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Bounds For The Inverses Of Diagonally Dominant Pentadiagonal Matrices*

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Abstract

In this paper, we give some new estimates for the lower and upper bounds of the inverse elements of strictly diagonally dominant pentadiagonal matrices.

1 Introduction

We consider the linear system

$$Ax = b, \tag{1}$$

where $x, b \in R^n$, and $A \in R^{n\times n}$ is an M-matrix. When the coefficient matrix A is ill-conditioned and n is large, i.e., the spectral condition number of A is $\kappa_2(A) \gg 1$, then it is 'expensive' to solve the linear system (1) (see [7, 8, 9]). For reducing the cost, we usually use the approximate inverse V of A for preconditioning the linear system $Ax = b$.

Band matrices are playing very important roles in numerical computing. It is very useful in many problems to have upper and lower bounds for the inverse entries of the matrices. Therefore, many researchers have attempted to investigate better estimates of bounds. Shivakumar et al. [1] showed upper and lower bounds for diagonally dominant tridiagonal matrices. Nabben [3] established decay rates for the entries of inverses of certain banded matrices. Then, Nabben in [2] and Peluso et al. [4] improved the upper and lower bounds in [1]. Later, Liu et al. [5] obtained upper and lower bounds for the inverse elements of diagonally dominant tridiagonal matrices that improved related results in [1-4].

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2 Estimates for the bounds of the inverse elements

Let us consider the following real pentadiagonal matrix with order $n \geq 5$:

$$\begin{pmatrix} a_1 & b_1 & d_1 & & & \\ c_1 & a_2 & b_2 & d_2 & & \\ e_1 & c_2 & a_3 & b_3 & d_3 & \\ & & & \ddots & \ddots & \ddots \\ & & & e_{n-2} & c_{n-1} & a_n \end{pmatrix}. \tag{2}$$

Assume A is diagonally dominant. Then the elements of A satisfy the condition:

$$|a_i| \geq |b_i| + |d_i| + |c_{i-1}| + |e_{i-2}|, \; i = 1, 2, \ldots, n,$$

with $a_1 \neq 0$, and $c_0 = e_0 = e_{-1} = d_{n-1} = d_n = b_n = 0$.

Let $X = A^{-1} = (x_{ij})_{n\times n}$ be the inverse of A, $x_j = (x_{1,j}, x_{2,j}, ..., x_{n,j})^T$ be the j-th column of X. First of all, we define

$$\begin{aligned}
&\tau_i = \frac{b_i - (\beta_{i-1}c_{i-1} - \tau_{i-2}e_{i-2}\beta_{i-1})}{\Gamma_i}, \; \tilde{\tau}_i = \frac{|b_i| + |\tilde{\beta}_{i-1}c_{i-1}| + |\tilde{\tau}_{i-2}e_{i-2}\tilde{\beta}_{i-1}|}{\tilde{\Gamma}_i}, \\
&\beta_i = \frac{d_i}{\Gamma_i}, \quad \tilde{\beta}_i = \frac{|d_i|}{\tilde{\Gamma}_i}, \quad \Gamma_i = a_i - (\tau_{i-1}c_{i-1} - \tau_{i-2}\tau_{i-1}e_{i-2} + \beta_{i-2}e_{i-2}), \\
&\tilde{\Gamma}_i = \Big||a_i| - |\tilde{\tau}_{i-1}c_{i-1}| - |\tilde{\tau}_{i-2}\tilde{\tau}_{i-1}e_{i-2}| - |\tilde{\beta}_{i-2}e_{i-2}|\Big|, \quad i = 1, \ldots, n-1; \\
&\gamma_i = \frac{c_{i-1} + \delta_{i+1}\gamma_{i+2}d_i - b_i\delta_{i+1}}{\Upsilon_i}, \quad \tilde{\gamma}_i = \frac{|c_{i-1}| + |\tilde{\delta}_{i+1}\tilde{\gamma}_{i+2}d_i| + |b_i\tilde{\delta}_{i+1}|}{\tilde{\Upsilon}_i}, \\
&\delta_i = \frac{e_{i-2}}{\Upsilon_i}, \quad \tilde{\delta}_i = \frac{|e_{i-2}|}{\tilde{\Upsilon}_i}, \quad \Upsilon_i = a_i + \gamma_{i+2}\gamma_{i+1}d_i - b_i\gamma_{i+1} - \delta_{i+2}d_i, \\
&\tilde{\Upsilon}_i = \Big||a_i| - |\tilde{\gamma}_{i+2}\tilde{\gamma}_{i+1}d_i| - |b_i\tilde{\gamma}_{i+1}| - |\tilde{\delta}_{i+2}d_i|\Big|, \quad i = 2, \ldots, n.
\end{aligned} \tag{3}$$

