Entropy of Tree Shift of Finite Type

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1. History and motivations: Entropy
   - 1-d SFT
   - 2-d SFT

2. Tree TSFT: SNRE
   - Definitions and notations
   - Systems of non-linear recurrence equations

3. Tree TSFT: Entropy
   - Complete characterization of $d = k = 2$
   - Realization problems and general cases
   - Boundary effect

4. Conclusion and open problems
Isotropic, space-invariant lattice dynamical systems $\mapsto$ subshift of finite type, other shifts.
Subshift of finite type: local rules on the lattice.
Measure the “complexity” of a given system.
Entropy on shifts:

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |B_n(X)|.$$ 

Theorem (Lind-Marcus 95’: 1-d SFT and sofic)

1. The entropy of a SFT is the logarithm of the maximal eigenvalue of the corresponding adjacent matrix.
2. The entropy of a irreducible right-resolving sofic shift is the logarithm of the maximal eigenvalue of the corresponding adjacent matrix.

Entropy = Perron values (Lind-Marcus 95’).
Question: How to compute the entropy for $\mathbb{Z}^d$-SFT for $d \geq 2$?

Theorem (Hochman-Meyerovitch 07’)

For $d \geq 2$ the class of $\mathbb{Z}^d$ SFTs coincides with the class of right-recursively-enumerable non-negative real numbers.

Theorem (Hochman-Meyerovitch 07’)

For $d \geq 1$ the topological entropy of any strongly irreducible $\mathbb{Z}^d$ SFT has to be a computable non-negative real number.

- Characterize the entropy in the computation perspective.
Entropy : 2-d

- No good algorithms!

Dynamical systems perspective:

1. Embedding theorem: Krieger 81’.
2. Subsystems dynamics: Littlewood 03’.
3. Dense sub-SFT or sub-sofics: Desai 06’.
4. Lower entropy factor theorem: Boyle 83’.
Given a basic set $\mathcal{B}$. Let $X = X(\mathcal{B})$ be the corresponding shift space.

Construct a sequence $\{A_n = A_n(X)\}$ in the sense B-Lin 05’.

**Theorem (B-Lin 05’)**

For $d = 2$, the entropy is the growth rate of the maximal eigenvalue of matrices $\{A_n = A_n(X)\}$, i.e.,

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log \rho(A_n),$$

where $\rho(A)$ is the maximal eigenvalue of $A$.

The recursive formula of $\{A_n(X)\}$ is established.
The sharp upper and lower bound can be computed via the “connecting” and “trace” operators.

Rigorous values: Symmetric structure!

Difficulties: Growth rate of the dimension of \( \{ A_n \} \) is exponential!

Other topics: B-Lin-Lin 08’ (3-d), B-Hu-Lin-Lin 14’ (mixing), B-Hu-Lin-Lin 14’ (Dynamical zeta functions).
Tree shift : Motivations

- Aubrun-Beal 12’ 13’ : Information and language viewpoint
  - William’s classification theorem.
  - Information theorem, language.
  - Cellular automata.

- B-Chang 15’ : Dynamical systems viewpoint
  - Topological dynamics : Chaos, irreducibility, mixing, matrix version.
  - Statistic properties : Entropy.
Tree shift : Blocks

- $\Sigma = \{0, 1, \ldots, d - 1\}$, $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$, $\Sigma^0 = \{\epsilon\}$
- Infinite tree : $t : \Sigma^* \rightarrow A$.
- Prefix-closed $L \subset \Sigma^* : \forall$ prefix of $L$ belongs to $L$.
- Pattern (or block) $u$ :
  - Defined on a finite prefix-closed subset $L$ with symbol set $A$.
  - Support : $L$.
- Block of height $n$ (or an $n$-block) : the support of $u$ is $\Sigma_{n-1}$, denoted by $\text{height}(u) = n$. 
Tree shifts: Transformation

- $t$ avoid $u$: $u$ is not a pattern of $t$.
- $u$ is an "allowed pattern" of $t$: $u$ is a pattern of $t$.
- $\mathcal{T} = A^{\Sigma^*}$: the set of all infinite trees on $A$.
- Shift transformations $\sigma_i$: For $i \in \Sigma$, $\forall$ tree $t \in \mathcal{T}$, $\sigma_i(t)$ is the tree rooted at the $i$-th child of $t$.
- Full tree-shift: $\mathcal{T}$ equipped with $\sigma_i$. 
Tree shifts: TSFT

