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Finite Fractal Dimension Of Pullback Attractors And Application To Non-Autonomous Reaction Diffusion Equations*

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Abstract

In this paper, we study the asymptotic behavior of dissipative non-autonomous PDEs in the framework of a process. In particular, we give sufficient conditions for the pullback attractor with finite fractal dimension. As an example, the result is applied to a non-autonomous reaction diffusion equation.

1 Introduction

In recent years, there is much literature on the study of the asymptotic behavior of non-autonomous PDEs (see [1-3, 8, 10]), and the theory of attractors for non-autonomous dynamical system is developed in the framework of evolutionary process $U(t, \tau)$. The solutions of non-autonomous dynamical systems depend on two time variables (the final time t and initial time τ). For stochastic PDEs, Crauel and Flandoli [9] developed the theory and introduced a more general concept of (random) pullback attractor. As a consequence, pullback attractors have been successfully used to study the asymptotic behavior of general non-autonomous and stochastic PDEs, and one of the main results refers to the finite dimensionality of pullback attractor. However, there are only a few results on their finite dimensionality. J. A. Langa in [1] studies the finite fractal dimension of a process, which needs the union of pullback attractors to be relatively compact [4,6,11], i.e., if $\hat{A} = \{A(t) : t \in R\}$ is a pullback attractor for a process $U(t, \tau)$, then $\bigcup_{\tau \le T} A(\tau)$ needs to be relatively compact. In fact, for general process, $\bigcup_{\tau \le T} A(\tau)$ is not necessary relatively compact, and even if $\bigcup_{\tau \le T} A(\tau)$ is relatively compact, it is difficult to provide a proof. Motivated by these problems, we present a new method to prove the finite dimensionality of pullback attractors. The method has been successfully applied to autonomous dynamical systems [6], but to our knowledge, it has not been applied to non-autonomous dynamical systems. We develop this theory and apply it to non-autonomous systems.

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2 Preliminaries

Let X be a complete metric space, and $U(t,\tau)$ be a process in X, i.e.,

(1) $U(t,s)U(s,\tau) = U(t,\tau), \forall t \geq s \geq \tau$, and

(2) $U(\tau,\tau) = Id$, is the identity operator, $\tau \in R$.

In general, we interpret $U(t,\tau)x_0$ as the solution of a non-autonomous equation at time t which was at x_0 in U at the initial date τ.

DEFINITION 1 ([7,8,10]). A bounded subset B_0 of X is called a uniformly pullback absorbing set for the process $\{U(t,\tau)\}_{t\geq\tau}$ if for every bounded set B of X, there exists a $\tau_0(B) \geq 0$ such that

$$U(t,t-\tau)B \subset B_0 \quad for \quad all \quad \tau \geq \tau_0,$$

here τ_0 does not depend on the choice of t.

DEFINITION 2 ([1,2,7,8,10]). The family $\hat{A} = \{A(t) : t \in R\}$ is said to be a pullback attractor for $U(t,\tau)$ if

(1) $A(t)$ is compact for all $t \in R$,

(2) $\hat{A}$ is invariant, i.e., $U(t,\tau)A(\tau) = A(t) \quad$ for all $t \geq \tau$,

(3) $\hat{A}$ is pullback attracting, i.e., $\lim\limits_{\tau\to-\infty} dist((U(t,\tau)B, A(t)) = 0$, for any bounded $B \subset X$, and all $t \in R$, where $dist(C,D) = \sup\limits_{y\in C}\inf\limits_{x\in D}\|y-x\|_X$ denotes the Hausdorff semidistance for arbitrary set $C, D \in X$,

(4) if $\{C(t)\}_{t\in R}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in R$.

We recall that the attracting sets $\{C(t)\}_{t\in R}$ is that for any bounded $B \subset X$,

$$\lim_{\tau\to-\infty} dist((U(t,\tau)B, C(t)) = 0.$$

Given a compact $K \subset X$, and $\varepsilon > 0$, we denote by $N(K,\varepsilon)$ the minimum number of open balls in X with radius ε which are necessary to cover K.