and $\tau_i = \beta_i = \gamma_{n+2+i} = \delta_{n+2+i} = \tilde{\tau}_i = \tilde{\beta}_i = \tilde{\gamma}_{n+2+i} = \tilde{\delta}_{n+2+i} = 0, \quad i = -1, 0.$

In this paper, we assume the elements of A satisfy the following conditions

$$\Gamma_i \neq 0, \quad \tilde{\gamma}_i \neq 0, \quad i = 1, 2, \ldots, n-1; \quad \Upsilon_i \neq 0, \quad \tilde{\Upsilon}_i \neq 0, \quad i = 2, 3, \ldots, n.$$

LEMMA 2.1. Let A be a diagonally dominant pentadiagonal matrix of the form (2), and $A^{-1} = X = (x_{ij})_{n\times n}$. Then

$$x_{i,j} = -\tau_i x_{i+1,j} - \beta_i x_{i+2,j}, i = 1, \ldots, n-1, \; j = i+1, \ldots, n.$$

PROOF. Note that $Ax_j = e_j$, where e_j is the j-th basis vector of R^n. Writing the first $j-1$ equations, with $j \geq 2$ and $i = 1, \ldots, j-1$, we have

$$\begin{cases} a_1x_{1,j} + b_1x_{2,j} + d_1x_{3,j} = 0, \\ c_1x_{1,j} + a_2x_{2,j} + b_2x_{3,j} + d_2x_{4,j} = 0, \\ e_1x_{1,j} + c_2x_{2,j} + a_3x_{3,j} + b_3x_{4,j} + d_3x_{5,j} = 0, \\ \qquad\qquad \cdots \\ e_{i-2}x_{i-2,j} + c_{i-1}x_{i-1,j} + a_ix_{i,j} + b_ix_{i+1,j} + d_ix_{i+2,j} = 0. \end{cases} \tag{4}$$

According to (4), we obtain

$$
\begin{aligned}
x_{1,j} &= -\frac{b_1}{a_1}x_{2,j} - \frac{d_1}{a_1}x_{3,j} = -\tau_1 x_{2,j} - \beta_1 x_{3,j},\\
x_{2,j} &= -\frac{b_2 - \frac{d_1}{a_1}c_1}{a_2 - \frac{b_1}{a_1}c_1}x_{3,j} - \frac{d_2}{a_2 - \frac{b_1}{a_1}c_1}x_{4,j} = -\tau_2 x_{3,j} - \beta_2 x_{4,j},\\
x_{3,j} &= -\frac{b_3 - (\beta_2 c_2 - \tau_1 e_1 \beta_2)}{a_3 - (\tau_2 c_2 - \tau_1\tau_2 e_1 + \beta_1 e_1)}x_{4,j} - \frac{d_3}{a_3 - (\tau_2 c_2 - \tau_1\tau_2 e_1 + \beta_1 e_1)}x_{5,j}\\
&= -\tau_3 x_{4,j} - \beta_3 x_{4,j},\\
&\qquad\cdots\\
x_{i,j} &= -\frac{b_i - (\beta_{i-1}c_{i-1} - \tau_{i-2}e_{i-2}\beta_{i-1})}{\Gamma_i}x_{i+1,j} - \frac{d_i}{\Gamma_i}x_{i+2,j}\\
&= -\tau_i x_{i+1,j} - \beta_i x_{i+2,j}.
\end{aligned}
\tag{5}
$$

Thus, we obtain our result.