- $X_\mathcal{F}$: all trees avoiding any element of $\mathcal{F}$.
- Tree-shift: $X \subseteq T$ and $X = X_\mathcal{F}$ for some $\mathcal{F}$.
- Tree-SFT: $X = X_\mathcal{F}$ and the forbidden set $\mathcal{F}$ is finite.
- $B_n(X)$: all blocks of height $n$ of $X$.
- $B(X)$: all blocks of $X$.
- The block $u \in B_n(X)$ is written as $u = (u_\epsilon, \sigma_0 u, \sigma_1 u)$.
  - For $d = 2$ a 2-block $u \in B$ is written as $(i, j, k)$.
- Tree SFT: An ‘intermediate class’ in between 1-d and 2-d SFTs.
Entropy of TSFT

- Entropy of $X^B$ ($h(X^B) = h(B)$):

$$h(B) = \lim_{n \to \infty} \frac{\ln^2 |B_n(X^B)|}{n},$$

(1)

where $\ln^2 = \ln \circ \ln$.

- "Hidden entropy" $\alpha$ (or sub-entropy): If $|B_n(X^B)| \approx c \exp(\alpha \kappa^n)$
  
  In this case $h = \ln \kappa$.

- ‘Doubly exponential’ comes from lattice structure and rules.

Question: How to compute $h$ and $\alpha$?
(d,k)-polynomial

Let $\mathcal{A} = \{x^{(1)}, \ldots, x^{(k)}\}$.

- Arrange $x = \prod_{i=1}^{n} x_i$ in lexicographical order for $x_i \in \mathcal{A}$.
- $f(x^{(1)}, \ldots, x^{(k)}) = \sum_{x \in \mathcal{A}^d} a_x x$: The ‘ordered summation’ of all entries in $\mathcal{A}^d$.
- $(d, k)$-polynomial $f(x^{(1)}, \ldots, x^{(k)})$: if $a_x \in \{0, 1\}$ $\forall x \in \mathcal{A}^d$.
- Full $(d, k)$-polynomial $K(d) = K(d)(x^{(1)}, \ldots, x^{(k)})$: if $a_x = 1$ $\forall x \in \mathcal{A}^d$.
- $P \leq K(d)$: if $P = \sum_{x \in \Lambda} x$ for some $\Lambda \subseteq A^d$.
- Indicator vector: $v_f = (a_x)_{x \in \mathcal{A}^d}$. 
System of nonlinear recurrence equations (SNRE)

\{a_1(n), \ldots, a_k(n)\}_{n \in \mathbb{N}} is defined by a “SNRE” : if \( \exists f_1, \ldots, f_k \) 
(d, k)-polynomials with

\[ a_i(n) = f_i(a_1(n-1), a_2(n-1), \ldots, a_k(n-1)), \quad n \geq 2, 1 \leq i \leq k, \]

and \( a_i(1) = a_i \) for \( 1 \leq i \leq k \).

- Doubly exponential sequences for single NRE \( a_{n+1} = a_n^2 + g_n \):
  - \( (\alpha, \kappa) \): Aho-Sloane 73’, Ionascu-Stanica 04’, Cox 86’, Fichtenholz 64’, Odlyzko 95’.
  - No general algorithm for SNRE!
Solving entropy by SNRE

- $X^B_i$: the collection of tree $t$ satisfying $t_ε = i$, where $1 \leq i \leq k$.
- $a_n = a_n(B) = |B_n(X_1^B)|$ and $b_n = b_n(B) = |B_n(X_1^B)|$ and $c_n = a_n + b_n$.

**Theorem (B-Chang 15': Solving entropy by SNRE)**

*Given a basic set $B$. Then $c_n = a_n + b_n$ and $a_n$ and $b_n$ satisfy the following SNRE of degree $(2,2)$:

$$
\begin{align*}
    a_n &= \sum_{(0,j,k) \in B} \Gamma_{n-1}(j)\Gamma_{n-1}(k), \\
    b_n &= \sum_{(1,j,k) \in B} \Gamma_{n-1}(j)\Gamma_{n-1}(k), \\
    a_1 &= |B_2(X_1^B)|, \\
    b_1 &= |B_2(X_2^B)|,
\end{align*}
$$

where $a_n(B) = |B_n(X_1^B)|$, $b_n(B) = |B_n(X_2^B)|$, $c_n(B) = |B_n(X^B)|$.