DEFINITION 3 ([4-6]). For any nonempty compact set $K \subset X$, the fractal dimension of K is the number

$$\dim_f(K) = \lim_{\varepsilon\to0}\sup\frac{\log\ N(K,\varepsilon)}{\log(1/\varepsilon)}. \tag{1}$$

3 Estimates of the Fractal Dimension

LEMMA 1 ([6]). Let B_r be a ball of the radius r in R^d equipped with Euclidean norm $|\cdot|$. Then for any $\varepsilon > 0$ there exist a finite set $\{x_k : k = 1,2,\ldots,n_\varepsilon\} \subset B_r$ such that $B_r \subset \bigcup\limits_{k=1}^{n_\varepsilon}\{x \in R^d : |x - x_k| < \varepsilon\}$ and $n_\varepsilon \leq (1+\frac{2r}{\varepsilon})^d$.

THEOREM 1. Let $U(t,\tau)$ be a process in a separable Hilbert space H, B be a uniformly pullback absorbing set in H, $\hat{A} = \{A(t) : t \in R\}$ be a pullback attractor for $U(t,\tau)$, if there exists a finite dimensional projection P in the space H such that

$$\|P(U(t,t-T_0)u_1 - U(t,t-T_0)u_2)\| \leq l(T_0)\|u_1-u_2\| \tag{2}$$

for all $u_1, u_2 \in B$ and some $T_0, l(T_0) > 0$, and

$$\|(I-P)(U(t,t-T_0)u_1 - U(t,t-T_0)u_2)\| \le \delta \|u_1 - u_2\| \tag{3}$$

for all $u_1, u_2 \in B$, where $\delta < 1$ and T_0 and $l(T_0)$ are independent on the choice of t, and $\|\cdot\|$ is the norm in H. Then the family of pullback attractors $\hat{A} = \{A(t) : t \in R\}$ possesses a finite fractal dimension, specifically

$$\dim_f(A(t)) \le \dim P \log\left(1 + \frac{8l(T_0)}{1-\delta}\right)\left[\log\frac{2}{1+\delta}\right]^{-1}, \ \forall t \in R. \tag{4}$$

We need the following Lemma 2 to prove the theorem.

LEMMA 2. Let $A(t-T_0) \in \hat{A}$ such that equation (2) and (3) hold. Then for any $q > 0$ and $\varepsilon > 0$ the following estimate holds

$$N(U(t,t-T_0)A(t-T_0),\varepsilon) \le \left(1 + \frac{4l}{q}\right)^n N\left(A(t-T_0), \frac{\varepsilon}{q+\delta}\right), \tag{5}$$

where $n = \dim P$ is the dimension of the projector P.

PROOF. Let $\varepsilon_0 = \frac{\varepsilon}{q+\delta}$, since $A(t-T_0)$ is compact, there exist finite closed subset $F_i \subset B$ (since B is uniformly pullback absorbing set in H, we can find a suitable B satisfying the condition) and $A(t-T_0) \subset \bigcup_{i=1}^{N(t-T_0,\ \varepsilon_0)} F_i$, with the diameter F_i does not exceed $2\varepsilon_0$. (2) implies that in PH there exist ball B_i with radius $2l\varepsilon_0$ such that $P(U(t,t-T_0)F_i \subset B_i$, by Lemma 1 there exists a covering $\{B_{ij}\}_{j=1}^{N_i}$ of the set B_i with balls of diameter $2q\varepsilon_0$, where $N_i \le (1+\frac{4l}{q})^n$, therefore, the collection

$$\{G_{ij} = B_{ij} + (I-P)U(t,t-T_0)F_i : i = 1,2,...,N(A(t-T_0),\varepsilon_0), j = 1,2,...,N_i\}$$

is a covering of the set $U(t,t-T_0)A(t-T_0)$.

Obviously that

$$diam\ G_{ij} \le diam\ B_{ij} + diam\ (I-P)U(t,t-T_0)F_i.$$

(3) implies that $diam(I-P)U(t,t-T_0)F_i \le 2\delta\varepsilon_0$. Therefore,

$$diam\ G_{ij} \le 2(q+\delta)\varepsilon_0.$$

Hence, $N(A(t),\varepsilon) = N(U(t,t-T_0)A(t-T_0),\varepsilon) \le (1+\frac{4l}{q})^n N(A(t-T_0),\frac{\varepsilon}{q+\delta})$.

