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New System Of General Nonconvex Variational Inequalities*

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Abstract

In this paper, we introduce and consider a new system of general nonconvex variational inequalities involving three different operators. Using the projection operator technique, we establish the equivalence between the system of general nonconvex variational inequalities and the fixed points problem. This alternative equivalent formulation is used to suggest and analyze some new explicit iterative methods for this system of nonconvex variational inequalities. We also study the convergence analysis of the new iterative method under certain mild conditions. Since this new system includes the system of nonconvex variational inequalities, variational inequalities and related optimization problems as special cases, results obtained in this paper continue to hold for these problems. Our results can be viewed as a refinement and improvement of the previously known results for variational inequalities.

1 Introduction

Variational inequalities, which was introduced and studied by Stampacchia [1] in the early sixties, can be considered as a natural and significant extension of the variational principles, the origin of which can be traced back to Fermat, Euler, Leibniz, Newton, Lagrange and Bernoulli brothers. The techniques and ideas of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. These activities have motivated to generalize and extend the variational inequalities and related optimization problems in several directions using new and novel techniques. In recent years, much attention has been given to study the system of variational inequalities involving different operators in Hilbert spaces. Using the projection technique, one may usually establish the equivalence between the system of variational inequalities and the fixed point problems. This alternative equivalent formulation has been used to suggest and analyze some iterative methods for solving the system of variational inequalities, see [2,3,4,5,6,7] and the references therein.

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We would like to emphasize that all the results regarding the iterative methods for solving the system of variational inequalities have been considered in the convexity setting. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. Noor [4,8,9,10,11] has introduced and studied a new class of variational inequalities, which is called the nonconvex variational inequality in conjunction with the uniformly prox-regular sets, which are nonconvex sets. Noor [4,8,9,10,11] has shown that the projection technique can be extended for the nonconvex variational inequalities.

Inspired and motivated by the ongoing research in this area, we introduce and consider a system of nonconvex variational inequalities involving three different operators. This class of system includes the system of nonconvex variational inequalities, considered by Moudafi [12] and the classical variational inequalities as special cases. Using essentially the technique of Noor [4,8,9,10,11] in conjunction with projection operator method, we establish the equivalence between the system of general nonconvex variational inequalities and fixed-point problems, which is lemma 3.1. This result can be viewed as the extension of a result of Noor [4,8,9,10,11]. We use this alternative equivalent formulation to suggest and analyze some iterative methods (Algorithm 31-Algorithm 3.4) for solving the system of general nonconvex variational inequalities. We also prove the convergence of the proposed iterative methods under suitable conditions, which is the main motivation of Theorem 4.1. Since the new system of general nonconvex variational inequalities includes the system of nonconvex variational inequalities, studied by Moudafi [12] and Noor [4] and related optimization problems as special cases, results proved in this paper continue to hold for these problems. Our result can be viewed as refinement and improvement of the previous results in this field.

2 Formulation and Basic Results

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively Let K be a nonempty closed and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [13,14].

DEFINITION 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

LEMMA 2.1. Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P$, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K.$$

Poliquin et al. [14] and Clarke et al. [13] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

DEFINITION 2.2. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N^P_{K_r}$, one has

$$\langle (\xi)/\|\xi\|, v-u\rangle \leq (1/2r)\|v-u\|^2, \quad \forall v \in K_r.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [13,14]. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N^P_{K_r}$ is closed as a set-valued mapping.

We now recall the well known proposition which summarizes some important properties of the uniform prox-regular sets.

LEMMA 2.2. Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K_r is uniformly prox-regular, then
i. $\forall u \in K_r, P_{K_r} \neq \emptyset$,
ii. $\forall r^{'} \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $K_{r'}$,
iii. The proximal normal cone is closed as a set-valued mapping.

For given nonlinear operators T_1, T_2, g, we consider the problem of finding $x^*, y^* \in K_r$ such that

$$\langle \rho T_1(y^*) + x^* - g(y^*), g(x) - x^*\rangle \geq 0, \quad \forall x \in H : g(x) \in K_r, \quad \rho > 0, \quad (1)$$
$$\langle \eta T_2(x^*) + y^* - g(x^*), g(x) - y^*\rangle \geq 0, \quad \forall x \in H : g(x) \in K_r, \quad \eta > 0, \quad (2)$$

which is called the system of general nonconvex variational inequalities (SGNVID). For the application of the system of variational inequalities in the setting of convexity, see [4,7].

We now discuss some special cases of the new system of general nonconvex variational inequalities.