LEMMA 2.2. Let A be a diagonally dominant pentadiagonal matrix of the form (2), and $A^{-1} = X = (x_{ij})_{n\times n}$. Then

$$x_{i,j} = -\gamma_i x_{i-1,j} - \delta_i x_{i-2,j}, i = 2,\ldots,n,\ j = 1,\ldots,i-1.$$

PROOF. Note that $Ax_j = e_j$, where e_j is the j-th basis vector of R^n. Writing the last $n-j$ equations, with $j \le n-1$ and $i = n,\ldots,j+1$, we have

$$
\begin{cases}
e_{n-2}x_{n-2,j} + c_{n-1}x_{n-1,j} + a_n x_{n,j} = 0,\\
e_{n-3}x_{n-3,j} + c_{n-2}x_{n-2,j} + a_{n-1}x_{n-1,j} + b_{n-1}x_{n,j} = 0,\\
\qquad\cdots\\
e_{i-2}x_{i-2,j} + c_{i-1}x_{i-1,j} + a_i x_{i,j} + b_i x_{i+1,j} + d_i x_{i+2,j} = 0.
\end{cases}
\tag{6}
$$

According to (6), we obtain

$$
\begin{aligned}
x_{n,j} &= -\frac{c_{n-1}}{a_n}x_{n-1,j} - \frac{e_{n-2}}{a_n}x_{n-2,j} = -\gamma_n x_{n-1,j} - \delta_n x_{n-2,j},\\
x_{n-1,j} &= -\frac{c_{n-2} - b_{n-1}\delta_n}{a_{n-1} - b_{n-1}\gamma_n}x_{n-2,j} - \frac{e_{n-3}}{a_{n-1} - b_{n-1}\gamma_n}x_{n-3,j}\\
&= -\gamma_{n-1}x_{n-2,j} - \delta_{n-1}x_{n-3,j},\\
x_{n-2,j} &= -\frac{c_{n-3} + \delta_{n-1}\gamma_n d_{n-2} - b_{n-2}\delta_{n-1}}{a_{n-2} + \gamma_n\gamma_{n-1}d_{n-2} - b_{n-2}\gamma_{n-1} - \delta_n d_{n-2}}x_{n-3,j}\\
&\quad - \frac{e_{n-4}}{a_{n-2} + \gamma_n\gamma_{n-1}d_{n-2} - b_{n-2}\gamma_{n-1} - \delta_n d_{n-2}}x_{n-4,j}\\
&= -\gamma_{n-2}x_{n-3,j} - \delta_{n-2}x_{n-4,j},\\
&\qquad\cdots\\
x_{i,j} &= -\frac{c_{i-1} + \delta_{i+1}\gamma_{i+2}d_i - b_i\delta_{i+1}}{\Upsilon_i}x_{i-1,j} - \frac{e_{i-2}}{\Upsilon_i}x_{i-2,j}\\
&= -\gamma_i x_{i-1,j} - \delta_i x_{i-2,j}, i = n-2,\ldots,j+1.
\end{aligned}
\tag{7}
$$

Thus, we obtain our result.

THEOREM 2.1. Let A be a diagonally dominant pentadiagonal matrix of the form (2), and $A^{-1} = X = (x_{ij})_{n\times n}$. For $i = 1, \ldots, n-1,\ j = i+1, \ldots, n$, we obtain

$$||\tau_i\mu_{i+1}| - |\beta_i\mu_{i+2}|||x_{jj}| \le |x_{ij}| \le (|\tilde{\tau}_i\tilde{\mu}_{i+1}| + |\tilde{\beta}_i\tilde{\mu}_{i+2}|)|x_{jj}|,$$

where

$$\mu_{j-1} = \frac{\beta_{j-1}\gamma_{j+1} - \tau_{j+1}}{1 - \delta_{j+1}\beta_{j-1}} \le \frac{|\beta_{j-1}\gamma_{j+1}| + |\tau_{j+1}|}{||1 - |\delta_{j+1}\beta_{j-1}||} \le \frac{|\tilde{\beta}_{j-1}\tilde{\gamma}_{j+1}| + |\tilde{\tau}_{j+1}|}{\left||1 - |\tilde{\delta}_{j+1}\tilde{\beta}_{j-1}|\right|} \overset{\text{def}}{=} \tilde{\mu}_{j-1},$$

$$\mu_{j-2} = -(\tau_{j-2}\mu_{j-1} + \beta_{j-2}) \le |\tilde{\tau}_{j-2}\tilde{\mu}_{j-1}| + |\tilde{\beta}_{j-2}| \overset{\text{def}}{=} \tilde{\mu}_{j-2},$$

$$\mu_i = -(\tau_i\mu_{i+1} + \beta_i\mu_{i+2}) \le |\tau_i\mu_{i+1}| + |\beta_i\mu_{i+2}| \overset{\text{def}}{=} \tilde{\mu}_i, \quad \mu_0 = \tilde{\mu}_0 = 0.$$

