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Generalized Solutions For Nonlinear Elliptic Equations*

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Abstract

In this paper we prove the existence of generalized solution, the so-called entropy solution, for a class of elliptic problems. We point out that the equations considered here have both quasilinear diffusion term growing quadratically in the gradient, and super linear absorption terms. The main novelty here is that the data belongs to $L^m(\Omega)$ with $\frac{2N}{N+2} \leq m < \frac{N}{2}$ and the nonlinearities have critical growth with respect to the gradient. Moreover, we show here a summability result of the solutions.

1 Introduction

We will be concerned with a class of elliptic equations containing critical growth with respect to the gradient and super linear absorption terms. The equation that we consider is the following

$$-div(A(x, \nabla u)) + a(x)u|u|^{r-1} = \beta(u)|\nabla u|^2 + f(x). \tag{1}$$

Alternatively, we study the limit of approximating equations of the form

$$-div(A(x, \nabla u_n)) + a_n(x, u_n)|u_n|^{r-1} = \beta_n(u_n)|\nabla u_n|^2 + f_n, \tag{2}$$

in a bounded open set $\Omega \in \mathbb{R}^N$, $N \geq 3$, coupled with a Dirichlet boundary condition. If a solution of (2) exists, we prove the convergence of u_n towards a solution u of (1).

There is a wide literature for this kind of problem (see e.g. [1, 4, 5, 7, 8, 9, 13] and the references therein). Let us now point out that if the nonlinearities depend on the gradient with $a = 0$, $\beta = c$, problem (1) is semi-linear, the existence of solutions have been obtained by the change of variable $v = e^u - 1$ provided $f \in L^{\frac{N}{2}}$ (see e.g. [7] and [9]). Existence and uniqueness results for L^1-data are studied in [12]. We recall that for f that belongs to some suitable Lebesgue spaces, existence results have been proved for the elliptic case by many authors, see for instance [1] and [3].

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In this work we consider parameters r, m and N with $r \geq 1$, $N > 2$, $\frac{2N}{N+2} \leq m < \frac{N}{2}$ and prove the existence of generalized solutions: the so-called entropy solution. We will use the techniques of a priori estimates and compactness of approximating solutions, to investigate the existence of solutions. We suppose that f is a measurable function such that $f \in L^m(\Omega)$ with $\frac{2N}{N+2} \leq m < \frac{N}{2}$, $\beta : \mathbb{R} \to \mathbb{R}$ a continuous, nonincreasing integrable function satisfying without loss of generality $\beta(0) = 0$, $a : \mathbb{R} \to \mathbb{R}$ is a real function satisfying $a \in L^\infty(\Omega)$. Moreover, we shall show some summability results of the solutions.

2 Assumptions and Main Results

Let us consider the following problem

$$(P)\begin{cases} -div(A(x,\nabla u)) + a(x)u|u|^{r-1} = \beta(u)|\nabla u|^2 + f(x) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^{\mathbb{N}}$, $N \geq 3$. $A(x;\xi) : R^N \times R^N \to R^N$ is measurable in $x \in R^N$ for any fixed $\xi \in R^N$ and continuous in $\xi \in R^N$ for a.e. $x \in R^N$. There exists a constant $c > 0$ such that for all ξ and a.e. x

$$A(x,\xi).\xi \geq c|\xi|^2. \tag{3}$$

There exist functions $b(x) \in L^2(\Omega)$, and $d(x) \in L^\infty(\Omega)$, such that for all ξ and a.e. x

$$|A(x,\xi)| \leq b(x) + d(x)|\xi|. \tag{4}$$

$$a(.) \in L^\infty(\Omega),\ a(x) \geq a_0 > 0 \text{ a.e in } \Omega. \tag{5}$$

$$\beta \text{ is continuous nonincreasing with } \beta \in L^1(\mathbb{R}). \tag{6}$$

$$f \in L^m(\Omega),\ \text{with } \frac{2N}{N+2} \leq m < \frac{N}{2},\ N \geq 3. \tag{7}$$

