**Qualified Examination: Mathematical Programming**
**February 2008**

1. Let $S$ be a nonempty closed convex subset of $R^n$ and $f : S \to R$ be a differentiable convex function. Consider the problem to minimize $f(x)$ subject to $x \in S$. Prove that a point $x \in S$ is an optimal solution to this problem if and only if $\langle f(x), y - x \rangle \geq 0$ for all $y \in S$.

2. Let $A$ be a $p \times n$ matrix and $B$ be a $q \times n$ matrix. Show that exactly one of the following systems has a solution.

   *System 1* $\quad Ax < 0,\ Bx = 0 \quad$ for some $x \in R^n$

   *System 2* $\quad A^t u + B^t v = 0 \quad$ for some $(u, v), u \neq 0, u \geq 0$.

3. Suppose that $\phi : R^n \to R$ is concave.

   a. Show that $\phi$ achieves its minimum at $u$ if and only if

   $$\text{maximum}\{\phi'(u; d) : \|d\| \leq 1\} = 0$$

   b. Show that $\phi$ achieves its maximum over the region $U = \{u : u \geq 0\}$ at $v$ if and only if

   $$\text{maximum}\{\phi'(v; d) : d \in D, \|d\| \leq 1\} = 0$$

   where $D$ is the cone of feasible directions of $U$ at $v$.

4. Consider the nonempty unbounded polyhedral set $S = \{x : Ax = b, x \geq 0\}$ where $A$ is an $m \times n$ matrix of rank $m$. Prove that $S$ has at least one extreme direction.