Ph.D. Qualifying Examination
Matrix Theory
Sep. 14, 2006

Please write down all the detail of your computation and answers.

1. (10%) Find the third column of the following product matrix

\[
\begin{bmatrix}
\pi & \sqrt{e} & \frac{1}{3} & \sqrt{2} \\
3.7 & 10^5 & 7 & 0 \\
\ln 2 & i & \sin 3 & -1
\end{bmatrix}
\begin{bmatrix}
-\sqrt{3} & 0 & \sqrt{3} \\
0.2 & -0.3 & 0.1 \\
2 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{bmatrix}
\begin{bmatrix}
-1 & 2.3 & 1 & -3 \\
\sqrt{5} & \frac{1}{2} & 1 & 2 \\
3 & -2 & 1 & \sqrt{2}
\end{bmatrix}.
\]

2. (15%) Let \( A \) be an diagonalizable matrix and \( t \) be a parameter. Consider the linear equation \((A - tI)x = b\). (1) Discuss the existence and uniqueness of \( x \) in terms of the values of \( t \) and \( b \), eigenvalues and eigenvectors of \( A \). (2) Find all of its solutions if they exist.

3. (15%) Let \( A \) and \( B \) be two real matrices. Without considering the multiplicity, show that \( AB \) and \( BA \) have the same eigenvalues.

4. (15%) (1) State the Fundamental Theorem of Linear Algebra. (2) Use it to prove the existence of singular value decomposition of any \( m \times n \) complex matrix.

5. (15%) What is the relation between eigenvalues and singular values of a square matrix \( A \) if \( A \) is (1) normal, (2) Hermitian, (3) Hermitian positive definite? State the reasons.

6. (15%) (1) State all the equivalent conditions you know for a matrix to be positive definite. (2) Prove all your conditions are equivalent.

7. (15%) (1) Use the Geršgorin Disk Theorem to prove that a strictly diagonally dominant matrix is nonsingular. (2) Use the nonsingularity of strictly diagonally dominant matrix to prove the Geršgorin Disk Theorem.