PROOF. According to (5) and (7), we have

$$\begin{cases} x_{j-1,j} = -\tau_{j-1}x_{j,j} - \beta_{j-1}x_{j+1,j}, \\ x_{j+1,j} = -\gamma_{j+1}x_{j,j} - \delta_{j+1}x_{j-1,j}, \end{cases} \quad j = 2, \ldots, n-1. \tag{8}$$

For $i = j-3, \ldots, 1$,

$$x_{i,j} = -\tau_i x_{i+1,j} - \beta_i x_{i+2,j} = -(\tau_i\mu_{i+1} + \beta_i\mu_{i+2})x_{jj} = \mu_i x_{jj}.$$

Notice that

$$||\tau_i\mu_{i+1}| - |\beta_i\mu_{i+2}|| \le |\tau_i\mu_{i+1} + \beta_i\mu_{i+2}| \le |\tau_i\mu_{i+1}| + |\beta_i\mu_{i+2}| \le |\tilde{\tau}_i\tilde{\mu}_{i+1}| + |\tilde{\beta}_i\tilde{\mu}_{i+2}|.$$

Thus, we obtain our result.

THEOREM 2.2. Let A be a diagonally dominant pentadiagonal matrix of the form (2), and $A^{-1} = X = (x_{ij})_{n\times n}$. For $i = 2, \ldots, n,\ j = 1, \ldots, i-1$, we have

$$||\gamma_i\nu_{i-1}| - |\delta_i\nu_{i-2}|||x_{jj}| \le |x_{ij}| \le (|\tilde{\gamma}_i\tilde{\nu}_{i-1}| + |\tilde{\delta}_i\tilde{\nu}_{i-2}|)|x_{jj}|,$$

where

$$\nu_{j+1} = \frac{\delta_{j+1}\tau_{j-1} - \gamma_{j+1}}{1 - \delta_{j+1}\beta_{j-1}} \le \frac{|\delta_{j+1}\tau_{j-1}| + |\gamma_{j+1}|}{||1 - |\delta_{j+1}\beta_{j-1}||} \le \frac{|\tilde{\delta}_{j+1}\tilde{\tau}_{j-1}| + |\tilde{\gamma}_{j+1}|}{\left||1 - |\tilde{\delta}_{j+1}\tilde{\beta}_{j-1}|\right|} \overset{\text{def}}{=} \tilde{\nu}_{j+1},$$

$$\nu_{j+2} = -(\gamma_{j+2}\mu_{j+1} + \delta_{j+2}) \le |\gamma_{j+2}\mu_{j+1}| + |\delta_{j+2}| \le |\tilde{\gamma}_{j+2}\tilde{\mu}_{j+1}| + |\tilde{\delta}_{j+2}| \overset{\text{def}}{=} \tilde{\nu}_{j+2},$$

$$\nu_i = -(\gamma_i\nu_{i-1} + \delta_i\nu_{i-2}) \le |\gamma_i\nu_{i-1}| + |\delta_i\nu_{i-2}| \le |\tilde{\gamma}_i\tilde{\nu}_{i-1}| + |\tilde{\delta}_i\tilde{\nu}_{i-2}| \overset{\text{def}}{=} \tilde{\nu}_i, \quad \nu_0 = \tilde{\nu}_0 = 0.$$

PROOF. According to (8), we have

$$x_{j+1,j} = \frac{\delta_{j+1}\tau_{j-1} - \gamma_{j+1}}{1 - \delta_{j+1}\beta_{j-1}} x_{jj} = \nu_{j+1}x_{jj}.$$

For $i = j+3, \ldots, n$, we further have

$$x_{i,j} = -\gamma_i x_{i-1,j} - \delta_i x_{i-2,j} = -(\gamma_i \nu_{i-1} + \delta_i \nu_{i-2}) x_{jj} = \nu_i x_{jj}.$$

Notice that

$$||\gamma_i \nu_{i-1}| - |\delta_i \nu_{i-2}|| \leq |\gamma_i \nu_{i-1} + \delta_i \nu_{i-2}| \leq |\gamma_i \nu_{i-1}| + |\delta_i \nu_{i-2}| \leq |\tilde{\gamma}_i \tilde{\nu}_{i-1}| + |\tilde{\delta}_i \nu_{i-2}|.$$

Thus, we obtain our result.

