Multistability for Delayed Neural Networks via Sequential Contracting

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This is a joint work with Chang-Yuan Cheng (NPTU), Kuang-Hui Lin (NCTU), and Chih-Wen Shih (NCTU).

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- We derive criteria from different geometric configurations which lead to disparate numbers of equilibria.
- We introduce a new approach, named sequential contracting, to conclude the global convergence (to multiple equilibrium points) of the system.

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  - Delays can modify the collective dynamics of neural networks; for example, they can induce oscillation or change the stability of the equilibrium point.

  - Taking time delay into account in mathematical models usually increases mathematical technicality.

Background: model

Hopfield-type neural network:

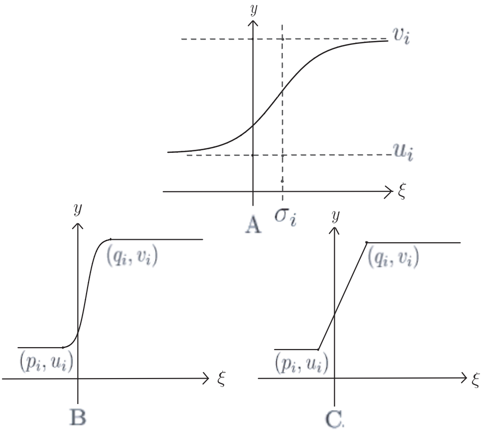
$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n}[\alpha_{ij} g_j(x_j(t)) + \beta_{ij} g_j(x_j(t - \tau_{ij}))] + I_i, \qquad (1)$$

$i = 1, 2, \cdots, n.$

- $\mu_i > 0$, α_{ij}, β_{ij}: connection weights, I_i: bias current sources
- $\tau_{ij} \geq 0$: time delays, bounded by τ_M
- g_j: activation/output function (introduced later)

Classes of activation functions

- Classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.



- We focus on class $\mathcal{A}$. Let $\rho_i := \max\{|u_i|,\ |v_i|\}$, $g_i'(\sigma_i) = L_i$

Background: the existing works

- Existence of multiple equilibrium points:

  - numbers of equilibria are in terms of n-power of the number of saturated (or near saturated) regions in a n-neuron system,
  e.g. 3^n, $(2r+1)^n$, etc.

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- Stability/convergence of dynamics:

  - common restriction 1: cooperative ($\alpha_{ij}, \beta_{ij} \geq 0$, $i \neq j$) or competitive ($\alpha_{ij}, \beta_{ij} < 0$, $i \neq j$) (monotone dynamics theory)

  - common restriction 2: restricted to the class of piecewise-linear activation functions.

Let us now present our approach to study the existence of equilibrium points for system (1)

Existence of equilibria for system (1)

- Recall system (1):

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} [\alpha_{ij} g_j(x_j(t)) + \beta_{ij} g_j(x_j(t - \tau_{ij}))] + I_i, \; i = 1, \ldots, n.$$

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- Consider the stationary equations for (1):

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0,\ i = 1, \ldots, n. \qquad (2)$$

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- Our approach combines a geometric formulation on $F_i(\mathbf{x})$ and the Brouwer's fixed-point theorem.

Brouwer's fixed-point theorem

Brouwer's fixed-point theorem.

Every continuous function from a convex compact subset K of a Euclidean space to K itself has a fixed point.

Existence of equilibria in system (1) - Idea

- Locate a region $K := K_1 \times \cdots \times K_n$, with each K_i an interval in $\mathbb{R}$, so that for an arbitrary $(\zeta_1, \ldots, \zeta_n) \in K$, for every $i = 1, \ldots, n$, there exists a solution $x_i \in K_i$ to

$$F_i(\zeta_1, \ldots, \zeta_{i-1}, x_i, \zeta_{i+1}, \ldots, \zeta_n) = 0.$$

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- $\bar{\mathbf{x}}$ is an equilibrium of system (1) (in K). If in addition that Φ is a contraction mapping, then $\bar{\mathbf{x}}$ the unique equilibrium in K.

How to locate region K: Upper and lower functions

- Recall the stationary equations:

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i, \tag{3}$$

where each $g_j(\cdot) \leq \rho_j$.

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- For $i = 1, 2, \cdots, n$, we define

$$\begin{aligned} \hat{f}_i(\xi) &:= -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + k_i^+, \\ \check{f}_i(\xi) &:= -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + k_i^-, \end{aligned}$$

where $k_i^{\pm} := \pm\sum_{j=1,j\neq i}^{n} \rho_j|\alpha_{ij} + \beta_{ij}| + I_i$.

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$$\check{f}_i(x_i) \leq F_i(\mathbf{x}) \leq \hat{f}_i(x_i), \ \ i = 1, \ldots, n,$$

  for all $\mathbf{x} = (x_1, \ldots, x_n)$.