I. If $T_1 = T_2 = T$, then the system of general nonconvex variational inequalities $SGVID$ is equivalent to finding $x^*, y^* \in K_r$ such that

$$\langle \rho T y^* + x^* - g(y^*), g(x) - x^*\rangle \geq 0, \quad \forall x \in H : g(x) \in K_r \quad (3)$$
$$\langle \eta T x^* + y^* - g(x^*), g(x) - x^*\rangle \geq 0, \quad \forall x \in H : g(x) K_r. \quad (4)$$

This system of general nonconvex variational inequalities (GSNVI) has been studied by Noor [4]. For $g = I$, the identity operator, the system of nonconvex variational inequalities(SNVI) has been considered by Moudafi [12].

II. If $\rho = 0, \quad x^* = y^*$, then system of general nonconvex variational inequalities (GSNVI) reduces to finding $x^* \in K_r$ such that

$$\langle Tx^*, g(x) - x^* \rangle \geq 0, \quad \forall x \in H : g(x) \in K_r, \tag{5}$$

which is known as the general nonconvex variational inequality (GNVI), introduced and studied by Noor [9,9] in recent years.

III. If $K_r \equiv K$, a convex set in H, then problem (GNVI) is equivalent to finding $x^* \in K$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in K, \tag{6}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [1] in 1964.

This shows that the system of mixed variational inequalities ($SNVID$) is more general and include several classes of variational inequalities and related optimization problems as special cases. For the recent applications, numerical methods and formulations of variational inequalities, see [1-26] and the references therein.

3 Iterative Algorithms

In this Section, we suggest some explicit iterative algorithms for solving the system of general nonconvex variational inequalities (SNVID). First of all, we establish the equivalence between the system of nonconvex variational inequalities and fixed point problems, which is the main motivation of our next result.

LEMMA 3.1. $x, y \in K_r$ is a solution of (1) and (2), if and only if, $x, y \in K_r$ satisfies the relation

$$\begin{aligned} x &= P_{K_r}[g(y) - \rho T_1(y)] && (7) \\ y &= P_{K_r}[g(x) - \eta T_2(x], && (8) \end{aligned}$$

where $\rho > 0$ and $\eta > 0$ are constants.

PROOF. Let $x, y \in K_r$ be a solution of (1) and (2). Then, we have

$$\begin{aligned} 0 &\in \rho T_1(y) + x - g(y) + N^P_{K_r}(x) = (I + N^P_{K_r})(x) - (g(y) - \rho T_1(y) \\ 0 &\in \eta T_2(x) + y - g(x) + N^P_{K_r}(y) = (I + N^P_{K_r}(y) - (g(x) - \eta T_2(x)), \end{aligned}$$

which implies that

$$\begin{aligned} x &= P_{K_r}[g(y) - \rho T_1(y)] \\ y &= P_{K_r}[g(x) - \eta T_2(x)], \end{aligned}$$

and conversely, where we have used the fact that $P_{K_r} = (I + N^P_{K_r})^{-1}$. The proof is complete.

Lemma 3.1 implies that the system of nonconvex variational inequalities (SNVID) is equivalent to the fixed point problem. This alternative equivalent formulation is used to suggest and analyze a number of iterative methods for solving system of nonconvex variational inequalities and related optimization problems.

Using Lemma 3.1, we can easily show that finding the solution $x^*, y^* \in K_r$ of SNVID is equivalent to finding $(x^*, y^*) \in K_r$ such that

$$x^* = (1 - a_n) + a_n P_{K_r}[g(y^*) - \rho T_1(y^*)], \tag{9}$$
$$y^* = P_{K_r}[g(x^*) - \eta T_2(x^*)], \tag{10}$$

where $a_n \in [0, 1]$, for all $n \geq 0$.

We use this alternative equivalent formulation to suggest the following explicit iterative method for solving the system of nonconvex variational inequalities(SNVID).

ALGORITHM 3.1. For arbitrarily chosen initial points $x_0, y_0 \in K_r$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = (1 - a_n)x_n + a_n P_{K_r}[g(y_n) - \rho T_1(y_n)] \tag{11}$$
$$y_{n+1} = P_{K_r}[g(x_{n+1}) - \eta T_2(x_{n+1})], \tag{12}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

If $T_1 = T_2 = T$, then Algorithm 3.1 reduces to the following.

ALGORITHM 3.2. For arbitrarily chosen initial points $x_0, y_0 \in K_r$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n P_{K_r}[g(y_n) - \rho T(y_n)], \\ y_{n+1} &= P_{K_r}[g(x_{n+1}) - \eta T(x_{n+1})], \end{aligned}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

If $T_1 = T_2 = T$ and $g = I$, the identity operator, then Algorithm 3.1 reduces to the following.

