Analysis

1. Let $\{f_n\}$ and $\{g_n\}$ be sequences of continuous functions on $E \subseteq \mathbb{R}$. Suppose that $f_n \to f$ and $g_n \to g$ uniformly on E.

(a) Does $f_n + g_n \to f + g$ uniformly on E?

(b) Does $f_n g_n \to fg$ uniformly on E?

2. Suppose that f is continuous on (a, b) with no local maximum and minimum. Show that f is monotone.

3. Let f be a continuous function on $(0, 1]$. Show that f is uniformly continuous if and only if $\lim_{x \to 0^+} f(x)$ exists.

4. Let f be a continuous function on $[0, +\infty)$. Suppose that

$$\lim_{x \to +\infty} f(x) = f(x_0)$$

for some $x_0 \geq 0$. Then f attains both maximum and minimum on $[0, +\infty)$.

5. Show that if f_n is continuous for all n and $f_n \to f$ uniformly on E, then $f_n(x_n) \to f(x)$ if $x_n \to x$ for some $x \in E$. Is the converse true?

6. Let S be a compact subset of $\mathbb{R}$. Show that $f : S \to \mathbb{R}$ is continuous if and only if $\Gamma(f) = \{(x, f(x)) : x \in S\}$ is compact in $\mathbb{R}^2$.

7. Let $a_n > 0$ and $\sum a_n = +\infty$.

(a) Show that $\sum \frac{\sqrt{a_n}}{n^p} < +\infty$ if $p > \frac{1}{2}$.

(b) Does (a) still hold if $p = \frac{1}{2}$?

8. Let $a_n > 0$ and $\sum a_n = +\infty$. Show that $\sum \frac{a_n}{1 + a_n} = +\infty$.

9. Let $a_n > 0$, $\sum a_n = +\infty$ and $s_n = a_1 + \cdots + a_n$. Show that

(a) $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$, $\forall N, k$.

(b) Use (a) to show that $\sum \frac{a_n}{s_n} = +\infty$.

10. Let

$$f(x) = \begin{cases} x & x \notin \mathbb{Q} \\ p \sin(1/q) & x = p/q,\ (p, q) = 1 \end{cases}.$$

Determine $\{x \in \mathbb{R} : \text{f is not continuous at } x\}$.

Linear Algebra

1. Find bases for (1) the column space and (2) the null space of the matrix

$$\begin{pmatrix} 2 & 3 & 1 & -1 \\ 5 & 2 & 1 & 3 \\ 1 & 7 & 2 & -6 \\ 6 & -2 & 0 & 8 \end{pmatrix}.$$

2. Determine whether the 3×3 matrix

$$\begin{pmatrix} 1 & 3 & -3 \\ 2 & 5 & -3 \\ -2 & 2 & -4 \end{pmatrix}$$

is invertible, and find its inverse if it is.

3. By the Gauss-Jordan reduction, one obtains the inverse of the 4×4 matrix A by a series of row operations: (1) interchanging the 1st row and the 3rd row (2) multiplying -3 to the 3rd row and add it to the 1th row (3) mutiplying the 2nd row by $\frac{1}{3}$ (4) add the 1st row to the 3th row and (5) interchanging the 1st row and the 4th row. Find A and A^{-1}.

4. Determine whether the 4×4 matrix A given by

$$\begin{pmatrix} -1 & 4 & 2 & -7 \\ 0 & 4 & -3 & 6 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is diagonalizable, and find a 4×4 matrix M such that $M^{-1}AM$ is diagonal if it is.

5. Find the rank of the 4×3 matrix A

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -3 \\ 1 & -3 & 4 \end{pmatrix}.$$

Can we find a 3×4 matrix B such that $BA = I_3$? Find such B if it can be done.

6. Let $W = \text{sp}([1, 2, 1, -1], [0, -1, 1, -1])$, a subspace of $\mathbb{R}^4$.

(1) Find $W^{\perp}$.

(2) Find orthonormal bases for W and $W^{\perp}$.

7. Find the Jordan canonical form J of the 6×6 matrix A

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

8. Find up to similarity all matrices with characteristic polynomial

$$(x-1)^3(x+1)^4(x-3)^5.$$