Classification of bifurcation curves for a multiparameter diffusive logistic problem with Holling type-III functional response

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1. Introduction

We study **exact multiplicity** and **bifurcation curves** of positive solutions for a multiparameter diffusive logistic problem with Holling type-III functional response

\[
\begin{aligned}
&\left\{ u''(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^p}{1 + u^p} \right] = 0, \quad -1 < x < 1, \\
&u(-1) = u(1) = 0,
\end{aligned}
\]  

(1.1)

where \( u \) is the population density of the species, \( f(u) = ug(u) \) is the growth rate,

\[
g(u) = r \left( 1 - \frac{u}{q} \right) - \frac{u^{p-1}}{1 + u^p},
\]  

(1.2)

is the growth rate per capita, \( p > 1, \) \( q, r \) are two positive dimensionless parameters, and \( \lambda > 0 \) is a bifurcation parameter.

On the right-hand side of (1.2), the first term \( r \left( 1 - \frac{u}{q} \right) \) is the **per capita birth rate** and the second term \( \frac{u^{p-1}}{1 + u^p} \) is the **per capita death rate**.
We define the bifurcation curve of (1.1)

\[ \bar{S} = \{ (\lambda, \| u_\lambda \|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)} \} . \]
(I) We say that the bifurcation curve $\tilde{S}$ is an S-shaped curve on the $(\lambda, ||u||_\infty)$-plane if $\tilde{S}$ consists of exactly one continuous curve with exactly two turning points at some points $(\lambda^*, ||u_{\lambda^*}||_\infty)$ and $(\lambda_*, ||u_{\lambda_*}||_\infty)$ such that

(i) $\lambda_* < \lambda^*$ and $||u_{\lambda^*}||_\infty < ||u_{\lambda_*}||_\infty$,

(ii) at $(\lambda^*, ||u_{\lambda^*}||_\infty)$ the bifurcation curve $\tilde{S}$ turns to the left,

(iii) at $(\lambda_*, ||u_{\lambda_*}||_\infty)$ the bifurcation curve $\tilde{S}$ turns to the right.

Note that, the upper stable branch represents outbreak states.
\[
\begin{align*}
\frac{d^2 u}{dx^2} + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^p}{1 + u^p} \right] &= 0, \quad -1 < x < 1, \\
u(-1) &= u(1) = 0.
\end{align*}
\]

(II) We say that the bifurcation curve \( \bar{S} \) is a broken S-shaped curve on the \( (\lambda, \|u\|_\infty) \)-plane if \( \bar{S} \) has two disjoint connected components such that

(i) the upper branch of \( \bar{S} \) has \textit{exactly one} turning point at some point \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) where the curve turns to the \textit{right},

(ii) the lower branch of \( \bar{S} \) is a monotone increasing curve.
Noy-Meir studied a grazing system of herbivore-plant interaction. He considered the differential equation

\[
\frac{dN}{dT} = G(N) - Hc(N),
\]

(1.3)

where \(N(T)\) is the vegetation biomass, \(G(N)\) is the growth rate of vegetation in absence of grazing, \(H\) is the herbivore population density, and \(c(N)\) is the per capita consumption rate of vegetation by the herbivore. For problem (1.3), if \(G(N)\) is given by the logistic function, and \(c(N)\) is the Holling type III function, then (1.3) takes the form

\[
\frac{dN}{dT} = r_N N \left(1 - \frac{N}{K_N}\right) - B \frac{N^p}{A^p + N^p},
\]

where \(p > 1\) and \(A, B, r_N, K_N > 0\), see (Shi and Shivaji 2006).


The **Holling type III functional response** was also considered in (Sugie et al. 1997) and (Sugie and Katagama 1999). They studied the existence of stable limit cycle and global asymptotic stability for a predator-prey system

\[
\begin{align*}
\frac{dx}{dt} &= rx \left( 1 - \frac{x}{K} \right) - \frac{x^p y}{A^p + x^p}, \\
\frac{dy}{dt} &= y \left( \frac{\mu x^p}{A^p + x^p} - d \right).
\end{align*}
\]

In addition, the Holling type III functional response also appears as the dynamics of lake eutrophication

\[ \frac{dN}{dT} = a - bN + B \frac{N^p}{A^p + N^p}, \]

where \( N(T) \) is the level of nutrients suspended in phytoplankton causing turbidity, \( a \) is the nutrient loading, \( b \) is the nutrient removal rate, and \( B \) is the rate of internal nutrient recycling, see (Carpenter et al. 1999) and (Scheffer et al. 2001).


The model of the diffusive logistic problem with Holling type-III functional response (without diffusion) is governed by the equation

$$\frac{dN}{dT} = r_N N \left( 1 - \frac{N}{K_N} \right) - B \frac{N^p}{A^p + N^p},$$

where $N$ is the population density of the species, and

1. the first term $r_N N \left( 1 - \frac{N}{K_N} \right)$ represents logistic growth, where $r_N$ is the linear birth rate of the species and $K_N$ is the carrying capacity,
The model of the diffusive logistic problem with Holling type-III functional response (without diffusion) is governed by the equation

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\]

where \( N \) is the population density of the species, and

1. the first term \( r_N N \left( 1 - \frac{N}{K_N} \right) \) represents logistic growth, where \( r_N \) is the linear birth rate of the species and \( K_N \) is the carrying capacity,
2. the second term \( B N^p / (A^p + N^p) \) represents predation of Holling type III functional response generated by predator, where \( B \) is a positive constant which represents the maximum predation rate of the predator and \( A \) is the species population when the predation rate is at half of the maximum, for \( p = 2 \), see (Ludwig, Jones and Holling 1978).

The model of the diffusive logistic problem with Holling type-III functional response (with diffusion) is governed by the equation

\[
\frac{\partial N}{\partial T} = D \frac{\partial^2 N}{\partial X^2} + r_N N \left( 1 - \frac{N}{K_N} \right) - B \frac{N^p}{A^p + N^p}
\]  

(1.4)

in spatial one dimension, where \( D > 0 \) is the diffusion (dispersion) coefficient characterizing the rate of the spatial dispersion of the species population, for \( p = 2 \), see (Ludwig, Aronson and Weinberger 1979). (Note that, for the sake of simplicity, in their paper, the habitat is taken as the infinite strip \( \left\{ (X, Y) : -\frac{L}{2} \sqrt{\frac{D}{r_N}} < X < \frac{L}{2} \sqrt{\frac{D}{r_N}}, -\infty < Y < \infty \right\} \) of width \( L \sqrt{\frac{D}{r_N}} \) and the species density is assumed to be independent of the \( Y \) coordinate.)