Configuration of upper and lower functions: Two cases, and eight subcases

Set $\mathcal{N} := \{1, 2, \cdots, n\}$.

$$
\begin{aligned}
\mathcal{M} &:= \{i \in \mathcal{N} | \max_{\xi \in \mathbb{R}} g_i'(\xi) \leq \frac{\mu_i}{\alpha_{ii} + \beta_{ii}}\}, \\
\mathcal{B} &:= \{i \in \mathcal{N} | \inf_{\xi \in \mathbb{R}} g_i'(\xi) < \frac{\mu_i}{\alpha_{ii} + \beta_{ii}} \\
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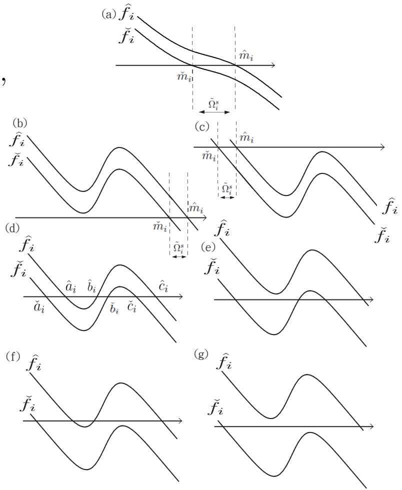
- (a) is of type $\mathcal{M}$; (b)-(g) are of type $\mathcal{B}$.
- (b)-(g) are of type $\mathcal{B}_{\mathrm{r}}^{\mathrm{r}}$, $\mathcal{B}_{\mathrm{l}}^{\mathrm{l}}$, $\mathcal{B}_3^3$, $\mathcal{B}_3^{\mathrm{r}}$, $\mathcal{B}_{\mathrm{l}}^3$, $\mathcal{B}_{\mathrm{l}}^{\mathrm{r}}$, respectively.



Existence of 3^k equilibria for system (1)

Theorem.

If $\mathcal{M} \cup \mathcal{B}_{\rm r}^{\rm r} \cup \mathcal{B}_{\rm l}^{\rm l} \cup \mathcal{B}_3^3 = \mathcal{N} := \{1, \ldots, n\}$, and $k = card(\mathcal{B}_3^3) \geq 1$, then there exist 3^k equilibria in system (1).

Sketch of Proof. We consider 3^k disjoint closed regions in $\mathbb{R}^n$:

$$\begin{aligned} \tilde{\Omega}^{\mathbf{w}} &= \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \\ \mathbf{w} &= (w_1, \cdots, w_n), \\ w_i &= \text{“l”}, \text{“m”}, \text{“r”}, \text{ for } i \in \mathcal{B}_3^3, \\ w_i &= \text{“s”}, \text{ for } i \in \mathcal{M} \cup \mathcal{B}_{\rm r}^{\rm r} \cup \mathcal{B}_{\rm l}^{\rm l}, \end{aligned} \tag{4}$$

where $\tilde{\Omega}_i^{\rm l} = [\check{a}_i, \hat{a}_i]$, $\tilde{\Omega}_i^{\rm m} = [\hat{b}_i, \check{b}_i]$, $\tilde{\Omega}_i^{\rm r} = [\check{c}_i, \hat{c}_i]$ and $\tilde{\Omega}_i^{\rm s} = [\check{m}_i, \hat{m}_i]$.

Existence of exact 3^k equilibria for system (1)

Theorem.

Assume that $\mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}} \cup \mathcal{B}_3^3 = \mathcal{N}$ with $k = card(\mathcal{B}_3^3) \geq 1$.

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$$\bar{L}_i := \begin{cases} \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}}, \\ L_i, & \text{if } i \in \mathcal{B}_3^3. \end{cases} \tag{5}$$

If $\theta_i > \sum_{j=1, j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}|$, and

$$g_i'(\xi) \begin{cases} < \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in [\check{m}_i, \hat{m}_i],\ i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}}, \\ < \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in (-\infty, \hat{a}_i] \cup [\check{c}_i, \infty),\ i \in \mathcal{B}_3^3, \\ > \frac{\mu_i + \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in [\hat{b}_i, \check{b}_i],\ i \in \mathcal{B}_3^3, \end{cases} \tag{6}$$

for all $i \in \mathcal{N}$,

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for all $i \in \mathcal{N}$, then there exist exactly 3^k equilibria in system (1), and each region $\tilde{\boldsymbol{\Omega}}^{\mathbf{w}}$, defined in (4), contains exactly one of these 3^k equilibria.

Global convergence to exactly 3^k equilibrium points for system (1)

Global convergence exact 3^k equilibria: Idea

- Fix an arbitrary initial condition ϕ.
  Its solution $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$ of system (1) is then a fixed function defined on $[t_0, \infty)$.

