(1) Let $A_1, A_2, \cdots$ be an independent sequence of events in a probability space $(\Omega, \mathcal{F}, P)$ and define the tail $\sigma$-field $T = \cap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \cdots)$. Prove that for each event $A$ in the tail $\sigma$-field $T$, $P(A)$ is either 0 or 1.(12%)

(2) Assume that $X_1, X_2, \cdots$ are iid random variables with $E(X_1) = \infty$. Prove $P(|X_n| \geq 0 \text{ i.o.}) = 1.(12\%)$

(3) Suppose that $A, B$ and $C$ are positive, independent random variables with distribution functions $F$. Show that the quadratic $Ax^2 + Bz + c$ has real zeros with probability(12%)

$$\int_0^\infty \int_0^\infty F(\frac{x^2}{4y})dF(x)dF(y).$$

(4) Suppose that $X_1, X_2, \cdots$ are identically distributed (not necessary independent). Show that $E[\max_{k \leq n} |X_k|] = o(n).(12\%)$

(5) Suppose that $\{X_n\}$ is an independent sequence and $E(X_n) = 0$. If $\sum_{n=1}^{\infty} Var(X_n) < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges with probability 1.(12%)

(6) Find the characteristic functions of the following density functions,

(a) $f(x) = 1 - |x|, -1 < x < 1;(6\%)$

(b) $f(x) = \frac{1 - \cos x}{\pi x^2}, -\infty < x < \infty.(7\%)$

(7) Let $X_1, X_2, \cdots$ be iid random variables with the distribution function $F(\cdot)$ and let $M_n = Max[X_1, X_2, \cdots, X_n]$. Find the limiting distributions of

(a) $M_n - \alpha^{-1} \log n$, when $F(x) = 1 - e^{-ax};(7\%)$

(b) $n^{\frac{1}{\alpha}} M_n$, when $F(x) = 1 - x^{-\alpha}$, if $x \geq 1; = 0$, otherwise.(7%) 

(8) Consider a discrete time Markov chain with states 0,1 and the transition matrix

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}$$

(a) Prove for $n \geq 1 (7\%)$

$$P^n = \frac{1}{a + b} \begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1 - a - b)^n}{a + b} \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.$$

(b) Find the stationary distribution of the chain if it exists.(6%)