The model of the diffusive logistic problem with Holling type-III functional response (with diffusion)

\[ \frac{\partial N}{\partial T} = D \frac{\partial^2 N}{\partial X^2} + r_N N \left( 1 - \frac{N}{K_N} \right) - B \frac{N^p}{A^p + N^p} \] (1.4)

Let

\[ w = \frac{N}{A}, \quad \tilde{t} = r_N T, \quad \tilde{x} = \sqrt{\frac{r_N}{D}} X, \quad r = \frac{r_N A}{B}, \quad q = \frac{K_N}{A}. \]

Then problem (1.4) takes the form

\[ \frac{\partial w}{\partial \tilde{t}} = \frac{\partial^2 w}{\partial \tilde{x}^2} + w \left( 1 - \frac{w}{q} \right) - \frac{1}{r} \frac{w^p}{1 + w^p}. \] (1.5)
The model of the diffusive logistic problem with Holling type-III functional response (with diffusion)

Assume that the habitat $-L/2 \leq \tilde{x} \leq L/2$ is surrounded by a totally hostile, outer environment. That is, Eq. (1.5) holds in the strip $|\tilde{x}| < L/2$ and

$$w(-L/2, \tilde{t}) = w(L/2, \tilde{t}) = 0, \quad \tilde{t} > 0.$$  \hspace{1cm} (1.6)

Let $v(x, t) = w(\tilde{x}, \tilde{t})$ with $x = \frac{2}{L} \tilde{x}$, $t = \left(\frac{2}{L}\right)^2 \tilde{t}$, and let $\lambda = \frac{1}{r} \left(\frac{L}{2}\right)^2$. Then problem (1.5), (1.6) takes the form

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \lambda \left[ rv \left(1 - \frac{v}{q}\right) - \frac{v^p}{1 + v^p} \right], \quad -1 < x < 1, \quad t > 0, \\ v(-1, t) = v(1, t) = 0, \quad t > 0. \end{array} \right.$$ \hspace{1cm} (1.7)
Let \( u(x) \) denote a steady-state (positive) population density of (1.7). Then \( u(x) \) satisfies

\[
\begin{aligned}
\begin{cases}
    u''(x) + \lambda \left( ru \left( 1 - \frac{u}{q} \right) - \frac{u^p}{1 + u^p} \right) = 0, & -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{cases}
\end{aligned}
\]  

(1.1)
Spruce budworm problem

For $p = 2$, problem

$$
\begin{cases}
  u''(x) + \lambda \left[ ru \left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}\right] = 0, \quad -1 < x < 1,
  \\
  u(-1) = u(1) = 0.
\end{cases}
$$

(1.8)

is a famous budworm problem in mathematical biology. For this budworm problem, roughly speaking, $r$ measures the foliage density while $q$ depends upon the properties of the budworm and the predators, see (Ludwig, Aronson and Weinberger 1979).

Problem (1.8) has been extensively studied by many authors, see (Murray 2002, 2003), (Lee, Sasi and Shivaji 2011) and (Wang and Yeh 2013).


The spruce budworm (Choristoneura fumiferana) is an very destructive native insect that lives in the spruce and fir forests of Northeastern United States and Canada.
Budworm

Pupa

Moth
Spruce tree

The foliage of the spruce tree
The spruce forest  
Spruce budworm defoliation
Spruce budworm defoliation at British Columbia
Normally the spruce budworm exists in low numbers in these forests, kept in check by the predators (primarily birds). However, every 40 years or so there is an outbreak of these insects and their numbers can defoliate and damage most of the spruce and fir trees in a forest in about 4 years. The budworm is capable of a \textit{five-fold} increase in density per year (under ideal conditions of food and weather), and the budworm can increase its density \textit{several hundred fold} in a few years during outbreaks. Outbreaks can last for several years or they may collapse after only 1 or 2 years. As a consequence, the dynamics of the forest is reversed and living conditions deteriorate, but for a while the budworm density remains relatively high before it returns to low numbers again.
\[
\begin{cases}
  u''(x) + \lambda \left[ r u \left( 1 - \frac{u}{q} \right) - \frac{u^2}{1 + u^2} \right] = 0, \quad -1 < x < 1, \\
  u(-1) = u(1) = 0.
\end{cases}
\] 

(1.8)
Numerical simulation of the bifurcation curve with $p=2$, $q=302$ and $r=2$

Numerical simulation shows a big jump from point $A$ to point $B$ with

$$\frac{\|v_{\lambda^*}\|_\infty}{\|u_{\lambda^*}\|_\infty} \approx 70.661$$

when $\lambda$ increases across $\lambda^* \approx 1.638$. So an outbreak occurs for the budworm population.
Numerical simulation of the bifurcation curve with $p=2$, $q=302$ and $r=0.7$

Numerical simulation shows a huge jump from point $A$ to point $B$ with

$$\frac{\|v_{\lambda^*}\|_{\infty}}{\|u_{\lambda^*}\|_{\infty}} \approx 203.527$$

when $\lambda$ increases across $\lambda^* \approx 11.700$. So an outbreak occurs hugely for the budworm population.
\[ \begin{cases} 
  u''(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^2}{1 + u^2} \right] = 0, \quad -1 < x < 1, \\
  u(-1) = u(1) = 0. 
\] 

(1.8)

Appropriate values for the parameters \( q \) and \( r \) in (1.8) have been estimated.

1. Based on basic ecological knowledge (Level II-general quantitative information in (Ludwig, Jones and Holling, 1978)), \( q \) ranges from 50 to 300 and \( r \) will range a minimum near 0 (for an infant forest) to a maximum of 1.07 to 3.84 (for a mature forest).

2. The studies in the more refinement from extensive field study of the forest lead to the parameters \( q = 302 \) and \( r \) ranges a minimum near 0 to a maximum of 0.994.
\[
\begin{aligned}
&u''(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^2}{1 + u^2} \right] = 0, \quad -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{aligned}
\] (1.8)

Accordingly, (Ludwig, Aronson and Weinberger 1979) studied the exact multiplicity of positive solutions and the shape of bifurcation curve \( \bar{S} \) of spruce budworm population problem (1.8) for various parameters \( q, r > 0 \). In particular, they chose parameter \( q = 302 \) and several various values of parameter \( r = 2, 0.7, 0.2, 0.015, 0.01 \). But their arguments are mostly not rigorous proofs.
\[
\begin{aligned}
&\left\{
\begin{array}{l}
u''(x) + \lambda \left[ ru \left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}\right] = 0, \quad -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{array}
\right.
\end{aligned}
\] (1.8)

Applying the quadrature method (time-map method), (Ludwig, Aronson and Weinberger 1979) showed that the rough bifurcation curve goes from a monotone curve with a unique small steady state, to a broken S-shaped curve, to an S-shaped curve, and finally a monotone curve with a unique large steady state, when \( r \) increases from \( 0^+ \) to a large value. Note that the results of evolutionary bifurcation curves in (Ludwig, Aronson and Weinberger 1979) are not exact, and it was only shown that the equation has at least three positive solutions but not exactly three.