ALGORITHM 3.3. For arbitrarily chosen initial points $x_0, y_0 \in K_r$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n P_{K_r}[y_n - \rho T(y_n)], \\ y_{n+1} &= P_{K_r}[x_{n+1} - \eta T(x_{n+1})], \end{aligned}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

We would like to emphasize that one can obtain a number of iterative methods for solving system of (nonconvex) variational inequalities and related optimization problems for appropriate choice of the operators and spaces. This shows that the Algorithm 3.1 is quite flexible and general.

DEFINITION 3.1. A mapping $T : H \to H$ is called r-strongly monotone, if there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq r||x - y||^2, \quad \forall x, y \in H.$$

DEFINITION 3.2. A mapping $T : H \to H$ is called relaxed γ-cocoercive, if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq -\gamma||Tx - Ty||^2, \quad \forall x, y \in H.$$

DEFINITION 3.3. A mapping $T : H \to H$ is called relaxed (γ, r)-cocoercive, if there exist constants $\gamma > 0, r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq -\gamma||Tx - Ty||^2 + r||x - y||^2 \quad \forall x, y \in H.$$

The class of relaxed (γ, r)-cocoercive mappings is more general than the class of strongly monotone mappings.

DEFINITION 3.4. A mapping $T : H \to H$ is called μ-Lipschitzian, if there exists a constant $\mu > 0$, such that

$$||Tx - Ty|| \leq \mu||x - y||, \quad \forall x, y \in H.$$

LEMMA 3.2 [27]. Suppose $\{\delta_n\}_{n=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n, \quad \forall \quad n \geq 0,$$

with $\lambda_n \in [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \delta_n = 0$.

4 Main Results

In this Section, we consider the convergence criteria of Algorithm 3.1 under some suitable mild conditions and this is the main motivation as well as main result of this paper.

THEOREM 4.1. Let (x^*, y^*) be the solution of SGVID. If $T_1(.) : H \to H$ is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian and $T_2(.) : H \to H$ is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian. Let g be relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitz. If

$$\begin{array}{l} \left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{\delta^2(r_1-\gamma_1\mu_1^2)^2 - \mu_1^2(\delta^2-(1-\delta k)^2)}}{\delta\mu_1^2} \\ \delta r_1 > \delta\gamma_1\mu_1^2 + \mu_1\sqrt{\delta^2 - (1 - \delta k)^2}, \end{array} \tag{13}$$

$$\begin{array}{l} \left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{\delta(r_2-\gamma_2\mu^2{}_2)^2 - \mu_2^2(\delta^2-(1-\delta k)^2)}}{\delta\mu_2^2}, \\ \delta r_2 > \delta\gamma_2\mu_2^2 + \mu_2\sqrt{\delta^2 - (1 - \delta k)^2}, \end{array} \tag{14}$$

where

$$k = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2 + \mu_3^2}, \tag{15}$$

and $a_n \in [0, 1]$, $\sum_{n=0}^{\infty} a_n = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in K$, x_n and y_n obtained from Algorithm 3.1 converge strongly to x^* and y^* respectively.

PROOF. To prove the result, we first evaluate $||x_{n+1} - x^*||$ for all $n \geq 0$. From (9), (11), and the Lipschitz continuity of the projection operator P_{K_r} with constant $\delta > 0$, we

$$\begin{aligned} & ||x_{n+1} - x^*|| \\ = \ & ||(1 - a_n)x_n + a_nP_{K_r}[g(y_n) - \rho T_1(y_n] - (1 - a_n)x^* - a_nP_{K_r}[g(y^*) - \rho T_1(y^*)]|| \\ \leq \ & (1 - a_n)||x_n - x^*|| + a_n||P_{K_r}[g(y_n) - \rho T_1(y_n)] - P_{K_r}[g(y^*) - \rho T_1(y^*)]|| \\ \leq \ & (1 - a_n)||x_n - x^*|| + a_n\delta||g(y_n) - g(y^*) - \rho[T_1(y_n) - T_1(y^*)]|| \\ & + a_n\delta\|y_n - y^* - (g(y_n) - g(y^*))\|. \end{aligned} \tag{16}$$

