Ph.D. Qualifying Examination: Algebra

Feb. 2008

<u>Answer all the problems and show all your works.</u>

We use the following notations:
$\mathbb{N}$: the set of all positive integers,
$\mathbb{Z}$: the set of all integers.

[20 pts] 1. (a) Show that no group of order 56 is simple.

(b) Let G be a finite group of order np with $p > n$ and p prime. If H is a subgroup of order p, show that H is normal in G. (Hint: Consider the actions of G on the set of all cosets of H.)

[10 pts] 2. For any $m \in \mathbb{N}$, let $\mathbb{Z}_m$ denote the set of all residue classes modulo m in $\mathbb{Z}$. Prove or disprove that there is an isomorphism of groups from $\mathbb{Z}_m \times \mathbb{Z}_n$ to $\mathbb{Z}_{[m,n]} \times \mathbb{Z}_{(m,n)}$ for all $m, n \in \mathbb{N}$, where (m, n) and $[m, n]$ are the greatest common divisor and least common multiple of m and n, respectively.

[15 pts] 3. Let $I_1, I_2, \cdots, I_n$ be ideals in a commutative ring R with identity.

(a) Show that if $I_i + I_j = R$, for all $i \neq j$, then $I_k + (\bigcap_{j \neq k} I_j) = R$, for all k.

(b) Let $\phi : R \to \prod_{i=1}^{n} R/I_i$ be a homomorphism defined by $\phi(x) = (x+I_1, x+I_2, \cdots, x+I_n)$. Show that is ϕ is surjective if and only if $I_i + I_j = R$, for all $i \neq j$.

[20 pts] 4. (a) Show that $\mathbb{Z}[\sqrt{-1}]$ is a Euclidean domain.

(b) Find a greatest common divisor of $2 + 6\sqrt{-1}$ and $39 + 52\sqrt{-1}$ in $\mathbb{Z}[\sqrt{-1}]$.

[15 pts] 5. Let D be a principal ideal domain.

(a) Show that every prime ideal in D is a maximal ideal.

(b) Show that every irreducible element in D is prime.

[20 pts] 6. Let E be a splitting field over $\mathbb{Q}$ of the equation $f(x) = x^4 - 5$, where $\mathbb{Q}$ is the field of all rational numbers.

(a) Find the splitting field E.

(b) Determine the Galois group of E over $\mathbb{Q}$.

(c) Find all the intermediate fields K between E and $\mathbb{Q}$ satisfying $[E : K] = 2$.

Total: 100 points