Recently, (Wang and Yeh 2013) gave a partial answer of this conjecture in (Ludwig, Aronson and Weinberger 1979). Assume that either $r \leq \rho_1 q$ and $(q,r)$ lies above the curve

$$\Gamma = \left\{(q,r) : q(a) = \frac{2a^3}{a^2 - 1}, r(a) = \frac{2a^3}{(a^2 + 1)^2}, 1 < a < \sqrt{3}\right\}$$

or $r \leq \rho_2 q$ for some constants $\rho_1 \approx 0.0939$ and $\rho_2 \approx 0.0766$. Then on the $(\lambda, ||u||_\infty)$-plane, they gave a classification of three qualitatively different bifurcation curves: an S-shaped curve, a broken S-shaped curve, and a monotone increasing curve. Their results settled rigorously a long-standing open problem in (Ludwig, Aronson and Weinberger 1979).

For \( p > 1 \), the exact multiplicity of positive solutions and the shape of bifurcation curve \( \tilde{S} \) for (1.1) and the \( n \)-dimensional problem of (1.1)

\[
\begin{align*}
\Delta u(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^p}{1+u^p} \right] &= 0, \quad x \in \Omega, \\
u(x) &= 0, \quad x \in \partial \Omega
\end{align*}
\]

remain mostly open since 1979, see (Jiang and Shi, 2009). One of the main difficulties is that the growth rate per capita \( g(u) = r(1 - \frac{u}{q}) - \frac{u^{p-1}}{1+u^p} \) could initially decrease, but then increases to a peak before falling to zero.

Note that, for $p = 1$, the function $\frac{u}{1+u}$ is called a Holling type-II function. A $n$-dimensional Dirichlet problem of (1.1) with $p = 1$

$$
\begin{cases}
\triangle u(x) + \lambda \left[r u \left(1 - \frac{u}{q}\right) - \frac{u}{1+u}\right] = 0, x \in \Omega, \\
u(x) = 0, x \in \partial \Omega
\end{cases}
$$

was considered by (Korman and Shi 2001). They obtained two qualitatively different bifurcation curves: a $\subset$-shaped curve and a monotone increasing curve, see (Korman and Shi 2001).

Fig. 1. Classified graphs of growth rate per capita \( g(u) = r(1 - \frac{u}{q}) - \frac{u^{p-1}}{1+u^p} \) on \((0, \infty)\), drawn on the first quadrant of \((q, r)\)-parameter plane.
\[
\begin{aligned}
\left\{ \begin{array}{c}
u''(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^p}{1 + u^p} \right] = 0, \quad -1 < x < 1, \\
u(-1) = u(1) = 0,
\end{array} \right.
\end{aligned}
\] (1.1)

For problem (1.1) with fixed \( p > 1 \), in Fig. 1, we divide the first quadrant of \((q, r)\)-parameter plane into the disjoint union of the three curves \( \Gamma_0, \Gamma_1, \Gamma_2 \) and four regions \( R_1, R_2, R_3, R_4 \) defined as follows:

\[ \Gamma_0 = \{(q, r) : r = m_p q > 0\}, \]

\[ \Gamma_1 = \left\{ (q, r) : q(a) = \frac{a[2a^p-(p-2)]}{a^p-(p-1)}, \ r(a) = \frac{a^{p-1}[2a^p-(p-2)]}{(a^p+1)^2}, \ \sqrt{p-1} < a < C_p^{**} \right\}, \]

\[ \Gamma_2 = \left\{ (q, r) : q(a) = \frac{a[2a^p-(p-2)]}{a^p-(p-1)}, \ r(a) = \frac{a^{p-1}[2a^p-(p-2)]}{(a^p+1)^2}, \ C_p < a < \infty \right\}, \]
\[ R_1 = \{(q,r) : 0 < r < m_p q \text{ and } (q,r) \text{ lies above the curve } \Gamma_1 \}, \]
\[ R_2 = \{(q,r) : 0 < r < m_p q, \text{ and } (q,r) \text{ lies between curves } \Gamma_1 \text{ and } \Gamma_2 \}, \]
\[ R_3 = \{(q,r) : 0 < r < m_p q \text{ and } (q,r) \text{ lies below the curve } \Gamma_2 \}, \]
\[ R_4 = \{(q,r) : r > m_p q > 0 \}, \]

where
\[ C_p^* = \left( \frac{p^2 + 3p - 4 + p\sqrt{p^2 + 6p - 7}}{4} \right)^{1/p} > \sqrt[p]{p - 1} > 0, \]
\[ m_p = \frac{(C_p^*)^{p-2}[1 - p + (C_p^*)^p]}{[1 + (C_p^*)^p]^2} > 0. \]
According to (Jiang and Shi, 2009), we classify all growth rate patterns according to the monotonicity of the growth rate per capita

\[ g(u) = r\left(1 - \frac{u}{q}\right) - \frac{u^{p-1}}{1+u^p} \] on \([0, \infty)\):

1. \(g(u)\) is of logistic type, if \(g(u)\) is strictly decreasing;
According to (Jiang and Shi, 2009), we classify all growth rate patterns according to the monotonicity of the growth rate per capita

\[ g(u) = r \left(1 - \frac{u}{q}\right) - \frac{u^{p-1}}{1+u^p} \text{ on } [0, \infty): \]

1. \( g(u) \) is of logistic type, if \( g(u) \) is strictly decreasing;
2. \( g(u) \) is of hysteresis type, if \( g(u) \) changes from decreasing to increasing then to decreasing again when \( u \) increases.

In the hysteresis case, if \( g(u) \) has three positive zeros, then it is strong hysteresis, otherwise it is weak hysteresis.
We have that:

(i) If \((q, r) \in \Gamma_0 \cup R_4\), then \(g\) is of logistic type. In this case \(g(u)\) has exactly one positive zero at some \(\beta_1\). The bifurcation curve \(\bar{S}\) of positive solutions of (1.1) is a monotone increasing curve since
\[
f(u) - uf'(u) = ug(u) - u(g(u) + ug'(u)) = -u^2g'(u) > 0 \text{ on } (0, \beta_1)
\]
except possibly at some value \(\beta_0 \in (0, \beta_1)\) when \((q, r) \in \Gamma_0\).

(ii) If \((q, r) \in \Gamma_2\), then \(g\) is of weak hysteresis type. In this case \(g(u)\) has exactly one positive zero at some \(\beta_1\). The bifurcation curve \(\bar{S}\) of positive solutions of (1.1) is a monotone increasing curve since
\[
f(u) - uf'(u) = ug(u) - u(g(u) + ug'(u)) = -u^2g'(u) > 0 \text{ on } (0, \beta_1)
\]
We have that:

(i) If \((q,r) \in \Gamma_0 \cup R_4\), then \(g\) is of logistic type. In this case \(g(u)\) has exactly one positive zero at some \(\beta_1\). The bifurcation curve \(\tilde{S}\) of positive solutions of (1.1) is a monotone increasing curve since
\[
f(u) - uf'(u) = ug(u) - u(g(u) + ug'(u)) = -u^2g'(u) > 0\] on \((0, \beta_1)\) except possibly at some value \(\beta_0 \in (0, \beta_1)\) when \((q,r) \in \Gamma_0\).

