Ph.D. Qualifying Examination
Functional Analysis

(2008.02)

1. Let $(X, \mathcal{T})$ be a topological vector space and M be a closed linear subspace of X. Define the quotient map $Q : X \to X/M$ by $Q(x) = x + M$ for all $x \in X$. Recall that the quotient topology $\mathcal{T}_M$ on X/M is the collection

$$\mathcal{T}_M \equiv \{U \subseteq X/M \ : \ Q^{-1}(U) \in \mathcal{T}\}.$$

   (a) Prove that Q is a continuous open mapping from X onto X/M. (5%)

   (b) Prove that $(X/M, \mathcal{T}_M)$ is a topological vector space. (10%)

2. Give both $C^{(1)}([0,1])$ and $C([0,1])$ the supremum norm.

   (a) If $D : C^{(1)}([0,1]) \to C([0,1])$ is defined by $Df = f'$ for all $f \in C^{(1)}([0,1])$, determine whether D is continuous. Verify your answer. (5%)

   (b) If $T : C([0,1]) \to C([0,1])$ is defined by $(Tf)(x) = \int_0^x f(t)dt$ for all $f \in C([0,1])$ and $x \in [0,1]$. Determine whether T is continuous. If it is, prove it and find $\|T\|$. (10%)

3. Let X be a normed space over $\mathbb{C}$ and Y be a finite-dimensional space over $\mathbb{C}$ with supremum norm. If K is a subspace of X and $f : K \to Y$ is a bounded linear operator, prove that f can be extended to a bounded linear operator F from X into Y such that $\|F\| = \|f\|$. (10%)

4. Let S be a linear subspace of $L^q([0,1])$ that is closed as a subspace of $L^p([0,1])$, where $1 < p < q < \infty$. Let $\{f_n\}_{n=1}^\infty$ be a sequence in S. Prove that $\{f_n\}_{n=1}^\infty$ is convergent in $(L^q([0,1]), \|\cdot\|_q)$ if and only if $\{f_n\}_{n=1}^\infty$ is convergent in $(L^p([0,1]), \|\cdot\|_p)$. (15%)

5. If T is a bounded linear operator on a complex Hilbert space H and $\alpha \in \mathbb{C}$, show that α is not in the spectrum of T if and only if the range of $T - \alpha I$ is dense in H and there exists $c > 0$ such that $\|(T - \alpha I)h\| \geq c\|h\|$ for all $h \in H$. (15%)

6. For each $n \geq 1$, let H_n be a complex Hilbert space and T_n a bounded linear operator on H_n. If $\sup_n \|T_n\| < \infty$ and $T = \oplus_{n=1}^\infty T_n$ on $H = \oplus_{n=1}^\infty H_n$.

   (a) Prove that $\|T\| = \sup_n \|T_n\|$. (5%)

   (b) Prove that T is compact if and only if each T_n is compact and $\|T_n\| \to 0$. (10%)

7. If $\mathcal{A}$ is a C^*-algebra with identity and a is a normal element of $\mathcal{A}$, show that $a = a^*$ if and only if $\exp(ia) = u$ is unitary. (15%)