- Compute the entropy by solving SNRE!
Example

Let \( A = \{1, 2\} \) and \( d = 2 \). Then \( (d, k) = (2, 2) \). Suppose the basic set

\[
B = \{(1, 1, 1), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 2, 2)\}.
\]

Then the corresponding SNRE

\[
\begin{align*}
a_n &= F(a_{n-1}, b_{n-1}) = a_{n-1}^2 + a_{n-1}b_{n-1} + b_{n-1}^2, \\
b_n &= G(a_{n-1}, b_{n-1}) = a_{n-1}^2 + b_{n-1}^2, \\
a_2 &= 3, b_2 = 2.
\end{align*}
\]

And the indicator vector \( v_F = (1, 1, 0, 1) \) and \( v_G = (1, 0, 0, 1) \).
The topological entropy of a tree-shift of finite type is realized as a system of $(d, k)$-SNRE for some $d, k$. Conversely, every $(d, k)$-SNRE is corresponding to the topological entropy of some tree-shift of finite type.

- Compare to 1-d and 2-d
  - New approach!
  - Exact number of $n$-block $\approx$ Doubly exponential
Complete characterization of $d = k = 2$

Question: How about the values of entropy for $d = k = 2$?

Theorem (B-Chang 15': Complete characterization)

Suppose $d = k = 2$. Then the entropy of the tree-shift of finite type generated from a basic set $\mathcal{B}$ is either $h(\mathcal{B}) = 0$ or $h(\mathcal{B}) = \ln 2$.

- General cases: Difficult!
- Topology perspective: Lattice and local rules.
- Computation perspective: SNRE = Coupled NREs!
Realization problems and general cases

Question : How large of the entropy values for general \((d, k)\)?

Theorem (B-Chang 15’)

Let \(\rho\) be the maximal root of \(x^n - \sum_{i=1}^{n-1} c_i x^i = 0\), \(c_i \in \mathbb{N} \cup \{0\}\) for \(i = 1, \ldots, n - 1\). Then there exist \(d, k \geq 2\) and \(B\) such that \(h(B) = \ln \rho\).

- The “multinacci number” of order \(n \in \mathbb{N}\) is the real number \(\gamma_n \in (1, 2)\) which is the unique positive real solution of the equation \(1 = x^{-1} + x^{-2} + \cdots + x^{-n}\).
  - The smallest multinacci number is the multinacci number of order 2 and is equal to the golden mean \(\frac{1 + \sqrt{5}}{2}\), and \(\gamma_{n+1} > \gamma_n\) for \(n \in \mathbb{N}\).
- The multinacci number is realizable.

Question :

1. What about the value \(h(B)\) : R-R-E, computable?
2. Realization of 2-d SFT entropy?
Example (Golden-Mean shift)

Let \( g = \frac{1+\sqrt{5}}{2} \): The maximal root of

\[ x^2 - x - 1 = 0. \]

- Construct the SNRE with \( h(\mathcal{B}) = \ln g \).

\[
\begin{align*}
  a_n^{(1)} &= 2a_{n-1}^{(1)}a_{n-1}^{(2)}b_{n-1}, \\
  a_n^{(2)} &= a_{n-1}^{(1)}b_{n-1}^2, \\
  b_n &= b_{n-1}^3, \\
  a_1^{(1)} &= 2, \\ b_1 &= a_1^{(2)} = 1.
\end{align*}
\]

- The possible basic sets:

\[ \mathcal{B} = \{(1, 1, 2, 3), (1, 2, 1, 3), (2, 1, 3, 3), (3, 3, 3, 3)\}. \]
SNRE \((d > 2)\):

\[
\begin{align*}
a^{(1)}_n &= F^{(1)}(a^{(1)}_{n-1}, \ldots, a^{(k)}_{n-1}), \\
&\quad \vdots \\
a^{(k)}_n &= F^{(k)}(a^{(1)}_{n-1}, \ldots, a^{(k)}_{n-1}), \\
a^{(i)}_1 &= a_i \text{ for } i = 1, \ldots, k.
\end{align*}
\]

**Indicator vector**: \(v^{(i)} := v_{F^{(i)}}\) and \(V := \{v^{(i)}\}^{k}_{i=1}\).

**Dominate vector** \(v^{(l)} : v^{(l)} \geq v^{(j)}, \forall 1 \leq j \leq k\).

**Dominate-type SNRE**: \(\exists\) a dominate vector \(v \in V\).
Entropy: Dominate type
Self-recurrent dominate vector

\( \nu^{(l)} \): Dominate vector

- Self-index: \( s_d = s_d^{(l)} \) such that \( s_d \)-term in \( F^{(l)} \) is \( (a^{(l)})^d \).
- \( s_n \): The \( s_n \)-term of \( F^{(l)} \) has \( n \)'s \( a^{(l)} \).