(ii) If \((q,r) \in \Gamma_2 \cup R_3\), then \(g\) is of weak hysteresis type. In this case \(g(u)\) has exactly one positive zero at some \(\beta_1\). The bifurcation curve \(\tilde{S}\) of positive solutions of (1.1) is a monotone increasing curve since
\[
f(u) - uf'(u) = -u^2g'(u) > 0\] on \((0, \beta_1)\).
(iii) If \((q, r) \in \Gamma_1 \cup R_2\), then \(g\) is of hysteresis type. In particular, \(g\) is of strong hysteresis type if \((q, r) \in R_2\). Notice that:

(a) For \((q, r) \in \Gamma_1, f(u) = ug(u)\) has exactly two positive zeros at some \(\beta_1 < \beta_3\) such that \(g(u) > 0\) on \((0, \beta_1) \cup (\beta_1, \beta_3)\), \(g(\beta_1) = g(\beta_3) = 0\) and \(g(u) < 0\) on \((\beta_3, \infty)\).

(b) For \((q, r) \in R_2\) with \(q\) fixed, \(f(u) = ug(u)\) has exactly three positive zeros at some \(\beta_1 < \beta_2 < \beta_3\) such that \(g(u) > 0\) on \((0, \beta_1) \cup (\beta_2, \beta_3)\), \(g(\beta_1) = g(\beta_2) = g(\beta_3) = 0\) and \(g(u) < 0\) on \((\beta_1, \beta_2) \cup (\beta_3, \infty)\).
In addition, for fixed 

\[ q > q^*(p) \equiv \frac{2(p+1+\sqrt{p^2+6p-7})}{p-1+\sqrt{p^2+6p-7}} \left( \frac{p^2+3p-4+p\sqrt{p^2+6p-7}}{4} \right)^{1/p}, \]

there exists \( \bar{r}_2 = \bar{r}_2(q) \in (r_2(q), r_1(q)) \) such that

\[
\int_{\beta_1}^{\beta_3} f(u) \, du < 0 \quad \text{for } r_2(q) < r < \bar{r}_2(q),
\]

\[
\int_{\beta_1}^{\beta_3} f(u) \, du = 0 \quad \text{for } r = \bar{r}_2(q),
\]

\[
\int_{\beta_1}^{\beta_3} f(u) \, du > 0 \quad \text{for } \bar{r}_2(q) < r < r_1(q).
\]
We thus define the curve

$$\tilde{\Gamma}_2 = \{(q,r) : q > q^*(p) \text{ and } r = \tilde{r}_2(q)\},$$

and regions

$$\tilde{R}_2 = \{(q,r) : 0 < r < m_p q, \text{ and } (q,r) \text{ lies between curves } \Gamma_1 \text{ and } \tilde{\Gamma}_2\},$$

$$\tilde{R}_2 = \{(q,r) : 0 < r < m_p q, \text{ and } (q,r) \text{ lies on curve } \tilde{\Gamma}_2 \text{ or between curves } \Gamma_2 \text{ and } \tilde{\Gamma}_2\}.$$
Notice that \( \int_{\beta_1}^{\beta_3} f(u) \, du > 0 \) for \( \tilde{r}_2(q) < r < r_1(q) \), then there exists a number \( \gamma \in (\beta_2, \beta_3) \) such that \( \int_{\beta_1}^{\gamma} f(u) \, du = 0 \).

(iv) If \( (q, r) \in R_1 \), then \( g \) is of weak hysteresis type. In this case \( g(u) \) has exactly one positive zero at some \( \beta_3 \), \( g(u) \) changes from decreasing to increasing then to decreasing on \( [0, \beta_3) \), \( g(u) > 0 \) on \( (0, \beta_3) \), \( g(\beta_3) = 0 \), and \( g(u) < 0 \) on \( (\beta_3, \infty) \).
Fig. 2. (a) S-shaped bifurcation curve $\tilde{S}$ of (1.1). (b)–(c) Broken S-shaped bifurcation curves $\bar{S}$ of (1.1).
For fixed $p > 1$, we define functions

\begin{align*}
I(u) &= \frac{pu^{p-2}[-(p-1)+(p+1)u^p]}{2(1+u^p)^3}, \\
J(u) &= \frac{u^{p-2}(1-p+u^p)}{(1+u^p)^2}, \\
K(u) &= \frac{4}{u^4}\left[\frac{u^2}{2} + \frac{u^2}{1+u^p} - 3\int_0^u \frac{t}{1+t^p}dt\right], \\
M(u) &= \frac{3}{u^3}\left[u + \frac{u}{1+u^p} - 2\int_0^u \frac{1}{1+t^p}dt\right].
\end{align*}
Graphs of functions \( \eta = I(u) \), \( \eta = J(u) \), \( \eta = K(u) \), \( \eta = \eta_1 \).

We first define two positive numbers \( \eta_{1,p} \) and \( \eta_{2,p} \) for \( p > 1 \) as follows:

(i) Let

\[ \eta_{1,p} \text{ be the unique positive intersection value of the two curves } \eta = I(u) \text{ and } \eta = K(u) \text{ for } u > 0. \]
Graphs of functions $\eta = I(u)$, $\eta = J(u)$, $\eta = M(u)$, $\eta = \eta_2$.

(ii) Let

$\eta_{2,p}$ be the unique positive intersection value of the two curves $\eta = I(u)$ and $\eta = M(u)$ for $u > 0$. 

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Remark

We know that for $p > 1$,

$$\eta_{1,p} < J(C_p^*) = m_p \text{ and } \eta_{2,p} < J(C_p^*) = m_p.$$
For $p > 1$ and $0 < \eta < m_p$, let $B_{1,p}(\eta)$ be the smallest positive root of $I(u) = \eta$ and $C_{2,p}(\eta)$ be the largest positive root of $J(u) = \eta$. We also define two positive numbers $\eta^*_{1,p}$ and $\eta^*_{2,p}$ for $p > 1$ as follows:

