PhD qualifying exam for analysis, Fall 2007

Part I: Real Analysis

1. Let $X$ be a compact Hausdorff space and $C(X)$ be the vector space of real-valued continuous functions on $X$ with norm

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$ 

(a) Show that $C(X)$ with the above norm is complete. (10pts)

(b) A linear functional $\Lambda$ on $C(X)$ is called positive if $\Lambda f \geq 0$ whenever $f \geq 0$. Show that if $\Lambda$ is positive, then $\Lambda$ must be bounded. (10pts)

2. Let $(X, \Omega, \mu)$ be a measure space with $\mu(X) = 1$ and $f$ be measurable on $X$.

(a) Show that $\|f\|_p \leq \|f\|_q$ if $1 \leq p \leq q \leq \infty$. (10pts)

(b) Show that either $\lim_{p \to \infty} \|f\|_p$ exists or $\|f\|_p$ diverges properly to $\infty$. (5pts)

(c) Show that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$ if $\|f\|_\infty < \infty$. (5pts)

(d) Does the conclusion in (c) still hold if $\|f\|_\infty = \infty$? (5pts)

3. Show that there exists a Lebesgue measurable function $f$ on $\mathbb{R}$ such that the set $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ has positive Lebesgue measure for every $g \in C(\mathbb{R})$. (10pts)

4. Let $\Omega$ be a $\sigma$-algebra of subsets of $X$. Show that if $\Omega$ has infinitely many elements, then $\Omega$ is uncountable. Hint: Show that the relation “∼” on $X$ defined by $x \sim y$ if, for each $U \in \Omega$, either $\{x, y\} \subseteq U$ or $\{x, y\} \cap U = \emptyset$, is an equivalent relation on $X$. (15pts)

5. Let $\Gamma$ be a set of real-valued functions on $X$. We say that $A, B \subseteq X$ are separated by $\Gamma$ if there exist $a < b$ and $\varphi \in \Gamma$ such that $\varphi(x) < a$ for all $x \in A$ and $\varphi(y) > b$ for all $y \in B$. We say that an outer measure $\mu^*$ on $X$ is a Carathéodory outer measure with respect to $\Gamma$ if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $A, B$ are separated by $\Gamma$. It is known that if $\mu^*$ is an Carathéodory outer measure on $X$ with respect to $\Gamma$, then every function in $\Gamma$ is $\mu^*$-measurable.

(a) Let $(X, \rho)$ be a metric space and $\mu^*$ be an outer measure on $X$ such that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $\rho(A, B) > 0$. Show that every closed set in $X$ is a $\mu^*$-measurable set. (10pts)
(b) Let \((X, \rho)\) be a metric space and \(\alpha > 0\). For each \(\epsilon > 0\), set
\[
\lambda_{\alpha}^{(\epsilon)}(E) := \inf \sum_i r_i^\alpha, \quad E \subseteq X,
\]
where \(\{r_i\}\) are radii of a sequence of balls \(\{B_i\}\) such that \(r_i < \epsilon\) and \(E \subseteq \bigcup B_i\). Show that \(\lambda_{\alpha}^{(\epsilon)}\) increases as \(\epsilon \to 0\). Defines
\[
m_{\alpha}^*(E) := \lim_{\epsilon \to 0} \lambda_{\alpha}^{(\epsilon)}(E), \quad E \subseteq X.
\]
Show that \(m_{\alpha}^*\) is an outer measure and induces a Borel measure \(m_{\alpha}\) on \(X\). The measure \(m_{\alpha}\) is called the Hausdorff \(\alpha\)-dimensional measure on \(X\). (10pts)

(c) Let \(E\) be a Borel set in \(X\). Show that \(m_\beta(E) = 0\) for all \(\beta > \alpha\) if \(m_\alpha(E) < \infty\) and \(m_\beta(E) = \infty\) for all \(0 < \beta < \alpha\) if \(m_\alpha(E) > 0\). (10pts)

Part II: Complex Analysis

1. Let \(p(z)\) be the polynomial given by
\[
p(z) = \alpha(z - \alpha_1)^{k_1}(z - \alpha_2)^{k_2} \cdots (z - \alpha_n)^{k_n}, \quad \alpha \neq 0,
\]
where \(k_1 + k_2 + \cdots + k_n = \deg p(z)\). Show that if \(\deg p(z) \geq 2\), then
\[
\oint_{\gamma} \frac{1}{p(z)} \, dz = 0
\]
for any piecewise smooth closed curve in \(\mathbb{C}\) if \(\alpha_1, \alpha_2, \ldots, \alpha_n \in \text{inside}(C)\). (10pts)

2. An entire function \(f\) is periodic if there exists \(z_0 \neq 0\) such that \(f(z + z_0) = f(z)\) for all \(z \in \mathbb{C}\). Show that if \(z_1\) and \(z_2\) are both periods of \(f\), then \(f\) is constant if \(z_1/z_2 \notin \mathbb{R}\). (10pts)

3. Show that if \(f(z)\) is analytic on a domain \(D\), then \(f(z)\) must be constant if any one of the following condition is satisfied
   (1) \(f(z)\) is real-valued for all \(z\) in \(D\); (5pts)
   (2) \(\overline{f(z)}\) is analytic on \(D\); (5pts)
   (3) \(|f(z)|\) is constant on \(D\). (5pts)
4. Let \( u(x, y) \) be harmonic on \( H = \{(x, y) : y > 0\} \), and is continuous and bounded upto the \( x \)-axis. Show that

\[
u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(t, 0)dt}{(t-x)^2 + y^2}.\] (10pts)

5. (a) Prove the Casoratti-Weierstrass theorem: If \( z_0 \) is an essential singularity of an analytic function \( g \), then \( g(\{z : |z - z_0| < \varepsilon, z \neq z_0\}) \) is dense for all \( \varepsilon > 0 \). (10pts)

(b) Let \( f \) be an entire function on \( \mathbb{C} \). Show that either \( f \) is constant or \( f(\mathbb{C}) \) is dense in \( \mathbb{C} \). (15pts)

6. Let \( C \) be the positively oriented rectangle

\[\{(x, y) : -3 \leq x \leq 5, \ y = \pm 3\} \cup \{(x, y) : -3 \leq y \leq 3, \ x = -3, 5\}.\]

Evaluate

(a) \( \int_C z^{3/2} \, dz \) (branch cut of \( z^{3/2} \) is \( \{x \leq 0\}) \); (6pts)

(b) \( \int_C \frac{dz}{z^2 + 16}. \) (4pts)

7. Let \( a \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \). Consider the function

\[\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}.\]

(a) Show that \( \varphi_a(z) \) is analytic on \( \mathbb{D} \) by computing \( \varphi'_a(z) \). (5pts)

(b) Show that \( |\varphi_a(z)| = 1 \) if \( |z| = 1 \). (5pts)

(c) Show that \( \varphi_a(z) \) sends \( \mathbb{D} \) one-to-one and onto \( \mathbb{D} \). (10pts)