學習漫談

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Mathematics

No Nobel Prize in Math!

algebra

number theory : many puzzles, seems naive!

mathematical physics : understand the physical world.

以自然为师

analysis

力學，电磁學，量子力學，統計力學

飛机，雷達

(math. methods)

釘打机

geometry . **Differential Geometry** : shape, ...

廣義相对論

motivation

由直象着手

一方 o.D.E 或 p.d.e 的物理意義

Number Theory : Pearl in Mathematics

質數 primes $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \cdots$

Fundamental Theorem in Arithmetic 算術基本定理

$\forall\ n \in \mathbb{Z}$ and $n \geq 2$

$\Rightarrow \quad n = \prod_{1}^{\ell} p_i^{\alpha_i}$ and the factorization is unique if we require $p_1 < p_2 < \cdots < p_\ell$

Euclid $\#$ primes $= \infty$ like elements in physics

$\{4n-1 : n = 1, 2, 3, \cdots\} = \infty$ $\{3, 7, 11, 15, 19, 23, \cdots\}$

$\#$ Primes in $\{4n+1 : n = 1, 2, 3, \cdots\} = \infty$ $\{5, 9, 13, 17, 21, \cdots\}$

But

Dirichlet (1837) $\#$ Primes in $\{an+b : n = 1, 2, \cdots\} = \infty$. $(a, b) = 1$ 互質

?? $\#$ Primes in $\{n^2+1 : n = 1, 2, \cdots\} = \infty$

分佈不規則

Define $\pi(x) \equiv \#\{p : p \le x\}$

Prime Number Theorem

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log_e x}} = 1$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \forall x$$

conjectured by Gauss in 1792, age 15.

eventually proved by Hadamand & Poussin in 1896

Riemann - Zeta Function : $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s \in \mathbb{C}$

複變函數論 (beautiful)

$\zeta(1) = \infty$. PNT holds $\Leftrightarrow$ $\zeta(1+it) \neq 0 \ \forall t \neq 0$

uniqueness thm

Riemann Hypothesis

$$\{ s : \zeta(s) = 0 \} \subseteq \{ \tfrac{1}{2} + it : t \in R \}$$

Goldbach, in a letter to Euler in 1742, conjectured

every even integer $2n$ = sum of two primes.

e.g $30 = 7 + 23$, $40 = 11 + 29$

best result so far: $n = p + \begin{cases} p' \\ p_1 \cdot p_2 \end{cases}$ 陳景潤 (1973)

(華羅庚的學生)
著有"數論導論"
(也有英文版)

Twin Prime Conjecture

$$\pi_2(x) = \#\{ 1 \le \overset{\text{prime}}{p} \le x : p, p+2 \text{ are primes} \}$$

e.g $(3,5)$, $(5,7)$, $(11,13)$, $(17,19)$, $(29,31)$, ...

$$\lim_{x \to \infty} \pi_2(x) = \infty \quad ??$$

conjecture: $\pi_2(x) \sim \text{const.} \dfrac{x}{(\log x)^2}$

Bertrand's Postulate (proved by Chebyshev)

By PNT,

$$\lim_{n} \frac{\pi(n)}{n} = 0$$

Primes are sparse.

Sieve

$\forall\ n \geq 1,\quad \exists$ prime p such that $n < p \leq 2n$

$\Rightarrow \quad \pi(2n) - \pi(n) \geq 1$ i.e. $\exists\ p \in (n, 2n]$ $\forall\ n \geq 1$

Hoheisel (1930) $\exists\ \theta < 1$ such that

$$\pi(n + n^{\theta}) - \pi(n) \sim \frac{n^{\theta}}{\log n}$$

$$\theta = 1 - \frac{1}{33000} + \varepsilon \quad (\text{for any } \varepsilon > 0)$$

Record: R. Baker & G. Harman (1995)

$$\theta = 0.535\cdots + \varepsilon$$

$3x+1$ Problem by Collatz

open for over 50 years

1. if $2 \mid n$ ~~then~~ then $n \to \frac{n}{2}$

   if n odd then $n \to 3n+1$

The rule is simple

everything is simple at the beginning.