(i) Let

$$\eta^*_{1,p} \equiv \sup \left\{ \eta : 0 < \eta \leq \eta_{1,p} \text{ and } N_1(B_{1,p}(\eta)) + N_2(C_{2,p}(\eta)) > 0 \right\},$$

where

$$N_1(u) \equiv \frac{u^2 \left[-12 + (p^2 - 5p - 24)u^p + (-p^2 - 5p - 12)u^{2p}\right]}{4(1+u^p)^3} + 6 \int_0^u \frac{t}{1 + t^p} dt$$

and

$$N_2(u) \equiv \frac{u^2 \left[6 + (p + 7)u^p + u^{2p}\right]}{2(1+u^p)^2} - 6 \int_0^u \frac{t}{1 + t^p} dt.$$
Let

\[ \eta_{2,p}^* \equiv \sup \{ \eta : 0 < \eta \leq \eta_{2,p} \text{ and } N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) > 0 \}, \]

where

\[ N_3(u) \equiv -\frac{[6u-(p^2-4p-12)u^{p+1}+(p^2+4p+6)u^{2p+1}]}{3(1+u^p)^3} + 2 \int_0^u \frac{1}{1+t^p} \, dt \]

and

\[ N_4(u) \equiv \frac{u((p+8)u^p+2u^{2p})}{3(1+u^p)^2} - 2 \int_0^u \frac{1}{1+t^p} \, dt. \]
Remark

Numerical simulations show that, for $p \in [1.01, 10]$,

$$N_1(B_{1,p}(\eta_{1,p})) + N_2(C_{2,p}(\eta_{1,p})) > 0$$

and

$$N_3(B_{1,p}(\eta_{2,p})) + N_4(C_{2,p}(\eta_{2,p})) > 0.$$

Hence we obtain that, for $p \in [1.01, 10]$,

$$\eta^*_{1,p} = \sup \{ \eta : 0 < \eta \leq \eta_{1,p} \text{ and } N_1(B_{1,p}(\eta)) + N_2(C_{2,p}(\eta)) > 0 \} = \eta_{1,p}$$

and

$$\eta^*_{2,p} = \sup \{ \eta : 0 < \eta \leq \eta_{2,p} \text{ and } N_3(B_{1,p}(\eta)) + N_4(C_{2,p}(\eta)) > 0 \} = \eta_{2,p}.$$ 

In particular, for $p = 2$, we obtain $\eta^*_{1,p} = \eta_{1,p} \approx 0.0939$ and $\eta^*_{2,p} = \eta_{2,p} \approx 0.0766$. 
Let $u_\lambda$ be a positive solution of (1.1) with $\alpha \equiv \|u_\lambda\|_\infty > 0$.

**Theorem (2.1)**

Consider (1.1) with $p > 1$. If $(q,r) \in R_1$ and $r \leq \eta^*_{1,p}q$, then

$$\lim_{\alpha \to 0^+} \lambda(\alpha) = \hat{\lambda} \equiv \frac{\pi^2}{4r}, \quad \lim_{\alpha \to \beta_3^-} \lambda(\alpha) = \infty,$$

and the bifurcation curve $\tilde{S}$ of (1.1) is an S-shaped curve on the $(\lambda, \|u\|_\infty)$-plane. More precisely, $\tilde{S}$ consists of a continuous curve with exactly two turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ such that $\hat{\lambda} < \lambda^*_* < \lambda^* < \infty$ and $0 < \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty < \beta_3$. 
Problem (1.1) has:

(i) exactly three positive solutions $w_\lambda$, $u_\lambda$, $v_\lambda$ with $w_\lambda < u_\lambda < v_\lambda$ for $\lambda_* < \lambda < \lambda^*$,

(ii) exactly two positive solutions $w_\lambda$, $u_\lambda$ with $w_\lambda < u_\lambda$ for $\lambda = \lambda_*$ and exactly two positive solutions $u_\lambda$, $v_\lambda$ with $u_\lambda < v_\lambda$ for $\lambda = \lambda^*$,

(iii) exactly one positive solution $w_\lambda$ for $\hat{\lambda} < \lambda < \lambda_*$ and exactly one positive solution $v_\lambda$ for $\lambda > \lambda^*$,

(iv) no positive solution for $0 < \lambda \leq \hat{\lambda}$. 
Theorem (2.1)

Furthermore,

\[ \lim_{\lambda \to \hat{\lambda}^+} \| w_\lambda \|_\infty = 0 \text{ and } \lim_{\lambda \to \infty} \| v_\lambda \|_\infty = \beta_3. \]
Theorem (2.2)

Consider (1.1) with \( p > 1 \). If \((q, r) \in \mathcal{R}_2\) and \( r \leq \eta_{2,p}^* q \), then

\[
\lim_{\alpha \to 0^+} \lambda(\alpha) = \hat{\lambda} \equiv \frac{\pi^2}{4r},
\]

\[
\lim_{\alpha \to \beta_1^-} \lambda(\alpha) = \lim_{\alpha \to \gamma^+} \lambda(\alpha) = \lim_{\alpha \to \beta_3^-} \lambda(\alpha) = \infty,
\]

and the bifurcation curve \( \tilde{S} \) of (1.1) is a broken S-shaped curve on the \((\lambda, \|u\|_\infty)\)-plane. More precisely, \( \tilde{S} \) has two disjoint connected components such that the upper branch of \( \tilde{S} \) has exactly one turning point \((\hat{\lambda}, \|u_{\hat{\lambda}}\|_\infty)\), with \( \hat{\lambda} < \lambda_* < \infty \) and \( \gamma < \|u_{\lambda_*}\|_\infty < \beta_3 \), where the curve turns to the right, and the lower branch of \( \tilde{S} \) is a monotone increasing curve starting at \((\hat{\lambda}, 0)\).
Theorem (2.2)

Problem (1.1) has:

(i) exactly three positive solutions $w_\lambda, u_\lambda, v_\lambda$ with $w_\lambda < u_\lambda < v_\lambda$ for $\lambda > \lambda^*$,
(ii) exactly two positive solutions $w_\lambda, u_\lambda$ with $w_\lambda < u_\lambda$ for $\lambda = \lambda^*$,
(iii) exactly one positive solution $w_\lambda$ for $\hat{\lambda} < \lambda < \lambda^*$,
(iv) no positive solution for $0 < \lambda \leq \hat{\lambda}$.
Theorem (2.2)

Furthermore,

\[ \lim_{\lambda \to (\hat{\lambda})^+} \| w_\lambda \|_\infty = 0, \quad \lim_{\lambda \to \infty} \| w_\lambda \|_\infty = \beta_1, \quad \lim_{\lambda \to \infty} \| u_\lambda \|_\infty = \gamma, \quad \text{and} \quad \lim_{\lambda \to \infty} \| v_\lambda \|_\infty = \beta_3. \]
Theorem (2.3)

Consider (1.1) with $p > 1$. If $(q, r) \in \Gamma_1$ and $r \leq \eta_{2_{q,p}}^*$, then

\[
\lim_{\alpha \to 0^+} \lambda (\alpha) = \hat{\lambda} \equiv \frac{\pi^2}{4r}, \\
\lim_{\alpha \to \beta_1^-} \lambda (\alpha) = \lim_{\alpha \to \beta_1^+} \lambda (\alpha) = \lim_{\alpha \to \beta_3^-} \lambda (\alpha) = \infty, \text{ and the bifurcation curve } \tilde{S} \text{ of (1.1) is a broken S-shaped curve on the } (\lambda, \|u\|_\infty)-\text{plane. More precisely, } \tilde{S} \text{ has two disjoint connected components such that the upper branch of } \tilde{S} \text{ has exactly one turning point } (\lambda_*, \|u_{\lambda_*}\|_\infty), \text{ with } \hat{\lambda} < \lambda_* < \infty \text{ and } \beta_1 < \|u_{\lambda_*}\|_\infty < \beta_3, \text{ where the curve turns to the right, and the lower branch of } \tilde{S} \text{ is a monotone increasing curve starting at } (\hat{\lambda}, 0).
Theorem (2.3)