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- For each $i \in \mathcal{N}$, the ith component $x_i(t)$ satisfies

$$\dot{x}_i(t) = -\mu_i x_i(t) + \alpha_{ii} g_i(x_i(t)) + \beta_{ii} g_i(x_i(t - \tau_{ii})) + w_i(t), \qquad (7)$$

  for all $t \geq t_0$, where

$$w_i(t) := \sum_{j \neq i} \{\alpha_{ij} g_j(x_j(t)) + \beta_{ij} g_j(x_j(t - \tau_{ij}))\} + I_i.$$

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- For later use, we define for each $i \in \mathcal{N}$,

$$w_i^{\max}(T) := \sup\{w_i(t) \mid t \geq T\}, w_i^{\min}(T) := \inf\{w_i(t) \mid t \geq T\}$$
$$w_i^{\max}(\infty) := \lim_{T \to \infty} w_i^{\max}(T), w_i^{\min}(\infty) := \lim_{T \to \infty} w_i^{\min}(T)$$

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Recall $\mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}} \cup \mathcal{B}_{3}^{3} = \mathcal{N}$ with $k = card(\mathcal{B}_{3}^{3}) \geq 1$.
We shall show that under some conditions,

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- for each $i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}}$, $x_i(t)$ converges to $[\underline{m}_i, \overline{m}_i]$, where

$$\overline{m}_i - \underline{m}_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i].$$

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- for each $i \in \mathcal{B}_3^3$, $x_i(t)$ converges to one of the three disjoint intervals: $[\underline{a}_i, \overline{a}_i]$, $[\underline{b}_i, \overline{b}_i]$, and $[\underline{c}_i, \overline{c}_i]$, where

$$\begin{aligned} 0 &\leq \overline{a}_i - \underline{a}_i, \overline{b}_i - \underline{b}_i, \overline{c}_i - \underline{c}_i \\ &\leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i]. \end{aligned}$$

Global convergence of dynamics in system (1)

Proposition

Let $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$ be a fixed solution of (1). Assume that for every $i \in \mathcal{N}$, there exists a compact interval J_i of length d_i, such that $x_i(t)$ converges to J_i and d_i satisfies

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for some $\eta_i > 0$, and there exist a compact interval $\tilde{J}_i$ and a $\tilde{L}_i \geq 0$, such that $J_i \subseteq \tilde{J}_i$ and

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$$g_i'(\xi) \leq \tilde{L}_i \text{ for all } \xi \in \tilde{J}_i.$$

Let $\mathbf{M} := [m_{ij}]_{1 \leq i,j \leq n}$ with $m_{ii} := \eta_i$, $m_{ij} := -(|\alpha_{ij}| + |\beta_{ij}|)\tilde{L}_j$ for $i \neq j$.

Global convergence of dynamics in system (1)

Proposition

Let $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$ be a fixed solution of (1). Assume that for every $i \in \mathcal{N}$, there exists a compact interval J_i of length d_i, such that $x_i(t)$ converges to J_i and d_i satisfies

$$d_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/\eta_i,$$

for some $\eta_i > 0$, and there exist a compact interval $\tilde{J}_i$ and a $\tilde{L}_i \geq 0$, such that $J_i \subseteq \tilde{J}_i$ and

$$g_i'(\xi) \leq \tilde{L}_i \text{ for all } \xi \in \tilde{J}_i.$$

Let $\mathbf{M} := [m_{ij}]_{1 \leq i,j \leq n}$ with $m_{ii} := \eta_i$, $m_{ij} := -(|\alpha_{ij}| + |\beta_{ij}|)\tilde{L}_j$ for $i \neq j$. If the Gauss-Seidel iteration for solving the linear system

$$\mathbf{Mv} = \mathbf{0}, \tag{8}$$

converges to zero, the unique solution of (8), then every d_i degenerates into zero and the solution $\mathbf{x}(t)$ of system (1) converges to a singleton.

Asymptotic behavior of $x_i(t)$, where $i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}}$

Proposition A.

Assume that conditions (M1)-(M3) hold for some $i \in \mathcal{N}$. Then $x_i(t)$ satisfying (7) converges to $[\underline{m}_i, \overline{m}_i]$, where

$$\overline{m}_i - \underline{m}_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i].$$

- Condition (M1):
  $|\beta_{ii}|\tau_{ii} < (|\alpha_{ii}| + |\beta_{ii}|)\rho_i/\{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0) - w_i^{\min}(t_0)]\}$.
- Condition (M2): There exists a $T_0 \geq t_0$ such that $\hat{f}_i^{(0)}(\cdot, T_0)$ and $\check{f}_i^{(0)}(\cdot, T_0)$ have unique zeros, $\hat{m}_i^{(0)}(T_0)$ and $\check{m}_i^{(0)}(T_0)$, respectively.
- Condition (M3): $g_i'(\xi) < (\mu_i - \theta_i)/(\alpha_{ii} + \beta_{ii})$ for all $\xi \in [\check{m}_i^{(0)}(T_0), \hat{m}_i^{(0)}(T_0)]$ for some $\theta_i \in (0, \mu_i)$.

