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1. If X and Y have joint density function

$$f(x,y) = \frac{1}{x^2y^2} \qquad x \geq 1, y \geq 1$$

(a) Compute the joint density function of $U = XY, V = X/Y$.

(b) What are the marginal densities?

(a) If $u = xy$, $v = xy$, then $J = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -2\frac{x}{y}$ and $y = \sqrt{u/v}$, $x = \sqrt{vu}$. Hence,

$$f_{u,v}(u,v) = \frac{1}{2v} f_{X,Y}(\sqrt{vy}, \sqrt{u/v}) = \frac{1}{2vu^2}, \quad u \geq 1, \frac{1}{u} < v < u$$

(b) $f_u(u) = \int_{1/u}^{u} \frac{1}{2vu^2}\, dv = \frac{1}{u^2} \log u, \quad u \geq 1.$

For $v > 1$

$$f_V(v) = \int_v^{\infty} \frac{1}{2vu^2}\, du = \frac{1}{2v^2}, \quad v > 1$$

For $v < 1$

$$f_V(v) = \int_{1/2}^{\infty} \frac{1}{2vu^2}\, du = \frac{1}{2}, \quad 0 < v < 1$$

2. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \quad 0 < \theta < \infty.$$

(a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.

(b) Find the method of moments estimator of θ.

(a)

$$\begin{aligned} f(\mathbf{x}|\theta) &= \prod_i \theta x_i^{\theta-1} = \theta^n \left(\prod_i x_i \right)^{\theta-1} = L(\theta|\mathbf{x}) \\ \frac{d}{d\theta} \log L &= \frac{d}{d\theta} \left[n \log \theta + (\theta - 1) \log \prod_i x_i \right] = \frac{n}{\theta} + \sum_i \log x_i \end{aligned}$$

Set the derivative equal to zero and solve for θ to obtain $\hat{\theta} = (-\sum_i \log x_i)^{-1}$. The second derivative is $-n/\theta^2 < 0$, so this is the MLE. To calculate the variance of $\hat{\theta}$, note that $Y_i = -\log X_i \sim$ exponential$(1/\theta)$, so $-\sum_i \log X_i \sim$ gamma$(n, 1/\theta)$.

Thus $\hat{\theta} = n/T$, where $T \sim \text{gamma}(n, 1/\theta)$. We can either calculate the first and second moments directly, or use the fact that $\hat{\theta}$ is inverted gamma. We have

$$\begin{aligned} \mathrm{E}\frac{1}{T} &= \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{\theta}{n-1} \\ \mathrm{E}\frac{1}{T^2} &= \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t^2} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)} \end{aligned}$$

and thus

$$\mathrm{E}\hat{\theta} = \frac{n}{n-1}\theta \quad \text{and} \quad \mathrm{Var}\,\hat{\theta} = \frac{n^2}{(n-1)^2(n-2)}\theta^2 \to 0 \text{ as } n \to \infty.$$

(b) Because $X \sim \text{beta}(\theta, 1)$, $\mathrm{E}X = \theta/(\theta+1)$ and the method of moments estimator is the solution to

$$\frac{1}{n}\sum_i X_i = \frac{\theta}{\theta+1} \Rightarrow \hat{\theta} = \frac{\sum_i X_i}{n - \sum_i X_i}.$$

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