Problem (1.1) has:

(i) exactly three positive solutions $w_\lambda, u_\lambda, v_\lambda$ with $w_\lambda < u_\lambda < v_\lambda$ for $\lambda > \lambda^*$,
(ii) exactly two positive solutions $w_\lambda, u_\lambda$ with $w_\lambda < u_\lambda$ for $\lambda = \lambda^*$,
(iii) exactly one positive solution $w_\lambda$ for $\hat{\lambda} < \lambda < \lambda^*$,
(iv) no positive solution for $0 < \lambda \leq \hat{\lambda}$.
Theorem (2.3)

Furthermore,

\[ \lim_{\lambda \to \hat{\lambda}} \|w_\lambda\|_\infty = 0, \quad \lim_{\lambda \to \infty} \|w_\lambda\|_\infty = \beta_1 = \lim_{\lambda \to \infty} \|u_\lambda\|_\infty, \text{ and } \lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \beta_3. \]
3. Lemma 3.1 and the time-map method

\[ \begin{cases} u''(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - \frac{u^p}{1+u^p} \right] = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{1.1} \]

For any \( \lambda > 0 \), let \( F(u) \equiv \int_0^u f(t)dt \). The time map formula which we take to prove Theorem 2.1 for problem (1.1) takes the form as follows:

\[ T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} du = \sqrt{\lambda} \text{ for } 0 < \alpha < \beta_3. \]

Positive solutions \( u_\lambda \) of (1.1) correspond to

\[ (T(\alpha))^2 = \lambda \text{ and } \|u_\lambda\|_\infty = \alpha. \]
Thus, the study of the exact number of positive solutions for problem (1.1) is equivalent to that of the shape of the time map $T(\alpha)$ on $(0, \beta_3)$.
Consider the bifurcation curve of positive solutions of the problem

\[
\begin{cases}
    u''(x) + \lambda \hat{f}(u(x)) = 0, & -1 < x < 1, \\
    u(-1) = u(1) = 0,
\end{cases}
\]

(3.1)

where \( \hat{f} \in C[0, \infty) \cap C^2(0, \infty) \), and \( \lambda > 0 \) is a bifurcation parameter. We define

\[
\hat{F}(u) = \int_0^u \hat{f}(t)dt, \quad \hat{\theta}(u) = 2\hat{F}(u) - uf(u), \quad \text{and} \quad \hat{H}(u) = 3 \int_0^u tf(t)dt - u^2\hat{f}(u)
\]

and assume that \( \hat{f} \) satisfies the following hypotheses (H1a) and (H2a):

(H1a) There exists a number \( \hat{\beta} > 0 \) such that \( \hat{f}(0) = \hat{f}(\hat{\beta}) = 0, \hat{f}(u) > 0 \) for \( u \in (0, \hat{\beta}) \), and \( \hat{f}(u) < 0 \) for \( u \in (\hat{\beta}, \infty) \).

(H2a) There exist numbers \( 0 < \hat{B}_1 < \hat{C}_1 < \hat{B}_2 < \hat{C}_2 < \hat{\beta} \) such that

\[
\hat{\theta}'(\hat{C}_1) = \hat{\theta}'(\hat{C}_2) = 0,
\]

\[
\hat{\theta}''(u) = -uf''(u) \begin{cases}
    > 0 & \text{on } (0, \hat{B}_1) \cup (\hat{B}_2, \hat{\beta}), \\
    = 0 & \text{for } u = \hat{B}_1 \text{ and } \hat{B}_2, \\
    < 0 & \text{on } (\hat{B}_1, \hat{B}_2),
\end{cases}
\]

\[
\hat{H}(\hat{B}_2) \leq 0, \quad \text{and} \quad 2\hat{H}(\hat{B}_1) - \hat{B}_1^2\hat{\theta}'(\hat{B}_1) \geq 2\hat{H}(\hat{C}_2).
\]
\[
\begin{aligned}
&\begin{cases}
  u''(x) + \lambda \hat{f}(u(x)) = 0, 
  -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\end{aligned}
\] (3.1)

To prove Theorem 2.1, we need the following key lemma: Lemma 3.1. We prove Lemma 3.1 by applying the time-mapping method (quadrature method) which was used by Ludwig, Aronson and Weinberger. The time map formula which we apply to study problem (3.1) takes the form as follows:

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\left[\hat{F}(\alpha) - \hat{F}(u)\right]^{1/2}} du = \sqrt{\lambda} \quad \text{for} \ 0 < \alpha < \hat{\beta}.
\]

So positive solutions \( u_\lambda \) of (3.1) correspond to \( \|u_\lambda\|_\infty = \alpha \) and \( T(\alpha) = \sqrt{\lambda} \). Thus, studying the exact number of positive solutions of (3.1) is equivalent to studying the number of roots of the equation \( T(\alpha) = \sqrt{\lambda} \) on \((0, \hat{\beta})\).
\[
\begin{aligned}
\left\{ 
\begin{array}{l}
u''(x) + \lambda \hat{f}(u(x)) = 0, 
-1 < x < 1, \\
u(-1) = u(1) = 0,
\end{array}
\right.
\end{aligned}
\] (3.1)

**Lemma (3.1)**

Consider (3.1). Suppose \( \hat{f} \in C[0, \infty) \cap C^2(0, \infty) \) satisfies (H1a) and (H2a), then

\[
\lim_{\alpha \to 0^+} T(\alpha) = \frac{\pi}{2 \sqrt{m_0}} \in [0, \infty), \quad \lim_{\alpha \to \beta^-} T(\alpha) = \infty,
\]

where \( 0 < \lim_{u \to 0^+} \hat{f}(u)/u = m_0 \leq \infty \), and \( T(\alpha) \) has exactly two positive critical points, \( \alpha^* < \alpha_\ast \), on \((0, \beta)\), such that \( T(\alpha^*) \) is a local maximum on \((0, \beta)\) and \( T(\alpha_\ast) \) is a local minimum on \((0, \beta)\).
3. Lemma 3.2 and the time-map method

\begin{equation}
\begin{cases}
    u''(x) + \lambda \left[ ru \left(1 - \frac{u}{q}\right) - \frac{u^p}{1+u^p} \right] = 0, \quad -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{cases}
\end{equation}

(1.1)

