注意事項：

1. 本試卷共四大題，每大題10分。
2. 考試時間： 9：30-11：00。
3. 請將姓名及報名編號寫在每頁頁尾指定處。
4. 請詳列計算過程書寫於題目下方空白處，並將答案書寫於右下方指定處。

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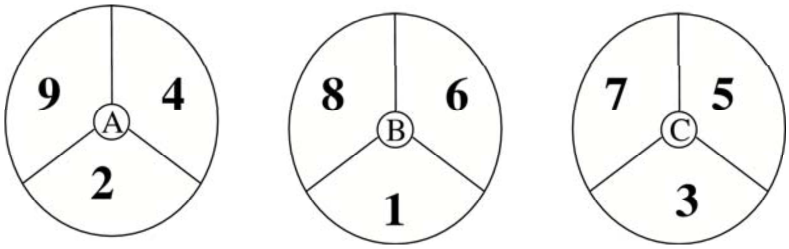
1. Two players play the following game. Player I chooses one of the three spinners pictured in the following figure, and then player II chooses one of the remaining two spinners. Both players then spin their spinner and the one that lands on the higher number is declared the winner. Assuming that each spinner is equally likely to land in any of its 3 regions, would you rather be player I or player II? Explain your answer! Ans: player II



解答: To be player II, the reason is as follows.

If player I chooses spinner A and player II chooses spinner B, then all results for spinner A and spinner B are

$$(9,8),(9,6),(9,1),(4,8),(4,6),(4,1),(2,8),(2,6),(2,1).$$

So the probability for player II being the winner is 4/9. Similarly, if player I chooses spinner A and player II chooses spinner C, then the probability for player II being the winner is 5/9. Therefore, if player I chooses spinner A, to choose spinner C is optimal for player II and the probability for player II being the winner is 5/9.

Similarly, if player I chooses spinner B, to choose spinner A is optimal for player II and the probability for player II being the winner is 5/9; if player I chooses C, to choose spinner B is optimal for player II and the probability for player II being the winner is 5/9. □

2. Let (X,Y) have pdf

$$f(x,y)=\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}x^{a-1}y^{b-1}(1-x-y)^{c-1},\quad 0<x<1, 0<y<1, 0<y<1-x<1,$$

where $a>0$, $b>0$, and $c>0$ are constants.

(a) Find the marginally distribution of X and Y. Ans: $X\sim \text{beta}(a,b+c)$; $Y\sim\text{beta}(b,a+c)$

(b) Find the conditional distribution of $Y|X = x$, and show that $\frac{Y}{1-X}$ is beta(b, c).
Ans: $\frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} y^{b-1}(1-x)^{1-b-c}(1-x-y)^{c-1}$

解答:

(a)

$$
\begin{aligned}
f_X(x) &= \int_0^{1-x} \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} y^{b-1} (1-x-y)^{c-1}\,\mathrm{d}y \qquad (\text{let } z = \tfrac{y}{1-x}) \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} (1-x)^{b+c-1} \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1} (1-x)^{b+c-1}, \quad 0 < x < 1
\end{aligned}
$$

Thus $X \sim$ beta$(a, b+c)$.

$$
\begin{aligned}
f_Y(y) &= \int_0^{1-y} \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} y^{b-1} (1-x-y)^{c-1}\,\mathrm{d}x \\
&= \int_0^{1-y} \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} y^{b-1} (1 - \frac{x}{1-y})^{c-1} (1-y)^{c-1} x^{a-1}\,\mathrm{d}x \quad (\text{let } z = \tfrac{x}{1-y}) \\
&= \int_0^{1-y} \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} y^{b-1} (1-z)^{c-1} (1-y)^{c-1} (1-y)^{a-1} z^{z-1} (1-y)\,\mathrm{d}z \\
&= \int_0^{1-y} \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} y^{b-1} (1-y)^{a+c-1} z^{a-1} (1-z)^{c-1}\,\mathrm{d}z \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} y^{b-1} (1-y)^{a+c-1} \frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)} \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(a+c)} y^{b-1} (1-y)^{a+c-1}, \quad 0 < y < 1
\end{aligned}
$$

Thus $Y \sim$ beta$(b, a+c)$.