2. Repeat step 1

Conjecture: starting from any $n \in N$, $n \to \cdots \to 4 \to 2 \to 1$

eg $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5$
$\to 16 \to 8 \to 4 \to 2 \to 1$

☆ P. Halmos :

if you cannot reach the n-th step of the ladder,
try the (n-1)th step first.

Normal Number

$x \in (0,1)$: $x = 0.a_1 a_2 a_3 \cdots\cdots$ 十進位展開 (p)

Let $N_n(x,j) = \frac{1}{n} \# \{ 1 \le k \le n : a_k(x) = j \}$ digit j 出現的頻率 $0 \le j \le 9$

Borel 幾乎所有的數都是 normal.

$$\left| \left\{ x \in (0,1) : \lim_{n\to\infty} N_n(x,j) = \frac{1}{10} \quad \forall\, 0 \le j \le 9 \right\} \right| = 1$$

Moreover, $\forall$ string $c_1 c_2 c_3 \cdots, c_{l+1}$ e.g. $c_1 c_2 \cdots c_{l+1} = \overbrace{1\,1 \cdots 1\,1}^{l+1}$

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ l+1 \le k \le l : a_k(x)\, a_{k+1}(x) \cdots\cdots a_{k+l+1}(x) = c_1 c_2 \cdots c_{l+1} \right\} = \frac{1}{10^{l+1}}$$

$\boxed{\text{Is } \pi \text{ normal?}}$

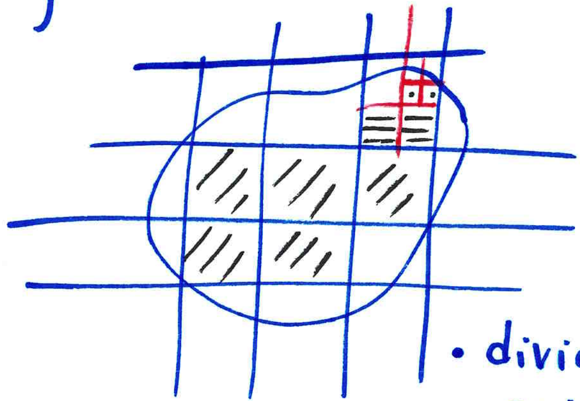
Conclusion Even a dog can write "The Shakespere Complete"!

(If the dog lives long enough!)

Strong Law of Large Numbers

⑥ Calculus $\longrightarrow$ Advanced Calculus $\longrightarrow$ complex variable analysis $\longrightarrow$ real analysis *

How to measure the length / area of a set ?



- divide and conquor
- by weight

measure

$m : A \longrightarrow \mathbb{R}^+$

① $0 \le m(A) \le \infty \quad \forall A \subseteq \mathbb{R}$ or $[0,1]$ or unit circle

② $m(A \dotplus x) = m(A)$

translation invariant

③ $m([a,b]) = b - a$

consistency

④ countable additivity :

if $A_1, A_2, \cdots$ are mutually disjoint, then

$$m\left(\bigcup_{1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i) = m(A_1) + m(A_2) + \cdots$$

① − ④ cannot hold simultaneously

② − ④ + ① for a special family of subsets.

Inner measure of a set A

$$m^*(A) = \inf \left\{ \sum_{j=1}^{\infty} |I_j| \;:\; I_j \text{ is an open interval, e.g. } I_j = (a_j, b_j),\ \bigcup_{j=1}^{\infty} I_j \supseteq A \right\}$$

greatest lower bound

an open covering

- $m^*\left(\begin{matrix}(a,b)\\ [a,b]\end{matrix}\right) = b-a$

- $m^*(A) \le m^*(B)$ if $A \subseteq B$ Monotonicity 單調性

- if $A_1, A_2, \ldots$ are mutually disjoint, then

$$m^*\left(\bigcup_{1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} m^*(A_j)$$

