ON THE GENERAL WISHART DISTRIBUTION ON HOMOGENEOUS CONES.

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The classical Wishart distributions were introduced by J. Wishart in 1928 as distributions of the maximum likelihood (ML) estimator of a completely unknown covariance matrix Σ in a sample of i.i.d. multivariate normal observations from R^I, where I is a finite set.. Thus the Wishart distributions live on $\mathbf{P}(I)$, the cone of positive definite $I \times I$ matrices. One specific Wishart distribution is denoted $W_{\lambda,\Sigma}$, where $\lambda > \frac{|I|-1}{2}$ is the *shape parameter* and $\Sigma \in \mathbf{P}(I)$ is the *multivariate scale parameter*. Thus the expectation $E(W_{\lambda,\Sigma}) = 2\lambda\Sigma$. When the degrees of freedom $f := 2\lambda$ is a noninteger there is no reference to a normal distribution and the Wishart distributions have a "life of their own".
Many classical and recently developed statistical models in multivariate statistical analysis involve inference in the covariance matrix, i.e., the parameter Σ is restricted to a subset $\Theta \subseteq \mathbf{P}(I)$. Some of the important features of Θ are that it is a subcone (or it can at least be nicely parametrized by a cone) and it is a homogeneous space under a natural action of a Lie group. This leads directly into a generalization of the classical Wishart distributions to the so-called general Wishart distributions on homogeneous cones. It is of course also important to investigate which results about the classical Wishart distributions can be generalized to this new class of distributions on homogeneous cones.

Finite dimensional cones are much more complicated than for example vector spaces. For each positive integer, the dimension n, there exist up to an isomorphism only one vector space R^n. For cones there exist non-isomorphic cones of the same dimension. This is even true for homogeneous cones. For example in dimension 11 all non isomorhic **homogeneous** cones can be parametrized by the real numbers. The classification of homogeneous cones is far from complete. Nevertheless the representation of homogeneous cones as the positive elements in a T-algebra provide the possibility to construct the homogeneous cones and their Wishart distributions in a more explicit way. This representation of **homogeneous cones** was developed by E.B. Vinberg, 1960-68.

Several specific homogeneous cones with the corresponding Wishart distributions are open for detailed studies and possible application in normal statistical models. For example several of the normal graphical models relates directly to the general Wishart distribution. Also the inverse general Wishart distribution provides a needed class of prior distributions in the graphical model framework.