Proof of Proposition A. -1

- Recall (7):

$$\dot{x}_i(t) = -\mu_i x_i(t) + \alpha_{ii} g_i(x_i(t)) + \beta_{ii} g_i(x_i(t - \tau_{ii})) + w_i(t),$$

Proof of Proposition A. -1

- Recall (7):

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- Define the upper and lower bounds for (7), respectively:

$$\hat{h}_i(\xi) := -\mu_i \xi + 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0), \quad (9)$$

$$\check{h}_i(\xi) := -\mu_i \xi - 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\min}(t_0). \quad (10)$$

Proof of Proposition A. -1

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- $\hat{h}_i$ and $\check{h}_i$ are linear decreasing functions, with unique zeros $\hat{A}_i^h$ and $\check{A}_i^h$, respectively.

Proof of Proposition A. -1

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- $\hat{h}_i$ and $\check{h}_i$ are linear decreasing functions, with unique zeros $\hat{A}_i^h$ and $\check{A}_i^h$, respectively.
- $$\check{h}_i(x_i(t)) + (|\alpha_{ii}| + |\beta_{ii}|)\rho_i \le \dot{x}_i(t) \le \hat{h}_i(x_i(t)) - (|\alpha_{ii}| + |\beta_{ii}|)\rho_i,$$

  for all $t \ge t_0$. Consequently, there exists a t_ϕ such that $x_i(t)$ enters and remains in interval $[\check{A}_i^h, \hat{A}_i^h]$ for $t \ge t_\phi$.

Proof of Proposition A.-2

- Accordingly, we can construct the second preliminary upper and lower bounds for (7):

$$\hat{f}_i^{(0)}(\xi, T) := \begin{cases} \hat{\gamma}_i(\xi, T) - \beta_{ii} L_i \tau_{ii} \check{h}_i(\hat{A}_i^h) & \text{if} \quad \beta_{ii} \geq 0, \\ \hat{\gamma}_i(\xi, T) - \beta_{ii} L_i \tau_{ii} \hat{h}_i(\check{A}_i^h) & \text{if} \quad \beta_{ii} < 0, \end{cases} \tag{11}$$

$$\check{f}_i^{(0)}(\xi, T) := \begin{cases} \check{\gamma}_i(\xi, T) - \beta_{ii} L_i \tau_{ii} \hat{h}_i(\check{A}_i^h) & \text{if} \quad \beta_{ii} \geq 0, \\ \check{\gamma}_i(\xi, T) - \beta_{ii} L_i \tau_{ii} \check{h}_i(\hat{A}_i^h) & \text{if} \quad \beta_{ii} < 0, \end{cases} \tag{12}$$

where

$$\hat{\gamma}_i(\xi, T) := -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + w_i^{\max}(T),$$
$$\check{\gamma}_i(\xi, T) := -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + w_i^{\min}(T).$$

Proof of Proposition A.-3

- Condition (M1) implies $|\alpha_{ii}| + |\beta_{ii}| > 0$, and thus

$$\check{h}_i(\xi) < \check{f}_i^{(0)}(\xi, t_0) \leq \check{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, t_0) < \hat{h}_i(\xi) \tag{13}$$

for all $T \geq t_0$ and $\xi \in \mathbb{R}$.

Proof of Proposition A.-3

- Condition (M1) implies $|\alpha_{ii}| + |\beta_{ii}| > 0$, and thus

$$\check{h}_i(\xi) < \check{f}_i^{(0)}(\xi, t_0) \leq \check{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, t_0) < \hat{h}_i(\xi) \tag{13}$$

  for all $T \geq t_0$ and $\xi \in \mathbb{R}$.

- For any $T \geq \max\{t_\phi + \tau, T_0\}$,

$$\check{f}_i^{(0)}(x_i(t), T) + \epsilon_i \leq \dot{x}_i(t) \leq \hat{f}_i^{(0)}(x_i(t), T) - \epsilon_i, t \geq T$$

  where $\epsilon_i := |\beta_{ii}|(|\alpha_{ii}| + |\beta_{ii}|)\rho_i L_i \tau_{ii}$.

Proof of Proposition A.-3

- Condition (M1) implies $|\alpha_{ii}| + |\beta_{ii}| > 0$, and thus

$$\check{h}_i(\xi) < \check{f}_i^{(0)}(\xi, t_0) \leq \check{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, t_0) < \hat{h}_i(\xi) \tag{13}$$

  for all $T \geq t_0$ and $\xi \in \mathbb{R}$.