Let us note that without loss of generality we can assume that $\beta(0) = 0$. We now introduce some notations and results which will be useful in the sequel. For $k > 0$, we will denote by $T_k(s)$ the truncature at level $\pm k$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\, sign(s) & \text{if } |s| > k. \end{cases} \tag{8}$$

For $k > 0$,we will denote by $G_k(s)$ the function

$$G_k(s) = s - T_k(s) = (|s| - k)^+ sign(s). \tag{9}$$

The gradient of u, denoted by $y = \nabla u$ is such that

$$\nabla(T_k(u)) = y \times 1_{\{|u| \leq k\}} \text{ a.e. in } \Omega. \tag{10}$$

We introduce $T_0^{1,2}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^{1,2}(\Omega)$ for all $k > 0$ (see e.g. [2]). We note that $T_0^{1,2}(\Omega) \cap L^\infty(\Omega) = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

In the following we denote by γ the function $\gamma : R \to R$ defined by

$$\gamma(s) = \int_0^s \beta(r)dr. \tag{11}$$

By a weak solution of (P) we mean a function $u \in W_0^{1,2}(\Omega)$ satisfying $a(.)u|u|^{r-1}$, $\beta(u)|\nabla u|^2 \in L^1(\Omega)$ such that the following equality

$$\int_\Omega A(x, \nabla u)\nabla\varphi + \int_\Omega a(x)u|u|^{r-1}\varphi = \int_\Omega \beta(u)|\nabla u|^2\varphi + \int_\Omega f(x)\varphi, \tag{12}$$

holds for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

An entropy solution of (P) is a function $u \in T_0^{1,2}(\Omega)$ satisfying $a(.)u|u|^{r-1}$, $\beta(u)|\nabla u|^2 \in L^1(\Omega)$ and

$$\begin{aligned} &\int_\Omega A(x, \nabla u)\nabla T_k(u-\varphi) + \int_\Omega a(x)u|u|^{r-1}T_k(u-\varphi) \\ = &\int_\Omega \beta(u)|\nabla u|^2 T_k(u-\varphi) + \int_\Omega f(x)T_k(u-\varphi) \end{aligned}$$

holds for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and for all $k > 0$.

THEOREM 1. Suppose that $f \in L^m(\Omega)$ with $\frac{2N}{N+2} \leq m < \frac{N}{2}$ and $r > 1$. Then there exists an entropy solution of the problem (P).

REMARK 2. This theorem is still valid when $r = 1$. That is the problem (P) turns to the following

$$\begin{cases} a(x)u - div(A(x, \nabla u)) = \beta(u)|\nabla u|^2 + f(x) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

We shall also prove the following summability result of the solution

THEOREM 3. If $\frac{2N}{N+2} \leq m < \frac{N}{2}$ and $r > 1$ then, the entropy solution satisfies

$$u \in L^r(\Omega), \text{ and } |\nabla u| \in L^q(\Omega), \text{ for any } q : 1 \leq q < \frac{N}{N-1}.$$

3 Estimates on General Problems

Let us consider the following regular problem

$$-div(A(x, \nabla u)) + a(x)u|u|^{r-1} = \beta(u)|\nabla u|^2 + f(x) \text{ in } \Omega, \tag{13}$$

$$u|_{\partial\Omega} = 0. \tag{14}$$

If u is a weak solution of (13)-(14), then for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have the following equality

$$\int_\Omega A(x,\nabla u)\nabla\varphi + \int_\Omega a(x)u|u|^{r-1}\varphi = \int_\Omega \beta(u)|\nabla u|^2\varphi + \int_\Omega f(x)\varphi. \tag{15}$$

