1. Let \( M_X(t) \) be the moment generating function of \( X \), and define \( S(t) = \log(M_X(t)) \). Show that
\[
\frac{d}{dt} S(t) \bigg|_{t=0} = \mathbb{E}X \quad \text{and} \quad \frac{d^2}{dt^2} S(t) \bigg|_{t=0} = \text{Var} X.
\]

2. Let \( X \) and \( Y \) be independent \( \mathcal{N}(0, 1) \) random variables, and define a new random variable \( Z \) by
\[
Z = \begin{cases} 
X & \text{if } XY > 0 \\
-X & \text{if } XY < 0.
\end{cases}
\]

(a) Show that \( Z \) has a normal distribution.

(b) Show that the joint distribution of \( Z \) and \( Y \) is not bivariate normal. (Hint: Show that \( Z \) and \( Y \) always have the same sign.)

3. Let \( X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)} \) be the ordered values of a set of \( n \) independent uniform \((0, 1)\) random variables. Prove that for \( 1 \leq k \leq n + 1 \),
\[
P\{X_{(k)} - X_{(k-1)} > t\} = (1 - t)^n
\]
where \( X_{(0)} = 0, X_{(n+1)} = t \).

4. Let \( X_1, \ldots, X_n \) be a random sample from a population with pdf
\[
f_X(x) = \begin{cases} 
1/\theta & 0 < x < \theta \\
0 & \text{otherwise}.
\end{cases}
\]
Let \( X_{(1)} < \cdots < X_{(n)} \) be the order statistics. Show that \( X_{(1)}/X_{(n)} \) and \( X_{(n)} \) are independent random variables.

5. Let \( X_1, \ldots, X_n \) be iid \( \mathcal{N}(\mu, \sigma^2) \). Find the best unbiased estimator of \( \sigma^p \), where \( p \) is a known positive constant, not necessarily an integer.
6. Let $X$ be an observation from the pdf

$$f(x|\theta) = \left( \frac{\theta}{2} \right)^{|x|} (1 - \theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \leq \theta \leq 1.$$ 

(a) Find the MLE of $\theta$.

(b) Define the estimator $T(X)$ by

$$T(X) = \begin{cases} 
2 & \text{if } x = 1 \\
0 & \text{otherwise}.
\end{cases}$$

Show that $T(X)$ is an unbiased estimator of $\theta$.

7. Let $X$ be one observation from a Cauchy($\theta$) distribution. Show that the test

$$\phi(x) = \begin{cases} 
1 & \text{if } 1 < x < 3 \\
0 & \text{otherwise}
\end{cases}$$

is most powerful of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Calculate the Type I and Type II Error probabilities.

8. Let $X_1, \ldots, X_n$ be iid observations from a beta($\theta, 1$) pdf and assume that $\theta$ has a gamma($r, \lambda$) prior pdf. Find a $1 - \alpha$ Bayes credible set for $\theta$.

9. Suppose that $X_1, \ldots, X_n$ are iid Poisson($\lambda$). Find the best unbiased estimator of $e^{-\lambda}$, the probability that $X = 0$.

10. A random sample $X_1, \ldots, X_n$ is drawn from a population with pdf

$$f(x|\theta) = \frac{1}{2}(1 + \theta x), \quad -1 < x < 1, \quad -1 < \theta < 1.$$ 

Find a consistent estimator of $\theta$ and show that it is consistent.