**Theorem (B-Chang 15' : Self-recurrent dominate vector)**

Given \( \mathcal{B} \), if the corresponding SNRE is of dominate-type with \( \nu^{(l)} \) being a self-recurrent dominate vector. Then \( h(\mathcal{B}) = \ln h \) or 0, where
\[
h := \max\{ n : s_n^{(l)} = 1 \}.
\]
Furthermore, if \( s_d = 1 \), then \( h(\mathcal{B}) = \ln d \).
Entropy: Symmetric type

- Decompose basic set \( B \) into \( B = \bigcup_{i=1}^{k} B^{(i)} \), where
  \[
  B^{(i)} = \{ \omega : \omega \text{ is labeled by } i \} \text{ for } 1 \leq i \leq k.
  \]
- Symmetric-type \( X^B \): if \( \pi(B^{(i)}) = \pi(B^{(j)}) \) \( \forall i \neq j \).

**Theorem (B-Chang 15’: Symmetric type)**

*If an SNRE is of symmetric-type with \( n^{(l)} \geq 2 \) for some \( l \geq 1 \). Then \( h(B) = \ln d \).*
n-Blocks with B.C.:

\[ B_n^P = \{ u \in B_n(X) : u_w = u_\epsilon \text{ for } |w| = n - 1 \}, \]
\[ B_n^{Di} = \{ u \in B_n(X) : u_w = i \text{ for } |w| = n - 1, \ 1 \leq i \leq k \}, \]
\[ B_n^N = \{ u \in B_n(X) : u_w = u_{\hat{w}} \text{ for } |w| = n - 1, \hat{w} = w_1 \cdots w_{n-2} \}, \]

Topological entropy with B.C.:

\[ h^t(X^B) = \lim_{n \to \infty} \frac{\ln^2 |B_n^t(X^B)|}{n} \]
Boundary effect: Neumann B.C.

Problem

\[ h = h^N = h^P = h^D ? \]

- Periodic B.C. (Barreira 96’): Topological pressure, avoiding Markov partition.

Theorem (Neumann B.C.)

Suppose \( X^B \) is a Markov tree-shift and \( h(X^B) > 0 \). Then

\[ h^N(X^B) = h(X^B) \text{ if and only if either (1) } \{(1, 1, 1), (2, 2, 2)\} \subseteq B \text{ or (2) } \{(1, i, i), (2, i, i)\} \subseteq B \text{ for some } i = 1, 2. \]
### Theorem (B-Chang 2015 : Dirichlet B.C.)

Suppose $X^B$ is a Markov tree-shift and $h(X^B) > 0$. Then, for $i = 1, 2$, $h^{D_i}(X^B) = h(X^B)$ if and only if either (1) $\{(1, i, i), (2, i, i)\} \subseteq B$ or (2) $\{((\bar{i}, i, i), (1, \bar{i}, \bar{i}), (2, \bar{i}, \bar{i})) \} \subseteq B$, where $i + \bar{i} = 3$.

### Theorem (B-Chang 2015 : Periodic B.C.)

Suppose $X^B$ is a Markov tree-shift and $h(X^B) > 0$. Then

1. $h^P(X^B) = h(X^B)$ if $v_i$ dominates $v_{\bar{i}}$ and $\{(1, i, i), (2, i, i)\} \subseteq B$ for some $i = 1, 2$, where $i + \bar{i} = 3$.

2. If $h^P(X^B) = h(X^B)$, then $\{(1, i, i), (2, i, i)\} \subseteq B$ for some $i = 1, 2$. 
Conclusion

- **Entropy without B.C.**
  - Complete characterization for $d = k = 2$.
  - Realization theorem.
  - General cases (dominate and symmetric types)

- **Entropy with B.C.**
  - Neumann B.C.
  - Dirichlet B.C.
  - Periodic B.C.
Open problems

- Entropy (Downarowicz 11’, Robinson 95’, Katok 97’)
  - Hidden entropy.
  - Sofic tree shift (Aubrun-Beal 12’ 13’).
  - Entropy conjugacy (Lind-Marcus 95’).
  - Dimension (Pesin 08’)
- Thermodynamics (Pesin 08’, Ruelle 78’)
  - Pressure.
  - Variational principle.