Let us consider $v = G_k(T_h(u))$ with $k \geq 0$ and $h \geq 0$. By taking $ve^{\gamma(T_k(u))}$ as a test function in (13), we obtain

$$\int_\Omega |\nabla G_k(u)|^2 \leq c\left(||f||_{L^m(\Omega)}, ||a||_{L^\infty(\Omega)}, |\Omega|\right). \tag{16}$$

On the other hand, by taking $\varphi = T_k(u)$ in the weak formulation we obtain

$$\left(\int_\Omega |\nabla T_k(u)|^2\right)^{\frac{1}{2}} \leq c(||f||_{L^m(\Omega)}). \tag{17}$$

Let us also substitute φ by $\frac{1}{k}T_k(u)$ in (15) and for k tending to zero, we obtain

$$\int_\Omega |u|^r dx \leq c\left(||f||_{L^m(\Omega)}, ||a||_{L^\infty(\Omega)}, |\Omega|\right). \tag{18}$$

Finally we see that

$$\int_\Omega |\beta(u)||\nabla u|^2 \leq c(||f||_{L^m(\Omega)}, ||a||_{L^\infty(\Omega)}, ||\beta||_{L^1(R)}, |\Omega|). \tag{19}$$

We consider

$$\varphi_k(s) = \gamma(G_k(s)) = \int_0^{G_k(s)} \beta(\sigma)d\sigma,$$

and

$$\psi_{k,h}(s) = \varphi_k(T_h(s)) = \gamma(G_k(T_h(s))).$$

Let $u_h = T_h(u)$. By taking $e^{\gamma(u_h)}\psi_{k,h}(u_h)$ as a test function in (13), we obtain

$$\begin{aligned} &\int_\Omega \beta(u_h)e^{\gamma(u_h)}\psi_{k,h}(u)A(x,\nabla u)\nabla u_h + \int_\Omega e^{\gamma(u_h)}A(x,\nabla u)\nabla\psi_{k,h}(u) \\ &+ \int_\Omega a(x)u|u|^{r-1}e^{\gamma(u_h)}\psi_{k,h}(u) \\ = \; &\int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u_h)}\psi_{k,h}(u) + \int_\Omega f(x)e^{\gamma(u_h)}\psi_{k,h}(u). \end{aligned} \tag{20}$$

We note that by the monotone convergence theorem and the hypothesis on A, we have

$$\lim_{h\to+\infty} \int_\Omega \beta(u_h)e^{\gamma(u_h)}\psi_{k,h}(u)A(x,\nabla u)\nabla u_h \geq c\int_\Omega \beta(u)|\nabla u|^2 e^{\gamma(u)}\varphi_k(u).$$

Letting h tends to infinity in (20) and applying the Lebesgue dominated convergence theorem, we then get

$$\int_\Omega A(x,\nabla u)\nabla\varphi_k(u)e^{\gamma(u)} \leq c_1 + \int_\Omega f(x)e^{\gamma(u)}\varphi_k(u).$$

Hence, it follows that

$$\int_\Omega |\nabla u|^2 \varphi_k'(u) \le c \int_\Omega f\, \varphi_k(u).$$

It yields that

$$\int_\Omega |\beta(u)||\nabla u|^2 \varkappa_{[|u|\ge k]} \le \int_\Omega f\, \varphi_k(u).$$

$$\int_\Omega |\beta(u)||\nabla u|^2 \le c||f||_{L^m(\Omega)} + \int_{\Omega\cap[|u|<k]} \beta(u)|\nabla u|^2$$

$$\le c + c\int_\Omega |\nabla T_k(u)|^2.$$

Hence (19) follows.