physics

daily phenomena

element

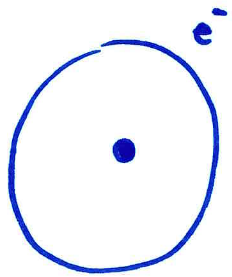
$\left\{\begin{array}{l}\text{Ordinary differential equations} \\ \text{partial d. e.}\end{array}\right.$

derivation : $\boxed{\text{from micro to macro}}$

电子, 质子, 中子

Quantum mechanics

Schrödinger equation



条件 elementary particles

the rules should be simple!

古典 analog

电子雲

$\left\{\begin{array}{l}\text{理論} \\ \text{实验 : 加速器}\end{array}\right.$

硬碰

Einstein

統一場論

电磁力, 弱/強 作用力

重力

$$\underline{\underline{i}}\,\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

Planck constant

$= 1.05457 \times 10^{-34}$ J·sec

not derivable

not a final theory

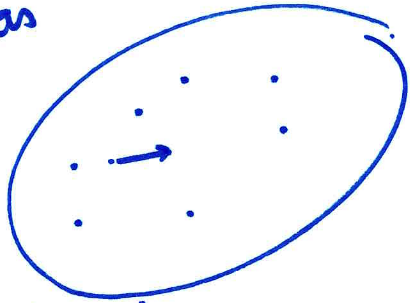
simple rules — complicated

From Micro $\longrightarrow$ Macro

壓力 體積 溫度

ideal gas equation $\quad PV = nRT$ 經驗公式

gas



統計力學

molecules

1 mole $= 6.02 \times 10^{27}$

In reality — Van der Waals $\longrightarrow$ 工程 \$

1910 Nobel Prize in physics

修正 P

table

by experiments !

經驗法則

physics   not separated from math in the past

Gauss 天文台台長. 測量學. 誤差處理,

$X_1, X_2, \cdots\cdots; X_n$ are independent identical distributed $\quad \text{Exp}(X_1) = 0$.

e.g 誤差

$\mathbb{P}(X_1 = \pm 1) = \frac{1}{2}$ in gambling

(遊戲; toss a coin

SLLN

$$\lim_n \frac{X_1 + X_2 \cdots + X_n}{n} = 0 \quad \text{almost surely}$$

modern probability 1933

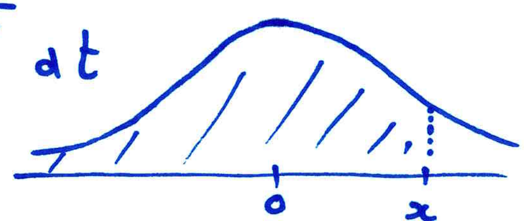
Kolmogorov

Weak LLN $\quad \forall\ \varepsilon > 0, \quad \mathbb{P}\left(\left| \frac{X_1 + X_2 + \cdots + X_n}{n} \right| \geq \varepsilon \right) \xrightarrow{n} 0.$

Central Limit Theorem. Assume $\text{Exp}(X_1^2) = \sigma^2$ as well

Gauss 分佈

$$\lim_n \mathbb{P}\left(\frac{X_1 + X_2 \cdots + X_n}{\sqrt{n\,\sigma^2}} \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$



X_n are i.i.d with $P\left(X_n = \begin{matrix} 1 \\ 0 \end{matrix}\right) = \begin{matrix} p \\ 1-p \end{matrix}$ (on fire)

$$\mathrm{Exp}(X_1) = 1 \cdot p + 0 \cdot (1-p) = p \Rightarrow \mathrm{Exp}(X_1 - p) = 0$$

$$\mathrm{Exp}\, X_1^2 = 1^2 \cdot p + 0^2 (1-p) = p \qquad \sigma^2 = \mathrm{Exp}(X_1 - p)^2 = p(1-p)$$

By SLLN

$$\lim_n \frac{X_1 + X_2 \cdots + X_n}{n} = p \Rightarrow \sum_1^n X_i \approx np$$

total # of houses on fire in 1 yr.

$$(p-\varepsilon)n \leq \sum_1^n X_i \leq (p+\varepsilon)n$$

for n large

How to calculate the 保險費 ?