- For any $T \geq \max\{t_\phi + \tau, T_0\}$,

$$\check{f}_i^{(0)}(x_i(t), T) + \epsilon_i \leq \dot{x}_i(t) \leq \hat{f}_i^{(0)}(x_i(t), T) - \epsilon_i, t \geq T$$

  where $\epsilon_i := |\beta_{ii}|(|\alpha_{ii}| + |\beta_{ii}|)\rho_i L_i \tau_{ii}$.

- Consequently $x_i(t)$ enters and remains in interval $[\check{m}_i^{(0)}(T), \hat{m}_i^{(0)}(T)]$ contained in $[\check{A}_i^h, \hat{A}_i^h]$ after certain time,
  where $\check{m}_i^{(0)}(T)$ (resp., $\hat{m}_i^{(0)}(T)$) is the unique zero of $\check{f}_i^{(0)}(\cdot, T) = 0$ (resp., $\hat{f}_i^{(0)}(\cdot, T) = 0$).

Proof of Proposition A.-4

- Iteratively applying arguments based on constructing finer upper $\hat{f}_i^{(k)}$ and lower bounds $\check{f}_i^{(k)}$ for (7) allows us to establish the convergence of $x_i(t)$ to some compact interval $[\underline{m}_i, \overline{m}_i]$, where

$$\overline{m}_i - \underline{m}_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i].$$

Convergence to one of three intervals

Proposition B.

Assume that conditions (B1)-(B3) hold for some $i \in \mathcal{N}$ and some $\theta_i \in (0, \mu_i)$. Then $x_i(t)$ satisfying (7) converges toone of the three disjoint intervals: $[\underline{a}_i, \overline{a}_i]$, $[\underline{b}_i, \overline{b}_i]$, and $[\underline{c}_i, \overline{c}_i]$, where

$$\begin{aligned} 0 &\leq \overline{a}_i - \underline{a}_i, \overline{b}_i - \underline{b}_i, \overline{c}_i - \underline{c}_i \\ &\leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i]. \end{aligned}$$

Conditions (B1)-(B3)

- Condition (B1): $L_i > \mu_i/(\alpha_{ii} + \beta_{ii}) > 0$,
  $|\beta_{ii}|\tau_{ii} < (|\alpha_{ii}| + |\beta_{ii}|)\rho_i/\{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0) - w_i^{\min}(t_0)]\}$.
  Notably, condition (B1) implies $L_i > \mu/(\alpha_{ii} + \beta_{ii})$. There hence exist exactly two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i < \sigma_i < \tilde{q}_i$, satisfying

$$g_i'(\tilde{p}_i) = g'(\tilde{q}_i) = \mu_i/(\alpha_{ii} + \beta_{ii}).$$

- Condition (B2): There exists a $T_0 \geq t_0$ such that $\check{f}_i^{(0)}(\tilde{q}_i, T_0) > 0$ and $\hat{f}_i^{(0)}(\tilde{p}_i, T_0) < 0$.
  Under condition (B2), there exist exactly three zeros $\hat{a}_i$, $\hat{b}_i$ and $\hat{c}_i$ (resp., $\check{a}_i$, $\check{b}_i$ and $\check{c}_i$) of $\hat{f}_i^{(0)}(\cdot, T_0) = 0$ (resp., $\check{f}_i^{(0)}(\cdot, T_0) = 0$), where $\check{a}_i \leq \hat{a}_i < \tilde{p}_i < \hat{b}_i \leq \check{b}_i < \tilde{q}_i < \check{c}_i \leq \hat{c}_i$. Let $\theta_i \in (0, \mu_i)$ be a fixed number.
- Condition (B3):

$$g_i'(\xi) \begin{cases} >(\mu_i + \theta_i)/(\alpha_{ii} + \beta_{ii}) & \text{if } \xi \in [\hat{b}_i, \check{b}_i], \\ <(\mu_i - \theta_i)/(\alpha_{ii} + \beta_{ii}) & \text{if } \xi \in (-\infty, \hat{a}_i] \cup [\check{c}_i, \infty). \end{cases}$$

Global convergence of dynamics in system (1)

Theorem

Assume that $\mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}} \cup \mathcal{B}_3^3 = \mathcal{N}$, (16) and (17) hold, and for each $i \in \mathcal{N}$

$$|\beta_{ii}|\tau_{ii} < \tau_{ii}^c, \tag{14}$$

and

$$g_i'(\xi) \begin{cases} < \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in [\check{m}_i^F, \hat{m}_i^F],\ i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}}, \\ < \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in (-\infty, \hat{a}_i^F] \cup [\check{c}_i^F, \infty),\ i \in \mathcal{B}_3^3, \\ > \frac{\mu_i + \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in [\hat{b}_i^F, \check{b}_i^F],\ i \in \mathcal{B}_3^3, \end{cases} \tag{15}$$

for some $\theta_i \in (0, \mu_i)$. **Then system** (1) **achieves global convergence to the** 3^k **equilibria** provided that the Gauss-Seidel iteration for the linear algebraic system (8) converges to zero, the unique solution.