Next, we use Lemma 2 to prove Theorem 1.

PROOF. The proof of (5) does not depend on t and by Definition 2, we get

$$A(t) = U(t,t-T_0)A(t-T_0),$$

$$A(t-T_0) = U(t-T_0,t-2T_0)A(t-2T_0),$$

so we have

$$\begin{aligned} N(A(t-T_0),\tfrac{\varepsilon}{q+\delta}) &= N(U(t-T_0,t-2T_0)A(t-2T_0),\tfrac{\varepsilon}{(q+\delta)}) \\ &\le \left(1+\tfrac{4l}{q}\right)^n N\left(A(t-2T_0),\tfrac{\varepsilon}{(q+\delta)^2}\right). \end{aligned}$$

It follows that

$$N(A(t),\varepsilon) \le \left(1+\frac{4l}{q}\right)^{nm} N\left(A(t-mT_0), \frac{\varepsilon}{(q+\delta)^m}\right).$$

We choose q and $m(\varepsilon)$, such that $q+\delta<1$, $\frac{\varepsilon}{(\delta+q)^m} > diamB$, since when $\frac{\varepsilon}{(\delta+q)^m} > diamB$, we only need one ball covering $A(t-mT_0)$, i.e.,

$$N\left(A(t-mT_0), \frac{\varepsilon}{(q+\delta)^m}\right) = 1.$$

Let $m(\varepsilon) = \left[\frac{\log\varepsilon - \log diamB}{\log(q+\delta)}\right] + 1$, where $[z]$ is an integer part of the number z. Consequently, we get

$$\begin{aligned} \dim_f(A(t)) &= \lim_{\varepsilon\to 0}\sup \frac{\log N(A(t),\ \varepsilon)}{\log(1/\varepsilon)} \\ &\le n\log(1+\tfrac{4l}{q}) \lim_{\varepsilon\to 0}\sup \frac{m(\varepsilon)}{\log(1/\varepsilon)} \\ &\le n\log(1+\tfrac{4l}{q}) \lim_{\varepsilon\to 0} \left(\frac{\log\varepsilon - \log diamB}{\log(q+\delta)} + 1\right) / \log\frac{1}{\varepsilon} \\ &= n\log(1+\tfrac{4l}{q})[\log\tfrac{1}{q+\delta}]^{-1}. \end{aligned}$$

Let $q=\frac{1-\delta}{2}$, we get $\dim_f(A(t)) \le \dim P\left(\log(1+\frac{8l}{1-\delta})\right)[\log\frac{2}{1+\delta}]^{-1}$.

4 Finite Fractal Dimension of Non-Autonomous Reaction Diffusion Equations

The purpose of this section is to apply the theoretical results from Section 3 to a non-autonomous reaction diffusion equation.

We consider the following non-autonomous differential equation

$$\begin{cases} u_t - \triangle u + f(u) = g(t), \quad x\in\Omega, \\ u|_{\partial\Omega} = 0, \\ u(\tau) = u_\tau, \end{cases} \tag{6}$$

where $f\in C^1(R,R)$, $g(\cdot)\in L^2_{loc}(R, L^2(\Omega))$, Ω is a bounded open subset of R^n and there exist $p\ge 2$, $c_i>0$, $i=1,...,5, l\in R$ such that

$$c_1|s|^p - c_2 \le f(s)s \le c_3|s|^p + c_4, \tag{7}$$

$$f'(s)\ge -l,\ f(0)=0,\ |f'(s)|\le c_5(1+|s|^{p-2}) \tag{8}$$

for all $s\in R$.

Denote $H=L^2(\Omega)$ with norm $|\cdot|$ and scalar product $(\cdot)$, $V=H^1_0(\Omega)$ with norm $\|\cdot\|$, $|\cdot|_k$ is the norm of $L^k(\Omega)$ and c is a constant which may change from line to line and even in the same line.

