x(n) = Y(n)P_1 x(1) + \sum_{k=1}^{n-1} Y(n)P_1 Y^{-1}(k+1)f(k) - \sum_{k=n}^{\infty} Y(n)P_2 Y^{-1}(k+1)f(k), \ n \geq 2.$$

Now, for a given $\varepsilon > 0$, there exists $n_1 \in N$ such that $\|\Psi(n)f(n)\| < \frac{\varepsilon}{2K}$, for $n \geq n_1$. Moreover, there exists $n_2 \in N$, $n_2 > n_1$, such that, for $n > n_2$,

$$|\Psi(n)Y(n)P_1| < \frac{\varepsilon}{2}\left[1 + \|x(1)\| + \sum_{k=1}^{n_1-1}\left\|Y^{-1}(k+1)f(k)\right\|\right]^{-1}.$$

Then, for $n > n_2$, we have

$$\begin{aligned}
\|\Psi(n)x(n)\| &\leq \|\Psi(n)Y(n)P_1x(1)\| \\
&\quad + \sum_{k=1}^{n_1-1}|\Psi(n)Y(n)P_1|\left\|Y^{-1}(k+1)f(k)\right\| \\
&\quad + \sum_{k=n_1}^{n-1}\left|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\right|\|\Psi(k)x(k)\| \\
&\quad + \sum_{k=n}^{\infty}\left|\Psi(n)Y(n)P_2Y^{-1}(k+1)\Psi^{-1}(k)\right|\|\Psi(k)x(k)\| \\
&\leq |\Psi(n)Y(n)P_1|\left[\|x(1)\| + \sum_{k=1}^{n_1-1}\left\|Y^{-1}(k+1)f(k)\right\|\right] \\
&\quad + \sum_{k=n_1}^{n-1}\left|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\right|\frac{\varepsilon}{2K} \\
&\quad + \sum_{k=n}^{\infty}\left|\Psi(n)Y(n)P_2Y^{-1}(k+1)\Psi^{-1}(k)\right|\frac{\varepsilon}{2K} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2K}\cdot K = \varepsilon.
\end{aligned}$$

This shows that $\lim_{n\to\infty}\|\Psi(n)x(n)\| = 0$.

The proof is now complete.

REMARK 3.1. If we do not have $\lim_{n\to\infty}\|\Psi(n)f(n)\| = 0$, then the solution x may be such that $\Psi(n)x(n) \nrightarrow 0$ as $n \to \infty$. For example, consider the equation (1) with

$$A(n) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \text{ and } f(n) = \begin{pmatrix} 2^n \\ 5^{-n} \end{pmatrix}.$$

A fundamental matrix for the equation (2) is

$$Y(n) = \begin{pmatrix} 1 & 0 \\ 0 & 4^{1-n} \end{pmatrix},\ n \in N.$$

Consider $\Psi(n) = \begin{pmatrix} 2^{-n} & 0 \\ 0 & 3^n \end{pmatrix}$, $n \geq 1$. The first hypothesis of the Theorem 3.2 is satisfied with

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } K = 17.$$

The second hypothesis of the Theorem 3.2 is satisfied with $T = 2$. In addition, $\|\Psi(n)f(n)\| = 1$, $n \in N$ (i.e. the function f is Ψ-bounded on N).

In the end, it is easy to see that

$$x(n) = \begin{pmatrix} 2^n \\ 4^{2-n} - 4 \cdot 5^{1-n} \end{pmatrix}, \ n \in N,$$

is a Ψ-bounded solution of (1) with

$$\Psi(n)x(n) = \begin{pmatrix} 1 \\ 16\left(\frac{3}{4}\right)^n - 20\left(\frac{3}{5}\right)^n \end{pmatrix} \nrightarrow 0 \text{ as } n \to \infty.$$

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