For any \( \lambda > 0 \), let \( F(u) \equiv \int_0^u f(t) \, dt \). The time map formula which we take to prove Theorem 2.2 for problem (1.1) takes the form as follows:

\[
T(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} \, du = \sqrt{\lambda} \quad \text{for} \quad \alpha \in (0, \beta_1) \cup (\gamma, \beta_3).
\]

Positive solutions \( u_\lambda \) of (1.1) correspond to

\[
(T(\alpha))^2 = \lambda \quad \text{and} \quad \|u_\lambda\|_\infty = \alpha.
\]
Thus, the study of the exact number of positive solutions for problem (1.1) is equivalent to that of the shape of the time map $T(\alpha)$ on $(0, \beta_1) \cup (\gamma, \beta_3)$. 

\[ (T(\alpha))^2 = \lambda \quad \text{and} \quad \|u_{\lambda_0}\|_\infty = \alpha \in (0, \beta_1) \cup (\gamma, \beta_3). \]
\[
\begin{aligned}
&u''(x) + \lambda \hat{f}(u(x)) = 0, \quad -1 < x < 1, \\
&u(-1) = u(1) = 0,
\end{aligned}
\]  
(3.1)

We define
\[
\hat{F}(u) = \int_0^u \hat{f}(t)dt, \quad \hat{\theta}(u) = 2\hat{F}(u) - u\hat{f}(u), \quad \text{and} \quad \hat{H}(u) = 3 \int_0^u t\hat{f}(t)dt - u^2\hat{f}(u)
\]

and assume that \(\hat{f}\) satisfies the following hypotheses (H1b), (H2b) and (H3):

(H1b) There exist numbers \(0 < \hat{\beta}_1 < \hat{\beta}_2 < \hat{\beta}\) such that
\[
\hat{f}(0) = \hat{f}(\hat{\beta}_1) = \hat{f}(\hat{\beta}_2) = \hat{f}(\hat{\beta}) = 0, \quad \hat{f}(u) > 0 \quad \text{on} \quad (0, \hat{\beta}_1) \cup (\hat{\beta}_2, \hat{\beta}), \quad \text{and} \quad \hat{f}(u) < 0 \quad \text{on} \quad (\hat{\beta}_1, \hat{\beta}_2) \cup (\hat{\beta}, \infty).
\]

(H2b) There exist numbers \(0 < \hat{B}_1 < \hat{C}_1 < \hat{E}_1 \leq \hat{B}_2 < \hat{C}_2 < \hat{E}_2 < \hat{\beta}\) such that
\[
\hat{\theta}(\hat{E}_1) = \hat{\theta}(\hat{E}_2) = \hat{\theta}'(\hat{C}_1) = \hat{\theta}'(\hat{C}_2) = 0 \quad \text{and}
\]
\[
\hat{\theta}''(u) = -u\hat{f}''(u) \begin{cases} > 0 & \text{on} \quad (0, \hat{B}_1) \cup (\hat{B}_2, \hat{\beta}), \\
= 0 & \text{for} \quad u = \hat{B}_1 \quad \text{and} \quad \hat{B}_2, \\
< 0 & \text{on} \quad (\hat{B}_1, \hat{B}_2).
\end{cases}
\]

Also, \(\hat{\theta}(\hat{B}_1) - \hat{B}_1 \hat{\theta}'(\hat{B}_1) - \hat{\theta}(\hat{C}_2) \geq 0\).

(H3) There exists a positive number \(\hat{\gamma} \in (\hat{\beta}_2, \hat{\beta})\) satisfies \(\int_{\hat{\beta}_1}^{\hat{\gamma}} \hat{f}(u)du = 0\).
\[
\begin{cases}
  u''(x) + \lambda \hat{f}(u(x)) = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\tag{3.1}
\]

To prove Theorem 2.2, we need the following key lemma: Lemma 3.2. We prove Lemma 3.2 by applying the time-mapping method (quadrature method) which was used by Ludwig, Aronson and Weinberger. The time map formula which we apply to study problem (3.1) takes the form as follows:

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{[\hat{F}(\alpha) - \hat{F}(u)]^{1/2}} du = \sqrt{\lambda} \quad \text{for } \alpha \in (0, \hat{\beta}_1) \cup (\hat{\gamma}, \hat{\beta}).
\]

So positive solutions \( u_\lambda \) of (3.1) correspond to \( \|u_\lambda\|_\infty = \alpha \) and \( T(\alpha) = \sqrt{\lambda} \). Thus, studying the exact number of positive solutions of (3.1) is equivalent to studying the number of roots of the equation \( T(\alpha) = \sqrt{\lambda} \) on \( (0, \hat{\beta}) \).
\[
\begin{aligned}
&\left\{
\begin{array}{l}
u''(x) + \lambda \hat{f}(u(x)) = 0, \quad -1 < x < 1, \\
u(-1) = u(1) = 0,
\end{array}
\right.
\end{aligned}
\]  
(3.1)

\textbf{Lemma (3.2)}

Consider (3.1). Suppose \( \hat{f} \in C[0, \infty) \cap C^2(0, \infty) \) satisfies (H1b), (H2b) and (H3), then

\[
\lim_{\alpha \to 0^+} T(\alpha) = \frac{\pi}{2 \sqrt{m_0}} \in [0, \infty), \quad \lim_{\alpha \to \hat{\beta}_1^-} T(\alpha) = \lim_{\alpha \to \gamma^+} T(\alpha) = \lim_{\alpha \to \hat{\beta}^-} T(\alpha) = \infty,
\]

where \( 0 < \lim_{u \to 0^+} \hat{f}(u)/u \equiv m_0 \leq \infty \), and \( T(\alpha) \) is strictly increasing on \((0, \hat{\beta}_1)\) and \( T(\alpha) \) has exactly one positive critical point at some \( \alpha_* \) on \((\gamma, \hat{\beta})\), such that \( T(\alpha_*) \) is a local minimum on \((\gamma, \hat{\beta})\).
For any \( \varepsilon > 0 \), let \( F(u) \equiv \int_0^u f(t) dt \). The time map formula which we take to prove Theorem 2.3 for problem (1.1) takes the form as follows:

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} du = \sqrt{\lambda} \text{ for } \alpha \in (0, \beta_1) \cup (\beta_1, \beta_3).
\]

Positive solutions \( u_\lambda \) of (1.1) correspond to

\[
(T(\alpha))^2 = \lambda \text{ and } \|u_\lambda\|_\infty = \alpha.
\]
\[(T(\alpha))^2 = \lambda\text{ and } \|u_\lambda\|_\infty = \alpha \in (0, \beta_1) \cup (\beta_1, \beta_3).\]

Thus, the study of the exact number of positive solutions for problem (1.1) is equivalent to that of the shape of the time map \(T(\alpha)\) on \((0, \beta_1) \cup (\beta_1, \beta_3)\).
Consider the bifurcation curve of positive solutions of the problem

\[
\begin{aligned}
& u''(x) + \lambda \hat{f}(u(x)) = 0, \quad -1 < x < 1, \\
& u(-1) = u(1) = 0,
\end{aligned}
\] 