Suppose that the function $g(t)$ is translation bounded in $L^2_{loc}(R;H)$ that is ,

$$|g|_b^2 = \sup_{t\in R}\int_h^{h+1} |g(s)|^2 ds < \infty. \tag{9}$$

THEOREM 2 ([7]). If $g(t)$ is translation bounded in $L^2_{loc}(R;H)$, $f(s)$ satisfies conditions (7) and (8) where $2 \le p < \infty$ for spatial dimensions $n \le 2$ and $2 \le p \le \frac{n}{n-2} + 1$ for spatial dimensions $n \ge 3$, then the process $U(t,\tau)$ corresponding to problem (6) possesses a uniformly pullback absorbing set B and a pullback attractors $\hat{A} = \{A(t) : t \in R\}$ in V.

We set $A = -\triangle$, since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to +\infty, \text{ as } j \to \infty,$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $H_0^1(\Omega)$ which are orthogonal in H such that

$$Ae_j = \lambda_j e_j \quad \forall j \in N.$$

Let $H_m = span\{e_1, e_2, ..., e_m\}$ in H and $P : H \to H_m$ be the orthogonal projection. For any $u \in H$ we write

$$u = Pu + (I - P)u \triangleq u_1 + u_2.$$

THEOREM 3. Assume that $g(t)$ and $f(s)$ satisfy conditions of Theorem 2 and B is the uniformly pullback absorbing set in V corresponding to problem (6). Then the pullback attractor $\hat{A} = \{A(t) : t \in R\}$ corresponding to problem (6) possesses a finite fractal dimension in V and

$$\dim_f(A(t)) \le n \log\left(1 + \frac{8l_0}{1-\delta}\right)\left[\log(\frac{2}{1+\delta})\right]^{-1},$$

where $l_0 = e^{2l}$, $\delta = e^{-\lambda_n} + \frac{c}{\lambda_n}$, we choose n large enough so that $\delta < 1$.

PROOF. Let $u(t)$ be the solution of equation (6) with initial data u_τ, taking inner product of (6) with $-\triangle u$ in H, we easily obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + |\triangle u|^2 + (f(u), -\triangle u) = (g(t), -\triangle u).$$

Since

$$|(g(t), -\triangle u)| \le |g(t)||\triangle u| \le \frac{1}{2}|g(t)|^2 + \frac{1}{2}|\triangle u|^2,$$

and using (8), we get

$$\frac{d}{dt}\|u(t)\|^2 \le 2l\|u(t)\|^2 + |g(t)|^2,$$

and consequently, by Gronwall's lemma,

$$\|u(t,\tau)\|^2 \le e^{2l(t-\tau)}\|u_\tau\|^2 + e^{2lt}\int_\tau^t e^{-2ls}|g(s)|^2 ds. \tag{10}$$

We set $u_1(t) = u(t,\tau)u_{1\tau}$ and $u_2(t) = u(t,\tau)u_{2\tau}$ to be solutions associated with equation (6) with initial data $u_{1\tau}, u_{2\tau} \in B$. Since B is the uniformly pullback absorbing set in V, there exists $M > 0$, such that $\|u_{i\tau}\|^2 \le M$ for $i = 1, 2$.

Let $w = u_1(t) - u_2(t)$, by (6), we get

$$w_t - \triangle w + f(u_1(t)) - f(u_2(t)) = 0. \tag{11}$$

Taking inner product of (11) with $-\triangle w$ in H, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + |\triangle w|^2 + (f(u_1) - f(u_2), -\triangle w) = 0,$$

from (8), we get

$$\frac{d}{dt}\|w\|^2 \le 2l\|w\|^2,$$

hence

$$\|w(t)\|^2 \le \|w(\tau)\|^2 e^{2l(t-\tau)}. \tag{12}$$

Let $w = w_1 + w_2$, where w_1 is the projection in PH, then

$$\|w_1(t)\|^2 \le \|w(\tau)\|^2 e^{2l(t-\tau)}.$$

Taking inner product of (11) with $-\triangle w_2$ in H, we have

$$\frac{1}{2}\frac{d}{dt}\|w_2\|^2 + |\triangle w_2|^2 + (f(u_1) - f(u_2), -\triangle w_2) = 0,$$

and

$$\begin{aligned}|(f(u_1) - f(u_2), -\triangle w_2)| &\le \int_\Omega |f(u_1) - f(u_2)||\triangle w_2| dx \\ &\le \frac{1}{2}|\triangle w_2|^2 + \frac{1}{2}\int_\Omega |f(u_1) - f(u_2)|^2 dx.\end{aligned}$$

