Department of Applied Mathematics
National Sun Yat-sen University

Ph. D. Qualifying Examination

Fall, 2006.

Real and Complex Analysis

Answer all of the following 10 questions.

1. Show that the most general Möbius transformation from the upper half plane to the unit disk can be written in the form

$$w = e^{i\theta}\frac{z-\mu}{z-\overline{\mu}}.$$

What is the most general Möbius transformation from the unit disk to the unit disk?

2. Prove that

$$I = \int_0^{+\infty} \frac{x^{-1/2}}{1+x}\,dx = \pi.$$

3. Find which quadrants contain the zeros of $p(z) = z^4 + 2z^3 + 3z^2 + z + 2$.

4. (a) Let f be a complex-valued function defined on a region D in $\mathbb{C}$. Prove that if the Cauchy-Riemann equations are satisfied for f and the real part $u(x, y)$ of $f(z)$ $(z = x + iy)$ has continuously differentiable first partial derivatives in D then f is differentiable in D.

   (b) Let $f(z) = \sqrt{|xy|}$ $(z = x + iy)$. Show that the Cauchy-Riemann equations are satisfied for f at $z = 0$, but $f'(0)$ does not exist.

5. Let f be an analytic function in a simply connected regoin D. Let C be the boundary of a rectangle R inside D. Prove that the integral

$$\int_C f(w)\,dw = 0,$$

where C is traced counterclockwisely.

6. Prove that if a subset A of $[0,1]$ has measure zero then

$$A^2 = \{x^2 : x \in A\}$$

has measure zero, too. If f is a continuously differentiable function on $[0,1]$, can you conclude again that $f(A)$ has measure zero? Justify your answer.

7. Suppose that f is a real-valued differentiable function on $[0,1]$. Prove that its derivative function f' is Lebesque measurable on $[0,1]$.

8. Prove that if a real-valued function f is integrable on $[a,b]$ and

$$\int_a^x f(t)\,dt = 0$$

for all x in $[a,b]$ then $f(t) = 0$ a.e. in $[a,b]$.

9. Use Zorn's Lemma, or its equivalences, to construct a non-measurable subset of $[0,1]$.

10. Let $I = [0,1]$ and $m^*(E)$ be the outer (Legesgue) measure of $E \subseteq \mathbb{R}$. Show that E is Lebesgue measurable if and only if

$$1 = m^*(E) + m^*(I \setminus E).$$