(1) The Cauchy density is \( C_u(x) = \frac{1}{\frac{1}{\pi} u^2 + x^2} \), \(-\infty < x < \infty\), for \( u > 0\).

(a) Show that \( C_u \ast C_v = C_{u+v} \), where \((C_u \ast C_v)(y) = \int_{-\infty}^{\infty} C_v(y-x)C_u(x)dx\) is the convolution of \( C_u \) and \( C_v \). (8pts)

(b) Show that if \( X_1, X_2, \ldots, X_n \) are independent and have density \( C_u \), then \((X_1 + X_2 + \cdots + X_n)/n\) has density \( C_u \) as well. (8pts)

(2) The triangular density is

\[
f(x) = \begin{cases} 
1 - |x| & \text{if } x \in (-1, 1) \\
0 & \text{otherwise.}
\end{cases}
\]

Find the characteristic function \( \phi(x) \) of the triangular distribution and show the following inversion formula holds,

\[
\int_{-b}^{b} f(x)dx = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-ibt}}{it} \phi(t)dt. \quad (16\text{pts})
\]

(3) Let \( S_n = \sum_{i=1}^{n} X_i, n = 1, 2, \ldots, \) where \( X_i \)'s are iid random variables. Assume \( \sup_{j \leq N} P(|S_N - S_j| > \alpha) = c < 1 \), show that

\[
P(\sup_{j \leq N} |S_j| > 2\alpha) \leq \frac{1}{1-c} P(|S_N| > \alpha). \quad (12\text{pts})
\]

(4) Assume \( X_i \)'s are iid random variables with \( E(X_i) = m, \) \( E(X_i^2) = \sigma^2 \) and \( E(X_i^4) < \infty \). Let \( S_n = \sum_{i=1}^{n} X_i \), show that

\[
P(\lim_{n \to \infty} \frac{S_n}{n} = m) = 1. \quad (12\text{pts})
\]

(5) If the \( X_i \)'s are uncorrelated and their second moments have a common bound, then \( \frac{S_n - E(S_n)}{n} \to 0 \) a.e., where \( S_n = \sum_{i=1}^{n} X_i \). (13pts)

(6) Let \( X_1, X_2, \ldots \) be iid random variables with the distribution function \( F(\cdot) \) and let \( M_n = \max\{X_1, X_2, \ldots, X_n\} \). Find the limiting distributions of

(a) \( M_n - \alpha^{-1} \log n \), when \( F(x) = 1 - e^{-\alpha x}, x > 0\). (8pts)

(b) \( n^{\frac{\alpha}{2}} M_n \), when

\[
F(x) = \begin{cases} 
1 - x^{-\alpha} & \text{if } x \geq 1 \\
0 & \text{otherwise.} \quad (8\text{pts})
\end{cases}
\]

(7) Consider a random walk on the integers such that \( P_{i,i+1} = p, P_{i,i-1} = q \), for all integer \( i \) \((0 < p < 1, p + q = 1)\).
(a) Show that
\[
P_{0,0}^{2m} = \binom{2m}{m} p^m q^m, \text{ and } P_{0,0}^{2m+1} = 0. (5 pts)
\]

(b) Show the generating function of \( u_n = P_{0,0}^n \), that is \( P(x) = \sum_{n=0}^{\infty} u_n x^n \) equals \( (1 - 4pqx^2)^{-1/2} \). (5 pts)

(c) Show that the generating function of the recurrence time from state 0 to state 0 is \( F(x) = 1 - \sqrt{1 - 4pqx^2} \). (5 pts)