Global convergence of dynamics in system (1)

Theorem. continued

where $m_{ii} = (1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i$ for $i \in \mathcal{N}$, $m_{ij} = -(|\alpha_{ij}| + |\beta_{ij}|)\bar{L}_j$ for $i, j \in \mathcal{N}$, $i \neq j$, and $\bar{L}_j$ is defined in (5), and

$$\theta_i > \sum_{j=1, j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}|, \tag{16}$$

$$\begin{cases} \check{F}_i(\tilde{p}_i) > 0 & \text{if } i \in \mathcal{B}^{\mathrm{r}}_{\mathrm{r}}, \\ \hat{F}_i(\tilde{q}_i) < 0 & \text{if } i \in \mathcal{B}^{\mathrm{l}}_{\mathrm{l}}, \\ \hat{F}_i(\tilde{p}_i) < 0, \ \check{F}_i(\tilde{q}_i) > 0 & \text{if } i \in \mathcal{B}^{3}_{3}, \end{cases} \tag{17}$$

Example: existence of 9 equilibria

Example 1

We consider system (1) with $n = 3$, under the parameters:

$$(\mu_i) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (\alpha_{ij}) = \begin{pmatrix} 1.8 & 0.05 & 0 \\ 0.05 & 1.9 & 0 \\ 0 & 0.05 & 0.6 \end{pmatrix},$$

$$(I_i) = \begin{pmatrix} 0.05 \\ 0 \\ 0.15 \end{pmatrix}, \quad (\beta_{ij}) = \begin{pmatrix} 0.2 & 0 & 0.05 \\ 0 & 0.1 & 0.05 \\ 0.05 & 0 & 0.1 \end{pmatrix}.$$

In addition, we set $\tau_{ii} = 0.1$, $\tau_{ij} = 12$ for $i, j = 1, 2, 3$, $i \neq j$.

- $i = 1, 2 \in \mathcal{B}_3^3$ and $i = 3 \in \mathcal{M}$
- $card(\mathcal{B}_3^3) = 2$

Example 1: 9 equilibria, where 4 ones are stable

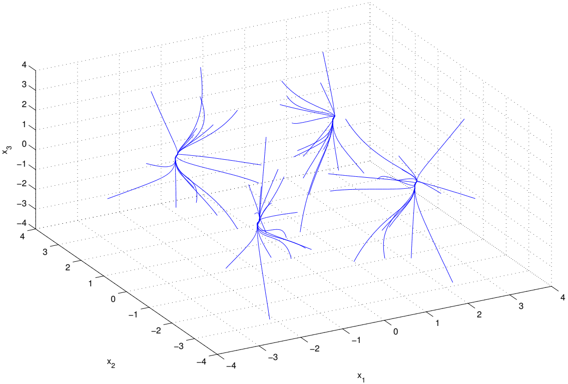


Figure: Numerical simulation for Example 1.

Other cases of multistability for (1) with $n = 2$

The upper and lower functions

- Recall the stationary equations:

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i, \tag{18}$$

where each $g_j(\cdot) \leq \rho_j$.

The upper and lower functions

- Recall the stationary equations:

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i, \tag{18}$$

  where each $g_j(\cdot) \leq \rho_j$.

- The upper and lower functions are now

$$\begin{aligned}
\hat{f}_1(\xi) &= -\mu_1 \xi + (\alpha_{11} + \beta_{11}) g_1(\xi) + |\alpha_{12} + \beta_{12}| \rho_2 + I_1, \\
\check{f}_1(\xi) &= -\mu_1 \xi + (\alpha_{11} + \beta_{11}) g_1(\xi) - |\alpha_{12} + \beta_{12}| \rho_2 + I_1, \\
\hat{f}_2(\xi) &= -\mu_2 \xi + (\alpha_{22} + \beta_{22}) g_2(\xi) + |\alpha_{21} + \beta_{21}| \rho_1 + I_2, \\
\check{f}_2(\xi) &= -\mu_2 \xi + (\alpha_{22} + \beta_{22}) g_2(\xi) - |\alpha_{21} + \beta_{21}| \rho_2 + I_2,
\end{aligned}$$

The upper and lower functions

- Recall the stationary equations:

