九十六學年度應用數學系碩士班丙組甄試

口試時請自選題目中的高等微積分(Advanced Calculus)一題及線性代數(Linear Algebra)一題講解，並回答口試委員所提的問題。若有不清楚之處可詢問帶領的人員。

Advanced Calculus

1. Find the Taylor series for $f(x) = xe^x$ about $x = 1$. Also, find the radius of convergence of that series.

2. Let $a_n > 0$ and $\sum a_n < \infty$. Prove or disprove: $\sum \frac{\sqrt{a_n}}{n^p} < \infty$ if $p > 1/2$.

3. Let, for $n = 1, 2, 3, \ldots$, $f_n(x) = x^n$, $x \in [0, 1]$. Does f_n converge pointwise on $[0, 1]$? Does f_n converge uniformly on $[0, 1]$?

4. Let f be a real continuous function on a metric space $\mathbf{S}$. Let $P(f) = \{s \in \mathbf{S} | f(s) > 0\}$. Is $P(f)$ a closed subset, open subset, or neither?

5. Let f be a continuous function on $[0, 1]$. Show that f is Riemann-integrable over $[0, 1]$.

Linear Algebra

Let $\Re$ be the set of all real numbers.

1. Let A, B be $n \times n$ matrix over $\Re$. We denote the rank of matrix X by rank(X), the transpose of matrix X by X^t. Prove or disprove the followings:
   (a) rank$(A + B) \leq$ rank(A)+ rank(B).
   (b) rank(AA^t)=rank(A).

2. Let V be a n-dimensional vector space orver $\Re$, A be an $n \times n$ nonsingular matrix over $\Re$, and I_V be the identity mapping on V. Prove that there exist ordered bases α and β for V such that the matrix representation of I_V in the ordered bases α and β is A.

3. Let $A, B \in M_{n \times n}(\Re)$ such that $AB = I_n$. Prove that $BA = I_n$.

4. Let T be a linear operator on $\Re^n$. Give a necessary and sufficient condiction for T being diagonalizable.

5. Suppose $(V, <, >)$ is an inner product space over $\Re$ and $\dim(V) = n$ and T is a linear operator on V. Prove or disprove: if $< T(u), v > = < u, T(v) >$ for all $u, v \in V$, then T is self-adjoint.