(b)

$$
\begin{aligned}
f(y|x) &= \frac{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} y^{b-1} (1-x-y)^{c-1}}{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1} (1-x)^{b+c-1}} \\
&= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} y^{b-1} (1-x)^{1-b-c} (1-x-y)^{c-1}
\end{aligned}
$$

$$
\text{Let} \begin{cases} U = \frac{Y}{1-X} \\ V = 1 - X \end{cases} \Rightarrow \begin{cases} X = 1 - V \\ Y = UV \end{cases} \Rightarrow |J| = \left\| \begin{matrix} 0 & -1 \\ v & u \end{matrix} \right\| = v.
$$

$$
\begin{aligned}
f_{U,V}(u,v) &= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} (1-v)^{a-1} u^{b-1} v^{b-1} (v-uv)^{c-1} v \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} (1-v)^{a-1} v^{b+c-1} u^{b-1} (1-u)^{c-1}, \quad 0 < uv < v < 1 \\
f_U(u) &= \int_0^1 \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} (1-v)^{a-1} v^{b+c-1} u^{b-1} (1-u)\,\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}(1-u)^{c-1}u^{b-1}\frac{\Gamma(b+c)\Gamma(a)}{\Gamma(a+b+c)} \\
&= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)}u^{b-1}(1-u)^{c-1}, \quad 0<u<1
\end{aligned}
$$

Thus $U \sim$ beta(b,c). □

3. Let X_1, X_2, X_3, X_4 be i.i.d. r.v. with common distribution $N(0,1)$. Find the moment generating function of $Z = X_1X_2 + X_3X_4$. Ans: $(1-t^2)^{-1}$, $t^2<1$

解答: Since X_1X_2 and X_3X_4 are i.i.d., so

$$E\big[e^{tZ}\big] = E\big[e^{t(X_1X_2+X_3X_4)}\big] = \big(E\big[e^{tX_1X_2}\big]\big)^2.$$

Moreover, X_1, X_2 are also i.i.d. with $N(0,1)$ distribution, we have

$$E\big[e^{tX_1X_2}\big] = E\big[E\big[e^{(tX_1)X_2}|X_1\big]\big] = E\Big[e^{t^2X_1^2/2}\Big].$$

Now let $Y = X_1^2$, then Y has χ_1^2 distribution, so

$$E\Big[e^{t^2X_1^2/2}\Big] = E\Big[e^{(t^2/2)Y_1}\Big] = \frac{1}{(1-2(t^2/2))^{1/2}} = \frac{1}{(1-t^2)^{1/2}}, \quad t^2<1.$$

It yields $E\big[e^{tZ}\big] = (1-t^2)^{-1}$, $t^2<1$. □

4. Consider a random sample $\{X_1, X_2, \ldots, X_n\}$ from a Pareto distribution with the following density function

$$f(x) = \kappa(1+x)^{-\kappa-1}, \quad x>0.$$

   (a) Find the maximum likelihood estimator of the parameter κ based on the sample. Ans: $n/\sum_{i=1}^n \ln(1+x_i)$

   (b) Find the Cramer-Rao lower bound (CRLB) of any unbiased estimator of κ.Ans: κ^2/n

解答:

(a) Since the log likelihood function of κ is

$$\ln L(x) = n\ln\kappa - (\kappa+1)\sum_{i=1}^n \ln(1+x_i),$$

by setting $\frac{\partial}{\partial x}\ln L(x) = 0$, we can obtain the MLE

$$\hat{\kappa} = n/\sum_{i=1}^n \ln(1+x_i).$$

(b) Since $\frac{\partial}{\partial x}\ln f(x) = 1/\kappa - \ln(1+x)$, CRLB $= 1/nE[1/\kappa - \ln(1+X)]^2$. Note that

$$Y = \ln(1+X) \sim \exp(1/\kappa)$$

$$\begin{aligned} E[\ln(1+X)] &= 1/\kappa E[1/\kappa - \ln(1+X)]^2 \\ &= \text{Var}(\ln(1+X)) = 1/\kappa^2 \end{aligned}$$

thus CRLB $= \kappa^2/n$. □

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