THEOREM 2.3. Let A be a diagonally dominant pentadiagonal matrix of the form (2), and $A^{-1} = X = (x_{ij})_{n\times n}$. Then

$$\frac{1}{|a_j| + h_j} \leq |x_{j,j}| \leq \frac{1}{|a_j| - h_j},\ j = 1, 2, \ldots, n,$$

where

$$\begin{aligned} &c_0 = e_0 = e_{-1} = d_{n-1} = d_n = b_n = 0, \\ &h_j = (|e_{j-2}||\tilde{\tau}_{j-2}| + |c_{j-1}|)\frac{|\tilde{\beta}_{j-1}||\tilde{\gamma}_{j+1}| + |\tilde{\tau}_{j-1}|}{1 - |\tilde{\delta}_{j+1}||\tilde{\beta}_{j-1}|} + |e_{j-2}||\tilde{\beta}_{j-2}| + |d_j||\tilde{\delta}_{j+2}| \\ &\qquad + (|b_j| + |d_j||\tilde{\gamma}_{j+2}|)\frac{|\tilde{\delta}_{j+1}||\tilde{\tau}_{j-1}| + |\tilde{\gamma}_{j+1}|}{1 - |\tilde{\delta}_{j+1}||\tilde{\beta}_{j-1}|}, j = 1, 2, \ldots, n. \end{aligned}$$

PROOF. According to $AA^{-1} = I$, we have

$$e_{j-2}x_{j-2,j} + c_{j-1}x_{j-1,j} + a_j x_{j,j} + b_j x_{j+1,j} + d_j x_{j+2,j} = 1, j = 1, 2, \ldots, n.$$

Furthermore,

$$\begin{aligned} |1 - a_j x_{j,j}| &= |e_{j-2}x_{j-2,j} + c_{j-1}x_{j-1,j} + b_j x_{j+1,j} + d_j x_{j+2,j}| \\ &\leq |e_{j-2}||x_{j-2,j}| + |c_{j-1}||x_{j-1,j}| + |b_j||x_{j+1,j}| + |d_j||x_{j+2,j}| \\ &\leq \Bigg[(|e_{j-2}||\tau_{j-2}| + |c_{j-1}|)\frac{|\beta_{j-1}||\gamma_{j+1}| + |\tau_{j-1}|}{1 - |\delta_{j+1}||\beta_{j-1}|} + |e_{j-2}||\beta_{j-2}| \\ &\quad + |d_j||\delta_{j+2}| + (|b_j| + |d_j||\gamma_{j+2}|)\frac{|\delta_{j+1}||\tau_{j-1}| + |\gamma_{j+1}|}{1 - |\delta_{j+1}||\beta_{j-1}|}\Bigg] |x_{j,j}| \\ &\leq h_j\, |x_{j,j}|\,. \end{aligned}$$

Then we obtain the desired bounds.

THEOREM 2.4. Let A be a nonsingular diagonally dominant pentadiagonal matrix of the form (2), and the comparison matrix $\mu(A) = (m_{ij})$ of A is defined as

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad \text{for } i \neq j.$$

Then the exact inverse of $\mu(A)$ is given by the upper bounds established in Theorem 2.1, Theorem 2.2 and Theorem 2.3.