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n}(\alpha_{ij} + \beta_{ij})g_j(x_j) + I_i, \tag{18}$$

  where each $g_j(\cdot) \leq \rho_j$.
- The upper and lower functions are now

$$\begin{aligned}
\hat{f}_1(\xi) &= -\mu_1\xi + (\alpha_{11} + \beta_{11})g_1(\xi) + |\alpha_{12} + \beta_{12}|\rho_2 + I_1, \\
\check{f}_1(\xi) &= -\mu_1\xi + (\alpha_{11} + \beta_{11})g_1(\xi) - |\alpha_{12} + \beta_{12}|\rho_2 + I_1, \\
\hat{f}_2(\xi) &= -\mu_2\xi + (\alpha_{22} + \beta_{22})g_2(\xi) + |\alpha_{21} + \beta_{21}|\rho_1 + I_2, \\
\check{f}_2(\xi) &= -\mu_2\xi + (\alpha_{22} + \beta_{22})g_2(\xi) - |\alpha_{21} + \beta_{21}|\rho_2 + I_2,
\end{aligned}$$

- For this two-neuron system, there are four basic types, as shown in the next slide.

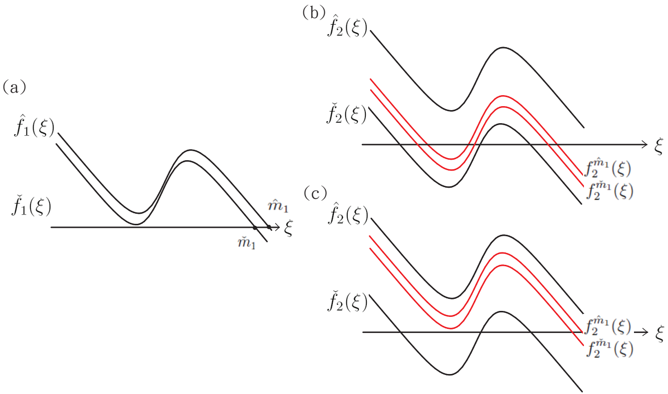
4 types for $n = 2$

We shall take the case $\binom{r}{r}\binom{r}{3}$ to introduce these notations.

Type	Subtype	Cases
$(\mathcal{M}, \mathcal{M})$	T_1	$\binom{m}{m}\binom{m}{m}$
$(\mathcal{M}, \mathcal{B})$	T_2	$\binom{m}{m}\binom{r}{r}, \binom{m}{m}\binom{l}{l}$
	T_3	$\binom{m}{m}\binom{3}{3}$
	T_4	$\binom{m}{m}\binom{r}{3}, \binom{m}{m}\binom{3}{l}$
	T_5	$\binom{m}{m}\binom{r}{l}$
$(\mathcal{B}, \mathcal{M})$	T_6	$\binom{r}{r}\binom{m}{m}, \binom{l}{l}\binom{m}{m}$
	T_7	$\binom{3}{3}\binom{m}{m}$
	T_8	$\binom{r}{3}\binom{m}{m}, \binom{3}{l}\binom{m}{m}$
	T_9	$\binom{r}{l}\binom{m}{m}$
$(\mathcal{B}, \mathcal{B})$	T_{10}	$\binom{r}{r}\binom{r}{r}, \binom{l}{l}\binom{l}{l}, \binom{r}{r}\binom{l}{l}, \binom{l}{l}\binom{r}{r}$
	T_{11}	$\binom{r}{r}\binom{3}{3}, \binom{l}{l}\binom{3}{3}, \binom{3}{3}\binom{r}{r}, \binom{3}{3}\binom{l}{l}$
	T_{12}	$\binom{r}{r}\binom{r}{3}, \binom{r}{r}\binom{3}{l}, \binom{l}{l}\binom{r}{3}, \binom{l}{l}\binom{3}{l}$
		$\binom{r}{3}\binom{r}{r}, \binom{r}{3}\binom{l}{l}, \binom{3}{l}\binom{r}{r}, \binom{3}{l}\binom{l}{l}$
	T_{13}	$\binom{r}{r}\binom{r}{l}, \binom{l}{l}\binom{r}{l}, \binom{r}{l}\binom{r}{r}, \binom{r}{l}\binom{l}{l}$
	T_{14}	$\binom{3}{3}\binom{3}{3}$
	T_{15}	$\binom{3}{3}\binom{r}{3}, \binom{3}{3}\binom{3}{l}, \binom{r}{3}\binom{3}{3}, \binom{3}{l}\binom{3}{3}$
	T_{16}	$\binom{3}{3}\binom{r}{l}, \binom{r}{l}\binom{3}{3}$
	T_{17}	$\binom{r}{3}\binom{r}{3}, \binom{3}{l}\binom{3}{l}$
	T_{18}	$\binom{r}{3}\binom{3}{l}, \binom{3}{l}\binom{r}{3}$
	T_{19}	$\binom{r}{3}\binom{r}{l}, \binom{3}{l}\binom{r}{l}, \binom{r}{l}\binom{r}{3}, \binom{r}{l}\binom{3}{l}$
	T_{20}	$\binom{r}{l}\binom{r}{l}$

Table: Subtypes in $(\mathcal{M}, \mathcal{M})$, $(\mathcal{M}, \mathcal{B})$, $(\mathcal{B}, \mathcal{M})$, and $(\mathcal{B}, \mathcal{B})$.

