科目：機率論【應數系甲組】

Do all problems in detail. 20 points for each problem.

1. Let $X_n \sim Geo(p_n)$, i.e. $P(X_n = r) = q_n^{r-1} p_n$, $r = 1, 2, \ldots$, where $0 < p_n < 1$ and $q_n = 1 - p_n$. Show that as $n \to \infty$ if $p_n \to 0$, then $p_n X_n \to^d X$, where $X \sim$ Exponential (1).

2. Let $\{X_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables with $EX_1 = 0$ and $EX_1^2 = 1$. Prove that

$$\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{j=1}^n X_j^2}} \longrightarrow^d N(0,1).$$

3. Let $\{X_i\}_{i \geq 1}$ be a sequence of independent and identically distributed random variables. Show that $\lim_{n \to \infty} \frac{X_n}{n} = 0$ with probability 1 if and only if $E|X_1| < \infty$.

4. Let $\{X_n\}_{n \geq 1}$ be a sequence of positive random variables such that for each $i = 1, 2$, $\{X_i, X_{i+2}, X_{i+4}, \ldots\}$ are independent and identically distributed. Furthermore, suppose that $E(|X_1| + |X_2|) < \infty$. Show that

$$\frac{1}{n} \sum_{i=1}^n X_i \longrightarrow \frac{EX_1 + EX_2}{2} \qquad \text{with probability 1.}$$

5. Let $X \sim^d Poi(\lambda)$ and $Y \sim^d Poi(\mu)$ be independent, and let $Z = X + Y$. Show that $E[X|Z] = \left(\frac{\lambda}{\lambda + \mu}\right) Z$ with probability 1.

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