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On The Controllability Under Constraints On The Control For Hyperbolic Equations*

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Abstract

Null-controllability for the wave equation is studied in the context of distributed linear constraints on the control.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$ with boundary Γ of class $\mathcal{C}^2$. For $T > 0$, we set $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$, and we consider the wave system

$$\left\{ \begin{array}{ccccc} \square_{a_0} q & = & g & \text{in} & Q, \\ q & = & 0 & \text{on} & \Sigma, \\ \left(q(T), \frac{\partial q}{\partial t}(T)\right) & = & (q_0, q_1) & \text{in} & \Omega, \end{array} \right. \tag{1}$$

where

$$\square_{a_0} = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + a_0\, I, \tag{2}$$

is the d'Alembertian with potential a_0. Here a_0 lies in $L^\infty(Q)$ and is real valued. It is well known that given $g \in L^1([0,T];\, L^2(\Omega))$ and $(q_0, q_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem (1) admits a unique solution q in the space $\mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];\, L^2(\Omega))$. Here, we state the problem of exact controllability for solutions of system (1). Let ω be an open subset of Ω; denote by $Q_\omega = (0,T) \times \omega$ the interior cylinder and χ_ω its characteristic function. Given $(q_0, q_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the goal is to find a source v in $L^2(Q_\omega)$ such that the unique solution q of (1) with $g = \chi_\omega v$ satisfies

$$q(0) = 0 \qquad \text{and} \qquad \frac{\partial q}{\partial t}(0) = 0 \qquad \text{in} \quad \Omega. \tag{3}$$

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This problem is by now well understood (see J. L. Lions [4], C. Bardos *et* al. [1] and A. Ruiz [6]). Indeed, assuming the geometric control condition (GCC) introduced by [1], on the couple (ω, T), one can establish an observation estimate which yields by the HUM method of Lions [4], to the existence of the control v. The (GCC) is a microlocal condition (i.e. a property in the cotangent bundle T^*Q) linking the couple (ω, T) and the bicharacteristic rays of the wave operator. Moreover, it is equivalent to exact controllability of the linear wave equation (with stability with respect to small perturbations of ω, T).

Let $\mathcal{K}$ be a real vector subspace of $L^2(Q_\omega)$ such that

$$\mathcal{K} \text{ is of finite dimension.} \tag{4}$$

The main question we ask in the present work is the following : find a control function $v \in L^2(Q_\omega)$ satisfying

$$v \in \mathcal{K}^\perp \tag{5}$$

such that the unique solution $q = q(t, x; v)$ of (1), satisfies (3). In addition, we impose that v satisfies a finite number of linear equations. This is a null-controllability problem with linear constraints on the control v. Obviously, the classical control without constraints corresponds to the case $\mathcal{K} = \{0\}$. In this article, we treat the more general case $\mathcal{K} \neq \{0\}$. This case was treated in the parabolic case by O. Nakoulima [5].

2 Main Results

Along this section, we denote by $\pi : L^2(Q_\omega) \to \mathcal{K}$ the orthogonal projection and let Φ be the solution of

$$\left\{ \begin{array}{cccc} \Box_{a_0}\Phi & = & 0 & \text{in} \quad Q, \\ \Phi & = & 0 & \text{on} \quad \Sigma, \\ (\Phi(T), \partial_t\Phi(T)) = (\Phi_0, \Phi_1) & \in & L^2(\Omega) \times H^{-1}(\Omega) & \end{array} \right. \tag{6}$$

(we recall that $\Phi \in \mathcal{C}([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega))$).

And now, we state the following hypothesis (we recall that $\mathcal{K}$ is the finite dimension linear subspace of $L^2(Q_\omega)$ defining the constraints):

A1. The couple (ω, T) satisfies the geometric control condition (GCC).

A2. The only element $k \in \mathcal{K}$ satisfying $\Box_{a_0} k = 0$ in Q is the trivial element $k \equiv 0$.

REMARK 1. We have already discussed the key role of the (GCC) condition in observation/control problems; it is almost necessary and sufficient. We first give an example of a non trivial subspace $\mathcal{K}$ of $L^2(Q_\omega)$, satisfying this condition A2: take $\omega_1, \omega_2, ..., \omega_N$, N open sets contained in ω, such that $\omega_i \cap \omega_j = \varnothing$ for $i \neq j$; moreover let $f_1, \cdots, f_N$, N smooth functions, with support$(f_j) \subset \omega_j$ and such that $\Box_{a_0} f_j \neq 0$. It is easy to check that the space $\mathcal{K} = \langle f_1, ..., f_N \rangle$ satisfies to condition A2.