Case $\binom{r}{r}\binom{\mathrm{r}}{3}$ in type $(\mathcal{B}, \mathcal{B})$



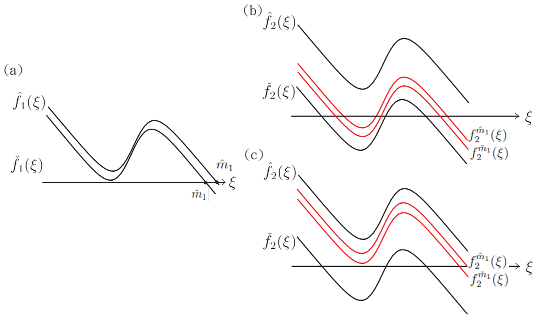
- Recall the F_i and $\check{f}_i$ and $\hat{f}_i$, $i = 1, 2$. If $\alpha_{21} + \beta_{21} > 0$, we consider

$$f_2^{\hat{m}}(\xi) = -\mu_2\xi + (\alpha_{22} + \beta_{22})g_i(\xi) + (\alpha_{21} + \beta_{21})g(\hat{m}_1) + I_2,$$
$$f_2^{\check{m}}(\xi) = -\mu_2\xi + (\alpha_{22} + \beta_{22})g_i(\xi) + (\alpha_{21} + \beta_{21})g(\check{m}_1) + I_2.$$

multistability for subcase $\binom{r}{r}\binom{r}{3}$

Theorem.

Consider system system (1) with $n = 2$ and the case $\binom{r}{r}\binom{r}{3}$. There exists one equilibrium if $K_2(\tilde{p}_2; S_1) > 0$, and three equilibria if $K_2(\tilde{p}_2; S_1) < 0$.



- We can further establish the convergence of dynamics for system (1).

Other cases of multistability: case $\binom{3}{3}\binom{r}{3}$

Criteria	$\sharp$ Equi.
$\alpha_{21} + \beta_{21} > 0$	
$K_2(\tilde{p}_2; A_1) > 0$	3
$K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$	5
$K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; C_1)$	7
$K_2(\tilde{p}_2; C_1) < 0$	9
$\alpha_{21} + \beta_{21} < 0$	
$K_2(\tilde{p}_2; C_1) > 0$	3
$K_2(\tilde{p}_2; C_1) < 0 < K_2(\tilde{p}_2; B_1)$	5
$K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; A_1)$	7
$K_2(\tilde{p}_2; A_1) < 0$	9

Table: Criteria for various numbers of equilibrium points for the case $\binom{3}{3}\binom{r}{3}$.

Other cases of multistability: case $\binom{3}{3}\binom{\mathrm{r}}{\mathrm{l}}$

Criteria	$\sharp$ Equi.
$\alpha_{21}+\beta_{21}>0$	
$K_2(\tilde{p}_2;A_1)>0$	3
$K_2(\tilde{p}_2;\bullet)<0$ and $K_2(\tilde{q}_2;\bullet)>0$ for $\bullet=A_1,B_1$ or C_1	5
$K_2(\tilde{p}_2;B_1)<0$, $K_2(\tilde{p}_2;C_1)>0$, $K_2(\tilde{q}_2;A_1)>0$	7
$K_2(\tilde{p}_2;C_1)<0$, $K_2(\tilde{q}_2;B_1)>0$, $K_2(\tilde{q}_2;A_1)<0$	7
$K_2(\tilde{p}_2;C_1)<0$, $K_2(\tilde{q}_2;A_1)>0$	9
$\alpha_{21}+\beta_{21}<0$	
$K_2(\tilde{q}_2;C_1)<0$	3
$K_2(\tilde{p}_2;\bullet)<0$ and $K_2(\tilde{q}_2;\bullet)>0$ for $\bullet=A_1,B_1$ or C_1	5
$K_2(\tilde{p}_2;B_1)<0$, $K_2(\tilde{p}_2;A_1)>0$, $K_2(\tilde{q}_2;C_1)>0$	7
$K_2(\tilde{p}_2;A_1)<0$, $K_2(\tilde{q}_2;B_1)>0$, $K_2(\tilde{q}_2;C_1)<0$	7
$K_2(\tilde{p}_2;A_1)<0$, $K_2(\tilde{q}_2;C_1)>0$	9

Table: Criteria for various numbers of equilibrium points for the case $\binom{3}{3}\binom{\mathrm{r}}{\mathrm{l}}$.

This is The End of The Presentation
And Thank You for Your Attention