Need to know better about $\sum_1^n X_i \Longrightarrow$ CLT

Demoivre - Laplace Theorem

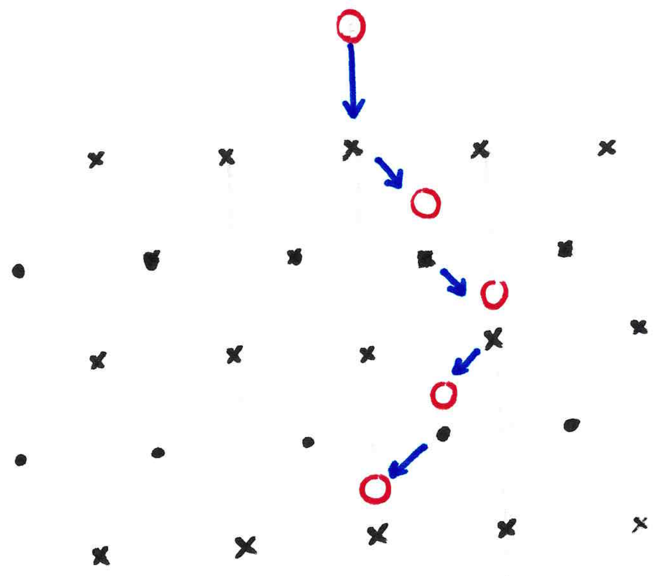
$$P\left(\sum_1^n X_i = k\right) = C\binom{n}{k} \cdot p^k (1-p)^{n-k},$$

k success in n trials

$$C\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)\cdots 1} \frac{(n-k)\cdots 3\cdot 2\cdot 1}{(n-k)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{n!}{k!(n-k)!}$$

stirling formula for $n!$



$X_1 = 1$

$X_2 = 1$

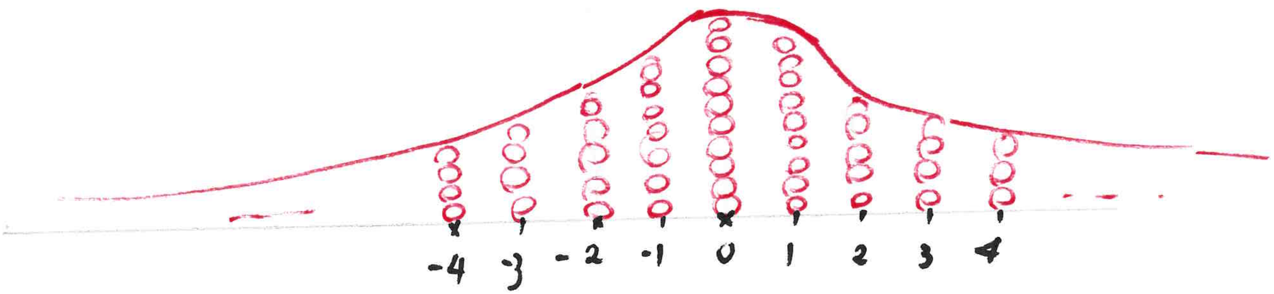
$X_3 = -1$

$X_4 = -1$

$$S_n = X_1 + X_2 + \cdots + X_n$$

position at time n

play many times :



Random Walk : Brownian Motion

Einstein (1905)

<u>Newton</u> inventor of Calculus. Motivation

$$\vec{F} = m\vec{a} \quad \& \quad \vec{F} = G\frac{m_1 m_2}{r^2}\cdot\hat{r} \quad \text{吸引力}$$