From the relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian definition of $T_1(.)$, we have

$$\begin{aligned}
& ||y_n - y^* - \rho[T_1(y_n) - T_1(y^*)]||^2 \\
= \quad & ||y_n - y^*||^2 - 2\rho\langle T_1(y_n) - T_1(y^*), y_n - y^*\rangle + \rho^2||T_1(y_n) - T_1(y^*)||^2 \\
\leq \quad & ||y_n - y^*||^2 - 2\rho[-\gamma_1||T_1(y_n) - T_1(y^*)||^2 + r_1||y_n - y^*||^2] \\
& +\rho^2||T_1(y_n) - T_1(y^*)||^2 \\
\leq \quad & ||y_n - y^*||^2 + 2\rho\gamma_1\mu_1^2||y_n - y^*||^2 - 2\rho r_1||y_n - y^*||^2 + \rho^2\mu_1^2||y_n - y^*||^2 \\
= \quad & [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]||y_n - y^*||^2. \qquad (17)
\end{aligned}$$

In a similar way, using the (γ_3, r_3)-cocoercivity and μ_3-Lipschitz continuity of the operator g, we have

$$\|y_n - y^* - (g(y_n) - g(y^*))\| \leq k\|y_n - y^*\|, \qquad (18)$$

where k is defined by (15). Set

$$\theta_1 = \delta\left\{k + [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]^{1/2}\right\} \qquad (19)$$

It is clear from the condition (13) that $\theta_1 < 1$. Hence from (18), (16) and (17), it follows that

$$\|x_{n+1} - x^*\| \leq (1 - a_n)\|x_n - x^*\| + a_n\theta_1\|y_n - y^*\|. \qquad (20)$$

Similarly, from the relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian of $T_2(.)$, we obtain

$$\begin{aligned}
& ||x_{n+1} - x^* - \eta[T_2(x_{n+1}) - T_2(x^*)]||^2 \\
= \quad & ||x_{n+1} - x^*||^2 - 2\eta\langle T_2(x_{n+1}) - T_2(x^*), x_{n+1} - x^*\rangle \\
& +\eta^2||T_2(x_{n+1}) - T_2(x^*)||^2 \\
\leq \quad & ||x_{n+1} - x^*||^2 - 2\eta[-\gamma_2||T_2(x_{n+1}) - T_2(x^*)||^2 + r_2||x_{n+1} - x^*||^2] \\
& +\eta^2||T_2(x_{n+1}) - T_2(x^*,)||^2 \\
= \quad & ||x_{n+1} - x^*||^2 + 2\eta\gamma_2||T_2(x_{n+1}) - T_2(x^*)||^2 - 2\eta r_2||x_{n+1} - x^*||^2 \\
& +\eta^2||T_2(x_{n+1}) - T_2(x^*)||^2 \\
\leq \quad & ||x_{n+1} - x^*||^2 + 2\eta\gamma_2\mu_2^2||x_{n+1} - x^*||^2 - 2\eta r_2||x_{n+1} - x^*||^2 \\
& +\eta^2\mu_2^2||x_{n+1} - x^*||^2 \\
= \quad & [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]||x_{n+1} - x^*||^2. \qquad (21)
\end{aligned}$$

Hence from (10), (12), (18), (20) and the Lipschitz continuity of the projection operator P_{K_r} with constant $\delta > 0$, we have

$$\begin{aligned}
||y_{n+1} - y^*|| \quad = \quad & ||P_{K_r}[g(x_{n+1}) - \eta T_2(x_{n+1}) - P_{K_r}[x^* - \eta T_2(x^*)]|| \\
\leq \quad & \delta||g(x_{n+1}) - g(x^*) - \eta(T_2(x_{n+1})] - T_2(x^*))|| \\
\leq \quad & \delta\|||x_{n+1} - x^* - \eta(T_2(x_{n+1})] - T_2(x^*))|| \\
& +\delta\|x_{n+1} - x^* - (g(x_{n+1}) - g(x^*))\| \\
\leq \quad & \theta_2||x_{n+1} - x^*||, \qquad (22)
\end{aligned}$$

where

$$\theta_2 = \delta \left\{ k + [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]^{1/2} \right\}$$

From (14), it follows that $\theta_2 < 1$.

From (20) and (22), we obtain that

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq (1 - a_n)||x_n - x^*|| + a_n\theta_1||y_n - y^*|| \\ &\leq (1 - a_n)||x_n - x^*|| + a_n\theta_1 \cdot \theta_2||x_n - x^*|| \\ &= [1 - a_n(1 - \theta_1\theta_2)]||x_n - x^*||. \end{aligned}$$

Since the constant$(1 - \theta_1\theta_2) \in (0, 1]$, and $\sum_{n=0}^{\infty} a_n(1 - \theta_1\theta_2) = \infty$, from Lemma 3.2, we have $\lim_{n\to\infty} ||x_n - x^*|| = 0$. Hence the result $\lim_{n\to\infty} ||y_n - y^*|| = 0$ is from (22). This completes the proof.

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