PROOF. Since $\mu(A)$ is a nonsingular diagonally dominant M-matrix, then

$$m_{ii} > 0, m_{ij} \leq 0, m_{ii} \geq -\sum_{j=1}^{n} m_{ij}.$$

By (3), we have

$$\begin{aligned}
&-1 < \tau_1 = \frac{-|b_1|}{|a_1|} = -\tilde{\tau}_1 \leq 0, -1 \leq \beta_1 = \frac{-|d_1|}{|a_1|} = -\tilde{\beta}_1 \leq 0,\\
&-1 \leq \tau_2 = \frac{-|b_2| - \beta_1(-|c_1|)}{|a_2| - \tau_1(-|c_1|)} = -\frac{|b_2| - \beta_1|c_1|}{|a_2| + \tau_1|c_1|} = -\frac{|b_2| + \tilde{\beta}_1|c_1|}{|a_2| - \tilde{\tau}_1|c_1|} = -\tilde{\tau}_2 \leq 0,\\
&-1 \leq \beta_2 = \frac{-|d_2|}{|a_2| - \tau_1(-|c_1|)} = -\frac{|d_2|}{|a_2| + \tau_1|c_1|} = -\frac{|d_2|}{|a_2| - \tilde{\tau}_1|c_1|} = -\tilde{\beta}_2 \leq 0,\\
&-1 \leq \tau_1 + \beta_1 \leq 0, \quad -1 \leq \tau_2 + \beta_2 \leq 0,\\
\tau_i &= \frac{-|b_i| - [\beta_{i-1}(-|c_{i-1}|) - \tau_{i-2}(-|e_{i-2}|)\beta_{i-1}]}{|a_i| - [\tau_{i-1}(-|c_{i-1}|) - \tau_{i-2}\tau_{i-1}(-|e_{i-2}|) + \beta_{i-2}(-|e_{i-2}|)]}\\
&= -\tilde{\tau}_i \qquad (0 \leq \tilde{\tau}_{i-2}|e_{i-2}|\tilde{\beta}_{i-1} + \tilde{\beta}_{i-2} \leq 1), \qquad i = 3, \ldots, n-1,\\
\beta_i &= \frac{-|d_i|}{|a_i| - [\tau_{i-1}(-|c_{i-1}|) - \tau_{i-2}\tau_{i-1}(-|e_{i-2}|) + \beta_{i-2}(-|e_{i-2}|)]}\\
&= -\tilde{\beta}_i, \qquad (0 \leq \tilde{\tau}_{i-2}\tilde{\tau}_{i-1} + \tilde{\beta}_{i-2} \leq 1) \qquad i = 3, \ldots, n-1.
\end{aligned}$$

$$-1 \leq \tau_i = -\tilde{\tau}_i \leq 0, \quad -1 \leq \beta_i = -\tilde{\beta}_i \leq 0, \quad -1 \leq \tau_i + \beta_i \leq 0.$$

Similar to the above proof, we have

$$\begin{aligned}
&-1 \leq \delta_i = -\tilde{\delta}_i \leq 0, \quad i = 2, \ldots, n,\\
&-1 \leq \gamma_i = -\tilde{\gamma}_i \leq 0, -1 \leq \delta_i + \gamma_i \leq 0, \qquad i = 2, \ldots, n.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mu_{j-1} = \tilde{\mu}_{j-1}, \mu_{j-2} = \tilde{\mu}_{j-2}, \mu_i = \tilde{\mu}_i, \mu_0 = \tilde{\mu}_0 = 0.\\
&\nu_{j+1} = \tilde{\nu}_{j+1}, \nu_{j+2} = \tilde{\nu}_{j+2}, \nu_i = \tilde{\nu}_i, \nu_0 = \tilde{\nu}_0 = 0.
\end{aligned}$$

Then we obtain the exact inverse of the comparison matrix $\mu(A)$ of A.

In [3, Theorem 3.12], R. Nabben obtained the following upper bounds of banded M-matrices: Let A be a $(2p+1)-$banded M-matrix. Let $A^{-1} = X = [x_{st}]$. Then for any s, t with $s \in \{(i-1)p+2, \ldots, ip+1\}$ and $t \in \{(j-1)p+2, \ldots, jp+1\}$ ($i = 1$ if $s = 1; j = 1$ if $t = 1$) with $i \neq j$

$$\theta_s^{-1} x_{st} \theta_t \leq \rho^{|i-j|} x_{tt}, \theta_s^{-1} x_{st} \theta_t \leq \rho^{|i-j|} x_{ss},$$

where $\rho = \rho(D^{-1}N)$ is the spectral radius of $D^{-1}N$ with $D = diag(A)$ and $N = D - A$, $\theta = \{\theta_1, \ldots, \theta_n\}$ denotes the eigenvector corresponding to ρ.

Since the bounds in [3, Theorem 3.12] are related to the spectral radius ρ and the eigenvector corresponding to ρ of nonnegative matrices $D^{-1}N$, it is very difficult to calculate the spectral radius and the eigenvector. Then, when A is a nonsingular diagonally dominant pentadiagonal M-matrix, the bounds in Theorem 2.4 are better than their counterparts in [3, Theorem 3.12].

