

每題 20 分，共 5 題，請詳列計算和推導過程書寫於題目下方空白處。

問題	1: 20 分	2: 20 分	3: 20 分	4: 20 分	5: 20 分	總分: 100 分
得分						

1. Let (X_1, \dots, X_n) be a random sample from the uniform distribution on the interval $[0, 1]$ and let $R = X_{(n)} - X_{(1)}$, where $X_{(i)}$ is the i th order statistic. Derive the probability density function of R and find the limiting distribution of $2n(1 - R)$.

2. Let (X_1, \dots, X_n) be a random sample from the uniform distribution on the interval $\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$, where $\theta \in \mathbb{R}$ is unknown. Let $X_{(j)}$ be the j th order statistic. Show that $(X_{(1)} + X_{(n)})/2$ is strongly consistent for θ and also consistent in mean squared error.

3. Let (X_1, \dots, X_n) be a random sample from a population having the probability density function

$$f_{\theta_1, \theta_2}(x) = \begin{cases} (\theta_1 + \theta_2)^{-1} e^{-x/\theta_1} & x > 0 \\ (\theta_1 + \theta_2)^{-1} e^{x/\theta_2} & x \leq 0 \end{cases}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are unknown. Obtain a moment estimator of (θ_1, θ_2) and its asymptotic distribution.

4. Let (x_1, \dots, x_n) be a random sample from a population on \mathbb{R} with probability density function f_θ . Let θ_0 and θ_1 be two constants. Find a UMP test of size α for testing $H_0: \theta = \theta_0$ versus $H_1: \theta = \theta_1$ in the following cases:
- (i) $f_\theta = e^{-(x-\theta)} I_{(\theta, \infty)}(x)$, $\theta_0 < \theta_1$;
 - (ii) $f_\theta(x) = \theta x^{-2} I_{(\theta, \infty)}(x)$, $\theta_0 \neq \theta_1$.

5. Let $X = (X_1, \dots, X_n)$ be a random sample from the Weibull distribution with probability density function $\frac{a}{\theta} x^{a-1} e^{-x^a/\theta} I_{(0,\infty)}(x)$, where $a > 0$ and $\theta > 0$ are unknown. Show that $R(X, a, \theta) = \prod_{i=1}^n (X_i^a/\theta)$ is pivotal. Construct a confidence set for (a, θ) with confidence coefficient $1 - \alpha$ by using $R(X, a, \theta)$.

~全卷完~

國立中山大學101學年度博士班統計組招生考試試題

科目：【機率論】

注意：每一題必須將過程清楚寫下來 每一題20分

1. Let X_1, X_2, \dots be independently identically distributed with mean μ (i.e. $EX_1 = \mu$) and variance σ^2 (i.e. $Var(X_1) = \sigma^2$). Please Prove

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}}} \sim N(0, 1),$$

where \bar{X} is the sample mean of X_1, \dots, X_n .

2. Let X_1, X_2, \dots , be a sequence of random variables and X be a random variable. Please answer the following questions:

(a) If $X_n \rightarrow X$ almost surely, prove $X_n \rightarrow X$ in probability.

(b) Give an example that $X_n \rightarrow X$ in probability but X_n does not converge to X almost surely.

3. Let X_1, X_2, \dots be independently identically distributed with density $f(x) = 2x^{-3}I_{(1,\infty)}(x)$, where I is an indicator function. Prove $\frac{X_n}{n} \rightarrow 0$ almost surely. (Hint: You can use Borel-Cantelli Lemma.)

4. Suppose that X_1 and X_2 are independent gamma variables, and the density functions of X_1 and X_2 are $\frac{1}{\Gamma(\alpha)}x_1^{\alpha-1}\exp(-x_1), 0 < x_1 < \infty$ and $\frac{1}{\Gamma(\beta)}x_2^{\beta-1}\exp(-x_2), 0 < x_2 < \infty$, respectively. Please prove that the distribution of $\frac{X_1}{X_1 + X_2}$ is beta distribution (You need to express the parameters of the beta distribution using α and β).

5. X is a non-negative random variable (i.e. $X \geq 0$) with cumulative distribution function $F(t)$ and expectation $E(X) < \infty$. Please Prove

$$E(X) = \int_0^{\infty} (1 - F(t))dt.$$

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Ph.D. Entrance Examination – Analysis (May, 2012)

Answer all the problems below. The total is 100%. The first problem carries 10%. Each of the rest carries 15%. Unless otherwise stated, f is a real-valued function defined on \mathbf{R} .

1. Show that if f and g are continuous real-valued functions on an open set $O \subset \mathbf{R}^n$, then the function h defined by $h(x) = \min\{f(x), g(x)\}$ is also continuous on O .

2. Let g_i ($1 \leq i \leq n$) be square integrable functions defined on $[a, b]$. Show that

$$\left[\sum_{i=1}^n \left(\int_a^b g_i \right)^2 \right]^{1/2} \leq \int_a^b \left[\sum_{i=1}^n g_i^2 \right]^{1/2},$$

and equality holds if and only if all g_i 's are multiples of some fixed function ϕ .

3. Let m^* be the Lebesgue outer measure on \mathbf{R} . Define

$$m_*(A) = \sup\{m^*(K) : K \text{ is a compact subset of } A\}.$$

Show that if A is measurable and $m(A) < \infty$, then $m_*(A) = m^*(A)$.

4. Give the definition of f being a measurable function. Suppose that for any $c \in \mathbf{R}$, $f^{-1}[\{c\}]$ is a measurable set. Is f a measurable function? Prove or disprove.

5. Let $\{g_n\}$ be a sequence of nonnegative integrable functions defined on a measurable set $E \subset \mathbf{R}$, which converges a.e. to an integrable function g . Let $\{f_n\}$ be a sequence of measurable functions on E such that $|f_n| \leq g_n$ and f_n converges to f a.e. $x \in E$. Show that if $\int_E g = \lim_{n \rightarrow \infty} \int_E g_n$, then

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n.$$

6. Let E be a measurable set and $1 < p < \infty$. Suppose that $\{f_n\}$ converges to f weakly. Show that there is a subsequence $\{f_{n_k}\}$ such that

$$\lim_{k \rightarrow \infty} \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_k}}{k} = f \quad \text{strongly in } L^p(E).$$

7. Show that any compact subset of a metric space has to be closed and bounded.

End of Paper

組合數學(Combinatorics)

May 2012

注意：每個問題必需證明或說明清楚。

1. Let a_n be the way to distribute n identical balls into five distinct boxes with the first box having at most two balls.
 - (a) Find the generating function for a_n . (10%)
 - (b) Find a_{505} . (10%)
2. Let $S=\{1,2,3,4,5\}$, A be the set of all permutations of S , and $B=\{(a,b,c,d,e)\in A : a\notin\{1,5\}, b\notin\{2,3\}, c\notin\{3,4\}, d\notin\{4,5\}, \text{ and } e\notin\{3,5\}\}$. Find the cardinality of the set B . (20%)
3. Let G be a simple planar graph. Prove that if every subgraph of G has a vertex of degree ≤ 5 , then G is 5-colorable. (20%)
4. Prove Hall's Theorem by two different methods. (20%)
5. Let n be a positive integer and $[n]$ be the set $\{1,2,\dots,n\}$. Define G as a graph with vertex set of G being all subsets of $[n]$, and edge set $\{AB : A \text{ is a subset of } B\}$. Find a maximum independent set of G . (20%)