4 Proof of Main Results

From standard results (see e.g. [11]), for all $n \in \mathbb{N}$, there exist $u_n \in H_0^1(\Omega)$ which solves the following problems

$$(P_n) \qquad -div(A(x,\nabla u_n) + a_n(x,u_n)|u_n|^{r-1} = \beta_n(u_n)|\nabla u_n|^2 + f_n \text{ in } \Omega, \tag{21}$$

$$u_n = 0 \text{ on } \partial\Omega, \tag{22}$$

where

$$\beta_n(s) = T_n(\beta(s)),\ f_n = T_n(f),\ \text{ and } a_n(x,s) = a(x)T_n(s).$$

From (17) the sequence u_n is bounded in $T_0^{1,2}(\Omega)$ independently of n. Indeed, Taking $\varphi = T_k(u_n)$ as test function in (P_n) we obtain

$$\int_\Omega |\nabla T_k(u_n)|^2)^{\frac{1}{2}} \le c(||f||_{L^m}. \tag{23}$$

Then, there exists $u \in T_0^{1,2}(\Omega)$ and we can extract a subsequence, still denoted by (u_n), such that

$$T_k(u_n) \to T_k(u) \text{ weakly in } H_0^1(\Omega).\ \forall\, k > 0. \tag{24}$$

$$u_n \to u \text{ a.e. in } \Omega. \tag{25}$$

LEMMA 1. For every $h > 0$, we have

$$\lim_{k\to+\infty} \int_\Omega |\nabla G_k(u_n - T_h(u))|^2 = 0, \tag{26}$$

uniformly on n.

PROOF. Let us note that

$$\int_\Omega |\nabla G_k(u_n - T_h(u))|^2 = \int_{[|u_n - T_h(u)|>k]} |\nabla(u_n - T_h(u))|^2.$$

$$\leq \int_{[|u_n|>k-h]} |\nabla(u_n - T_h(u))|^2.$$

$$\leq 2\left[\int_{[|u_n|>k-h]} |\nabla u_n|^2 + \int_{[|u_n|>k-h]} |\nabla u|^2\right].$$

Now, we deduce that

$$\int_\Omega |\nabla G_k(u_n - T_h(u))|^2 \leq c\left(\int_{[|u_n|>k-h]} |f|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}} + 2\int_{[|u_n|>k-h]} |\nabla u|^2.$$

There then follows (26).

LEMMA 2. For any $k > 0$, we have

$$\lim_{h\to+\infty} \lim_{n\to+\infty} \int_\Omega |\nabla T_k(u_n - T_h(u))|^2 = 0. \tag{27}$$

PROOF. Taking $\psi_{l,k,h}(u) = e^{\gamma(T_l(u))} T_k(u_n - T_h(u))$ as a test function in (21), we obtain

$$\begin{aligned} & \int_\Omega \beta_n(T_l(u_n)) e^{\gamma(T_l(u_n))} \psi_{l,k,h}(u_n) A(x,\nabla u_n)\nabla u_n \\ & + \int_\Omega e^{\gamma(u_l)} A(x,\nabla u_n)\nabla T_k(u_n - T_h(u)) + \int_\Omega a(x) u_n |u_n|^{r-1} \psi_{l,k,h}(u_n) \\ = & \int_\Omega \beta_n(u_n)|\nabla u_n|^2 \psi_{l,k,h}(u_n) + \int_\Omega f_n \psi_{l,k,h}(u_n). \end{aligned}$$

From Lebesgue's dominated theorem and monotone convergence theorem we have, for l tending to infinity

$$\begin{aligned} & \int_\Omega \beta_n(u_n) e^{\gamma(u_n)} T_k(u_n - T_h(u))|\nabla u_n|^2 + \int_\Omega e^{\gamma(u_n)} \nabla u_n \nabla T_k(u_n - T_h(u)) \\ & + \int_\Omega a(x) u_n |u_n|^{r-1} e^{\gamma(u)} T_k(u_n - T_h(u)) \\ \leq & \int_\Omega \beta_n(u_n)|\nabla u_n|^2 e^{\gamma(u)} T_k(u_n - T_h(u)) + \int_\Omega f_n e^{\gamma(u_n)} T_k(u_n - T_h(u)). \end{aligned} \tag{28}$$