PROPOSITION 1. There exists a positive constant $C > 0$, such that any solution of (6) satisfies :

$$\|(\Phi_0, \Phi_1)\|^2_{L^2 \times H^{-1}} \leq C \int_0^T \int_\omega |(I - \pi)\, \Phi|^2 \, dxdt. \tag{7}$$

PROOF. We argue by contradiction and we suppose that there exists a sequence of data $(\Phi_0^n, \Phi_1^n)_n$, such that

$$\|(\Phi_0^n, \Phi_1^n)\|_{L^2 \times H^{-1}} = 1, \quad \text{for all} \quad n \in \mathbf{N}^* \tag{8}$$

and the associated solutions Φ_n satisfy

$$\int_0^T \int_\omega |(I-\pi)\Phi_n|^2 \, dxdt < 1/n. \tag{9}$$

The sequence $(\Phi_n)_n$ is bounded in $L^2(Q)$, so it admits a subsequence still denoted by Φ_n satisfying $\Phi_n \rightharpoonup \Phi$ weakly in $L^2(Q)$; therefore, $(I-\pi)\Phi_n \rightharpoonup (I-\pi)\Phi$ weakly in $L^2(Q_\omega)$. Moreover, from (9), we deduce that $(I-\pi)\Phi_n \to 0$ strongly in $L^2(Q_\omega)$. Thus $(I-\pi)\Phi = 0$ in $L^2(Q_\omega)$, i.e. $\Phi \in \mathcal{K}$. And thanks to assumption *A2,* this implies $\Phi = 0$. So, $\Phi_n \rightharpoonup 0$ weakly, and then $\pi\Phi_n \to 0$ strongly, since the projection operator π is compact (the finite dimension of $\mathcal{K}$). This yields

$$\Phi_n \to 0 \quad \text{strongly in} \quad L^2(Q_\omega). \tag{10}$$

Now let μ be a microlocal defect measure attached to the sequence (Φ_n) in $L^2(Q)$ (see Gérard [3] for details on these measures, in particular the propagation of their support, see also B. Dehman *et* al. [2]). By the strong convergence (10), we know that $\mu = 0$ on the interior cylinder Q_ω. Therefore, by propagation along the bicharacteristic flow of the wave operator and using the geometric control condition (assumption *A1*), we deduce that $\mu = 0$ on the whole cylinder Q, which means that $\Phi_n \to 0$, strongly in $L^2(Q)$. But this is in contradiction with our assumption $\|(\Phi_0^n, \Phi_1^n)\| = 1$.

COROLLARY 1. Under assumptions A1 and A2 and for every $(q_0, q_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a control function $v \in L^2(Q_\omega)$ satisfying to constraints (5) such that the state solution q to problem

$$\begin{cases} \Box_{a_0} q = \chi_\omega v & \text{in} \quad Q, \\ q = 0 & \text{on} \quad \Sigma, \\ (q(T), \partial_t q(T)) = (q_0, q_1) & \text{in} \quad \Omega, \end{cases} \tag{11}$$

satisfies $(q(0), \partial_t q(0)) = (0, 0)$. Moreover, v is optimal in the following sense:

$$\min\{\|g\|_{L^2(Q_\omega)}, \ g \text{is a control vector in } \mathcal{K}^\perp\} \quad \text{is achieved at} \quad g = v. \tag{12}$$

PROOF. We will run a suitable version of the HUM method. Consider the system

$$\begin{cases} \Box_{a_0} q = \chi_\omega (I-\pi)\Phi & \text{in} \quad Q, \\ q = 0 & \text{on} \quad \Sigma, \\ (q(0), \partial_t q(0)) = (0, 0) & \text{in} \quad \Omega. \end{cases} \tag{13}$$

The function $(I-\pi)\Phi$ will play the role of the control vector v; it obviously satisfies to constraints (5). Multiplying by Φ and integrating by parts we easily get the fundamental identity

$$\langle (\Phi_0, \Phi_1), \Lambda(\Phi_0, \Phi_1) \rangle = \int_0^T \int_\omega |(I-\pi)\,\Phi|^2 \, dxdt \tag{14}$$

where the operator Λ is defined by

$$\begin{array}{rccl} \Lambda & : & L^2 \times H^{-1} & \rightarrow & L^2 \times H_0^1 \\ & & (\Phi_0, \Phi_1) & \mapsto & (\partial_t q(T), -q(T)). \end{array}$$

Now, from the above proposition, there exists a positive constant $C > 0$, such that (7) holds true, the operator Λ is an isomorphism and thus $v = (I - \pi)\,\Phi$, with $(\Phi_0, \Phi_1) = \Lambda^{-1}(q_1, -q_0)$. This finally gives the right control vector satisfying to constraints (5).

REMARK 2. Notice here that the hypothesis **A2** is in fact necessary and sufficient.

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