Let $\vec{X}(t) = (x(t),\ y(t),\ z(t))$: position of a particle (an object) at time t

average velocity at time t. $= \dfrac{x(t+\Delta) - x(t)}{\Delta} \qquad \dfrac{\text{位移}}{\text{時間}}$

instaneously velocity at time $t \equiv \lim\limits_{\Delta\to 0} \dfrac{x(t+\Delta)-x(t)}{\Delta} \equiv x'(t) \equiv \dfrac{dx}{dt} \equiv v_x(t)$

average acceleration at time t in x-direction : $= \dfrac{v_x(t+\Delta) - v_x(t)}{\Delta} \xrightarrow{\Delta\to 0} x''(t) \equiv a_x(t)$

Under gravitation only :

$$\vec{F} = m(0,\ 0,\ -g) = m\vec{a} = m(x''(t),\ y''(t),\ z''(t))$$

$9.80\ m/\sec^2$ O.D.E.

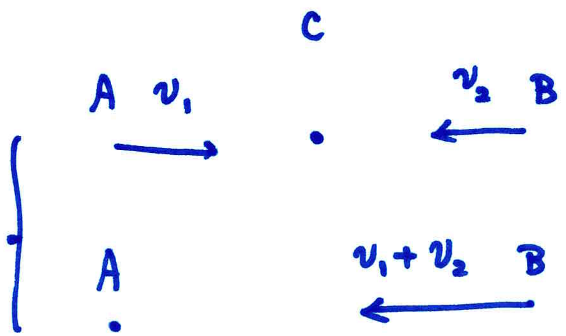
$$\left.\right\} \quad z''(t) = -g \Rightarrow z(t) = z_0 + v_{0,z}t - \frac{gt^2}{2}$$

Einstein's Special Theory of Relativity

Postulates 1. Physical Laws are the same in all inertial reference systems.

e.g $\vec{F} = m\vec{a}$

2. 光速 c is a universal constant



古典 or $v_1, v_2 << c$

◎ space and time : Lorentz transformation

ℓ

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

. . . .

$E = mc^2$

H. Poincare credit

D. Hilbert

23 problems proposed in 1900 at ICM

◎ General Theory of Relativity

Geometry & finite universe ?

Minkovsky (天才)

金庸：射鵰英雄傳

macro → micro

生物學 Cell → DNA

Not the end!

$E=mc^2$ → 核武 → ------ 世界大戰

成住壞空

DNA → 治病 → 複製人 ...

age of the universe

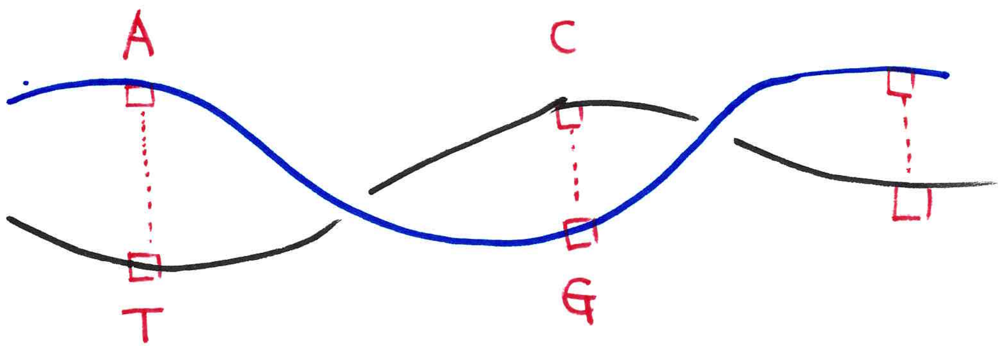
⇓

人類的文明

energy ← 太陽 fusion

生命 $< \infty$

Don't worry! 太久遠之事



only 4 symbols

molecular biology is easy!

double helix

more secure

RNA

finer structure (coiled)

儀器 based on physics

Life is complicated!

matter 身

mind 心

AI artificial intelligence $\xrightarrow{\text{if yes}}$ 机器会敵

scientists
strong will

将"不可能"变成"可能"

a thrilling challenge

many benefits!

Feynman : elements

心经