Let us consider $\lambda = \sup e^{|\gamma|}$. We remark that

$$\begin{aligned} \lambda \int_\Omega |\nabla T_k(u_n - T_h(u))|^2 &= \lambda \int_{[u_n - T_h(u)|\geq 0]} |\nabla T_k(u_n - T_h(u))|^2 \\ & \quad + \lambda \int_{[u_n - T_h(u)|<0]} |\nabla T_k(u_n - T_h(u))|^2, \\ &\leq \int_\Omega e^{\gamma(u_n)} |\nabla T_k(u_n - T_h(u))|^2. \end{aligned}$$

We get

$$\begin{aligned}\int_\Omega |\nabla T_k(u_n - T_h(u))|^2 &\leq \frac{1}{\lambda}\int_\Omega e^{\gamma(u_n)}\nabla u_n \nabla T_k(u_n - T_h(u)) \\ &\quad -\frac{1}{\lambda}\int_\Omega e^{\gamma(u_n)}\nabla T_h(u)\nabla T_k(u_n - T_h(u)).\end{aligned}$$

We deduce that there exists a constant $c > 0$ such that

$$\begin{aligned}\int_\Omega |\nabla T_k(u_n - T_h(u))|^2 &\leq c\int_\Omega |f_n||T_k(u_n - T_h(u))| \\ &\quad + c\int_\Omega |\nabla T_h(u)||\nabla T_k(u_n - T_h(u))| \\ &\quad - \int_\Omega a(x)u_n|u_n|^{r-1}e^{\gamma(u_n)}T_k(u_n - T_h(u)) \\ &\leq c\left[\int_\Omega |f||u_n - T_h(u)|\right] + \int_\Omega |\nabla T_h(u)||\nabla T_k(u_n - T_h(u))| \\ &\quad + \int_\Omega |u_n|^r|u_n - T_h(u)|.\end{aligned}$$

Since

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\int_\Omega |u_n|^r|u_n - T_h(u)| = 0,$$

and on the other hand, we have $f \in L^{\frac{2N}{N-2}}(\Omega)$ and

$$u_n \to u \text{ weakly in } H_0^1(\Omega),$$

from (24) and Sobolev's embedding theorem, we deduce that

$$u_n \to u \text{ weakly in } L^{\frac{2N}{N-2}}(\Omega).$$

Now, by (25), it yields that

$$|u_n - T_h(u)| \to |u - T(u)| \text{ weakly in } L^{\frac{2N}{N-2}}(\Omega).$$

$$\lim_{n\to+\infty}\int_\Omega |f||u_n - T_h(u)| = \int_\Omega |f||u - T_h(u)|.$$

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\int_\Omega |f||u_n - T_h(u)| = 0.$$

On the other hand, since

$$\begin{aligned}\int_\Omega |\nabla T_h(u)||\nabla T_k(u_n - T_h(u))| &\leq \int_{[|u_n - T_h(u)|<k]} |\nabla T_h(u)||\nabla T_k(u_n - T_h(u))|, \\ &\leq \int_{[|u_n|<k+h]} |\nabla T_h(u)||\nabla T_k(u_n - T_h(u))|,\end{aligned}$$

then from (24), it yields that

$$\lim_{n\to+\infty}\int_\Omega |\nabla T_h(u)||\nabla T_k(u_n - T_h(u))| = 0.$$

Hence (27) is proved.

We remark that

$$\nabla(u_n - u) = \nabla T_k(u_n - T_h(u)) + \nabla G_k(u_n - T_h(u)) - \nabla(u - T_h(u)).$$