Taking into account (8) and Hölder inequality, it is immediate to see that

$$\begin{aligned}\int_\Omega |f(u_1) - f(u_2)|^2 dx &= \int_\Omega |f'(u_1 + \theta(u_2 - u_1))|^2 |u_1 - u_2|^2 dx \\ &\le c\int_\Omega (1 + |u_1|^{p-2} + |u_2|^{p-2})^2 |u_1 - u_2|^2 dx \\ &\le c(\int_\Omega (1 + |u_1|^{2(p-1)} + |u_2|^{2(p-1)} dx)^{\frac{p-2}{p-1}} (\int_\Omega |u_1 - u_2|^{2(p-1)})^{\frac{1}{p-1}} \\ &\le c(1 + |u_1|_{2(p-1)}^{2(p-2)} + |u_2|_{2(p-1)}^{2(p-2)})|w|_{2(p-1)}^2.\end{aligned}$$

Since $2 \le p < \infty$ ($n \le 2$), $2 \le p \le \frac{n}{n-2} + 1$ ($n \ge 3$), using Sobolev embedding theorem, we get

$$\int_\Omega |f(u_1) - f(u_2)|^2 dx \le c(1 + \|u_1\|^{2(p-2)} + \|u_2\|^{2(p-2)})\|w\|^2.$$

Since

$$\lambda_n\|w_2\|^2 \le |\triangle w_2|^2,$$

it is immediate that

$$\frac{d}{dt}\|w_2\|^2 + \lambda_n\|w_2\| \le c(1+\|u_1\|^{2(p-2)} + \|u_2\|^{2(p-2)})\|w\|^2,$$

then, by Gronwall's lemma, we have

$$\|w_2(t)\|^2 \le e^{-\lambda_n(t-\tau)}\|w(\tau)\|^2 + ce^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}(1+\|u_1(s)\|^{2(p-2)}+\|u_2(s)\|^{2(p-2)})\|w(s)\|^2.$$

Let $T_0 = t-\tau = 1$, from (12), we get

$$\begin{aligned} e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|w(s)\|^2 ds &\le e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}e^{2l(s-\tau)}\|w(\tau)\|^2 ds \\ &\le e^{2l}e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}ds\|w(\tau)\|^2 \\ &\le \frac{c}{\lambda_n}\|w(\tau)\|^2, \end{aligned}$$

$$\begin{aligned} e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|u_i(s)\|^{2(p-2)}\|w(s)\|^2 ds \quad &\le \quad e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}(e^{2l(s-\tau)}\|u_{i\tau}\|^2 \\ &\quad + e^{2ls}\int_\tau^s e^{-2lr}|g(r)|^2 dr)^{(p-2)}e^{2l(s-\tau)}\|w(\tau)\|^2 ds. \end{aligned}$$

for $i = 1, 2$, and

$$e^{2ls}\int_\tau^s e^{-2lr}|g(r)|^2 dr \le e^{2ls}\int_{s-1}^s e^{-2l(s-1)}|g(r)|^2 dr \le c.$$

So

$$\begin{aligned} e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}\|u_i\|^{2(p-2)}\|w\|^2 ds &\le e^{-\lambda_n t}\int_\tau^t e^{\lambda_n s}(e^{2l}\|u_{i\tau}\|^2 + c)^{p-2}e^{2l}\|w(\tau)\|^2 ds \\ &\le \frac{c}{\lambda_n}\|w(\tau)\|^2 \end{aligned}$$

for $i = 1, 2$. We easily obtain

$$\|w_2(t)\|^2 \le (e^{-\lambda_n} + \frac{c}{\lambda_n})\|w(\tau)\|^2.$$

Since $\lambda_n \to +\infty$, $e^{-\lambda_n} + \frac{c}{\lambda_n} < 1$ when n is sufficiently large.

Obviously

$$\|w_1(t)\|^2 \le l_0\|w_\tau\|^2; \quad \|w_2(t)\|^2 \le \delta\|w_\tau\|^2.$$

Here $l_0 = e^{2l}, \delta = e^{-\lambda_n} + \frac{c}{\lambda_n}, T_0 = 1.$ We get that the process generated by (6) satisfies all conditions of Theorem 1.

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