On the other hand, by the following numerical examples, the obtained upper bounds are the exact elements of the inverse of A in Theorem 2.4. Moreover, the upper bounds we obtained are better than the results in [3, Theorem 3.12].

EXAMPLE. We consider the 5×5 nonsingular diagonally dominant pentadiagonal M-matrix with $a_i = 10$ $(i = 1, \ldots, 5)$, $b_i = -2,\ d_i = -3$ $(i = 1, \ldots, 4)$, $c_i = -1,\ e_i = -2$ $(i = 1, \ldots, 3)$.

Denote the upper bounds in [3] and the upper bounds in this paper of A^{-1} by $\hat{V} = (\hat{v}_{i,j})_{5\times 5}$ and $\tilde{V} = (\tilde{v}_{i,j})_{5\times 5}$ respectively. Then we get

$$\begin{aligned} A^{-1} &= \tilde{V} = \begin{pmatrix} 0.1122 & 0.0310 & 0.0454 & 0.0201 & 0.0176 \\ 0.0192 & 0.1165 & 0.0375 & 0.0445 & 0.0201 \\ 0.0278 & 0.0257 & 0.1263 & 0.0375 & 0.0454 \\ 0.0079 & 0.0274 & 0.0257 & 0.1165 & 0.0310 \\ 0.0064 & 0.0079 & 0.0278 & 0.0192 & 0.1122 \end{pmatrix} \\ &< \hat{V} = \begin{pmatrix} 0.1122 & 0.1132 & 0.1104 & 0.0812 & 0.1100 \\ 0.1154 & 0.1165 & 0.1136 & 0.0836 & 0.1132 \\ 0.1284 & 0.1295 & 0.1263 & 0.0930 & 0.1258 \\ 0.0485 & 0.0489 & 0.0477 & 0.1165 & 0.1576 \\ 0.0345 & 0.0348 & 0.0340 & 0.0829 & 0.1122 \end{pmatrix} \end{aligned}$$

3 Numerical Examples

In this section, we consider some examples for different matrices A and compare the entries of A^{-1} with our bounds. Denote the upper bounds in [3] and the upper bounds in this paper of A^{-1} by $\hat{V} = (\hat{v}_{i,j})_{n\times n}$, and $\tilde{V} = (\tilde{v}_{i,j})_{n\times n}$ respectively. Then denote $\tilde{\delta} = \max\{\tilde{v}_{i,j} - |x_{i,j}|\}$, and $\delta = \max\{|x_{i,j}| - v_{i,j}\}$.

EXAMPLE 3.1. In this case, we consider the bounds for a strictly diagonally dominant matrix with $a_i = 10$ $(i = 1, \ldots, n)$, $b_i = -1,\ d_i = -1$ $(i = 1, \ldots, n-1)$, $c_i = 3,\ e_i = -4$ $(i = 1, \ldots, n-2)$. The lower and upper bounds are given as follows:

n	5	50	100	
$\tilde{\delta}$	2.279×10^{-2}	3.800×10^{-2}	3.800×10^{-2}	
δ	$2.226.7766 \times 10^{-2}$	2.500×10^{-2}	2.500×10^{-2}	

EXAMPLE 3.2. In this case, we consider the bounds for a strictly diagonally dominant matrix with $a_i = 5\ (i = 1, \ldots, n)$, $b_i = 1.1, d_i = 0.9\ (i = 1, \ldots, n-1)$, $c_i = 0.9,\ e_i = 1.1\ (i = 1, \ldots, n-2)$. The lower and upper bounds are given as follows:

n	5	50	100	
$\tilde{\delta}$	5.910×10^{-2}	8.360×10^{-2}	8.360×10^{-2}	
δ	6.500×10^{-2}	7.460×10^{-2}	7.460×10^{-2}	

EXAMPLE 3.3. In this case, we consider the bounds for a strictly diagonally dominant matrix with $a_i = 16$ $(i = 1, \ldots, n)$, $b_i = -4.5$, $d_i = 4.5$ $(i = 1, \ldots, n-1)$, $c_i = 3$, $e_i = -3$ $(i = 1, \ldots, n-2)$. The lower and upper bounds are given as follows:

n	5	50	500	
$\tilde{\delta}$	3.990×10^{-2}	7.080×10^{-2}	7.080×10^{-2}	
δ	6.5×10^{-3}	1.200×10^{-2}	1.200×10^{-2}	

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