From (17), we have

$$||\nabla u - \nabla T_h(u)||_2^2 = \int_{[|u|>h]} |\nabla u|^2 \le c.$$

Then

$$\lim_{h\to+\infty} ||\nabla(u - T_h(u))||_2^2 = 0.$$

Moreover, since

$$\lim_{k\to+\infty} ||\nabla G_k(u_n - T_h(u))||_2^2 = \lim_{k\to+\infty} |\nabla G_k(u_n - T_h(u))|^2 = 0,$$

and

$$\lim_{h\to+\infty}\lim_{n\to+\infty} ||\nabla T_k(u_n - T_h(u))||_2^2 = \lim_{h\to+\infty}\lim_{n\to+\infty}\int_\Omega |\nabla T_k(u_n - T_h(u))|^2 = 0,$$

we obtain

$$\lim_{n\to+\infty} ||\nabla u_n - \nabla u||_2^2 = 0.$$

As a consequence, we have

$$\nabla u_n \to \nabla u \text{ in } L^2(\Omega)^N. \tag{29}$$

It follows that

$$\nabla u_n \to \nabla u \text{ in measure in } \Omega.$$

So that, up to a subsequence

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{30}$$

From (29) and the inequality $|\nabla T_k(u_n)|^2 \le |\nabla u_n|^2$ we deduce that the sequence $(|\nabla T_k(u_n)|^2)_n$ is equi-integrable. Then, from (30) and Vitali's theorem we deduce that

$$T_k(u_n) \to T_k(u) \text{ strongly in } H_0^1(\Omega),\ \forall\, k > 0. \tag{31}$$

As $T_k(u)$ is in $H_0^1(\Omega)$, for all $k > 0$, we deduce that u is in $T_0^{1,2}(\Omega)$.

Finally, using (18) and (25) we deduce by Vitali's theorem that

$$a(x)u_n|u_n|^{r-1} \to a(x)u|u|^{r-1} \text{ in } L^1(\Omega).$$

Having in mind (25) and (30), we obtain

$$\beta_n(u_n)|\nabla u_n|^2 \to \beta(u)|\nabla u|^2 \text{ a.e. in } \Omega.$$

Now, we prove the equi-integrability of the sequence $\beta_n(u_n)|\nabla u_n|^2$. Taking $\varphi_k(u_n) = e^{\gamma(T_h(u_n))}\gamma(G(T_h(u_n)))$ as test function in (P_n) we obtain

$$\int_\Omega \beta_n(u_n)|\nabla u_n|^2 \leq c(||f||_m)(1 + \int_\Omega |\nabla T_K(u_n)|^2.$$

Using (17) we obtain the equi-integrability of the sequence $\beta_n(u_n)|\nabla u_n|^2$.

So we have

$$\beta_n(u_n)|\nabla u_n|^2 \to \beta(u)|\nabla u|^2 \text{ in } L^1(\Omega).$$

We consider $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and take $\varphi = T_k(u_n - v)$ as a test function in the weak formulation of (P_n) . We obtain, for all $n \in \mathbb{N}$, the following equality

$$\begin{aligned} & \int_\Omega A(x, \nabla u_n)\nabla T_k(u_n - v) + \int_\Omega a_n(x)u_n|u_n|^{r-1}T_k(u_n - v) \\ = & \int_\Omega \beta_n(u_n)|\nabla u_n|^2 T_k(u_n - v) + \int_\Omega f_n T_k(u_n - v). \end{aligned}$$

Let us consider $h = k + ||v||_\infty$. Then

$$A(x, \nabla u_n)\nabla T_k(u_n - v) = A(x, \nabla T_h(u_n))\nabla T_k(T_h(u_n) - v).$$

Since

$$T_k(u_n) \to T_k(u) \text{ strongly in } H_0^1(\Omega) \text{ for all } k > 0,$$

we obtain

$$\nabla T_k(T_h(u_n) - v) \to \nabla T_k(T_h(u) - v) \text{ in } L^2(\Omega)^N.$$

$$\nabla T_h(u_n) \to \nabla T_h(u) \text{ in } L^2(\Omega)^N.$$

Hence

$$A(x, \nabla T_h(u_n))\nabla T_k(T_h(u_n) - v) \to A(x, \nabla T_h(u))\nabla T_k(T_h(u) - v) \text{ in } L^1(\Omega).$$

and the proof of the results is complete.

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