(3.1)

where \( \hat{f} \in C[0, \infty) \cap C^2(0, \infty) \), and \( \lambda > 0 \) is a bifurcation parameter. We define

\[
\hat{F}(u) = \int_0^u \hat{f}(t)dt, \quad \hat{\theta}(u) = 2\hat{F}(u) - uf'(u), \quad \text{and} \quad \hat{H}(u) = 3 \int_0^u t\hat{f}(t)dt - u^2\hat{f}(u)
\]

and assume that \( \hat{f} \) satisfies the following hypotheses hypotheses (H1c) and (H2c):

(H1c) There exist numbers \( 0 < \hat{\beta}_1 < \hat{\beta} \) such that \( \hat{f}(0) = \hat{f}(\hat{\beta}_1) = \hat{f}(\hat{\beta}) = 0 \), and \( \hat{f}(u) > 0 \) for \( u \in (0, \hat{\beta}_1) \cup (\hat{\beta}_1, \hat{\beta}) \), and \( \hat{f}(u) < 0 \) for \( u \in (\hat{\beta}, \infty) \).

(H2c) There exist numbers \( 0 < \hat{B}_1 < \hat{C}_1 < \hat{E}_1 \leq \hat{B}_2 < \hat{C}_2 < \hat{E}_2 < \hat{\beta} \) such that \( \hat{\theta}(\hat{E}_1) = \hat{\theta}(\hat{E}_2) = \hat{\theta}'(\hat{C}_1) = \hat{\theta}'(\hat{C}_2) = 0 \) and

\[
\hat{\theta}''(u) = -uf''(u) \begin{cases} > 0 & \text{on } (0, \hat{B}_1) \cup (\hat{B}_2, \hat{\beta}), \\ = 0 & \text{for } u = \hat{B}_1 \text{ and } u = \hat{B}_2, \\ < 0 & \text{on } (\hat{B}_1, \hat{B}_2). \end{cases}
\]

Also, \( \hat{\theta}(\hat{B}_1) - \hat{B}_1 \hat{\theta}'(\hat{B}_1) - \hat{\theta}(\hat{C}_2) \geq 0 \).
\[
\begin{aligned}
\begin{cases}
  u''(x) + \lambda \hat{f}(u(x)) = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0,
\end{cases}
\end{aligned}
\]

(3.1)

To prove Theorem 2.3, we need the following key lemma: Lemma 3.3. We prove Lemma 3.3 by applying the time-mapping method (quadrature method) which was used by Ludwig, Aronson and Weinberger. The time map formula which we apply to study problem (3.1) takes the form as follows:

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\left[ \hat{F}(\alpha) - \hat{F}(u) \right]^{1/2}} du = \sqrt{\lambda} \quad \text{for } \alpha \in (0, \hat{\beta}_1) \cup (\hat{\gamma}, \hat{\beta}).
\]

So positive solutions \( u_\lambda \) of (3.1) correspond to \( \| u_\lambda \|_\infty = \alpha \) and \( T(\alpha) = \sqrt{\lambda} \). Thus, studying the exact number of positive solutions of (3.1) is equivalent to studying the number of roots of the equation \( T(\alpha) = \sqrt{\lambda} \) on \( (0, \hat{\beta}) \).
\begin{equation}
\begin{cases}
    u''(x) + \lambda \hat{f}(u(x)) = 0, \quad -1 < x < 1, \\
    u(-1) = u(1) = 0,
\end{cases}
\tag{3.1}
\end{equation}

Lemma (3.3)

Consider (3.1). Suppose \( \hat{f} \in C[0, \infty) \cap C^2(0, \infty) \) satisfies \((H1c)\) and \((H2c)\), then

\[
\lim_{\alpha \to 0^+} T(\alpha) = \frac{\pi}{2 \sqrt{m_0}} \in [0, \infty), \quad \lim_{\alpha \to \hat{\beta}_1^-} T(\alpha) = \lim_{\alpha \to \hat{\beta}_1^+} T(\alpha) = \lim_{\alpha \to \hat{\beta}^-} T(\alpha) = \infty,
\]

where \( 0 < \lim_{u \to 0^+} \hat{f}(u)/u \equiv m_0 \leq \infty, \) and \( T(\alpha) \) is strictly increasing on \((0, \hat{\beta}_1)\) and \( T(\alpha) \) has exactly one positive critical point at some \( \alpha_* \) on \((\hat{\beta}_1, \hat{\beta})\), such that \( T(\alpha_*) \) is a local minimum on \((\hat{\beta}_1, \hat{\beta})\).
Thanks for your attention
Consider

\[
\begin{cases}
u''(x) + \lambda f(u(x)) = 0, & -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{cases}
\]  

(7.1)

The time-map formula which we apply to study (7.1) takes the form as follows:

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_{0}^{\alpha} \frac{1}{[F(\alpha) - F(u)]^{1/2}} du = \sqrt{\lambda} \text{ for } 0 < \alpha < \beta \text{ for some } \beta \leq \infty.
\]  

(7.2)

On the \((\lambda, \|u\|_{\infty})\)-plane, we define the bifurcation curve of (7.1)

\[
S = \{ (\lambda, \|u_{\lambda}\|_{\infty}) : \lambda > 0 \text{ and } u_{\lambda} \text{ is a positive solution of (7.1)} \}.
\]
\[
\begin{cases}
  u''(x) + \lambda f(u(x)) = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0.
\end{cases}
\tag{7.1}
\]

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{[F(\alpha) - F(u)]^{1/2}} du = \sqrt{\lambda} \quad \text{for} \quad 0 < \alpha < \beta.
\tag{7.2}
\]

It can be easily proved that positive solutions \( u_\lambda \) of (7.1) correspond to

\[ \| u_\lambda \|_\infty = \alpha \quad \text{and} \quad T(\alpha) = \sqrt{\lambda}. \]

Thus to study the number of positive solutions of (7.1) is equivalent to study the shape of the time map \( T(\alpha) \) on \((0, \beta)\).
\[
\begin{aligned}
\left\{
\begin{array}{l}
\begin{align*}
& u''(x) + \lambda f(u(x)) = 0, \quad -1 < x < 1, \\
& u(-1) = u(1) = 0.
\end{align*}
\end{array}
\right.
\end{aligned}
\]
(7.1)

Derivation of the time-map formula \( T(\alpha) \).

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{[F(\alpha) - F(u)]^{1/p}} du = \sqrt{\lambda} \quad \text{for} \quad 0 < \alpha < \beta.
\]
(7.2)
\[
\begin{cases}
   u''(x) + \lambda f(u(x)) = 0, \quad -1 < x < 1, \\
   u(-1) = u(1) = 0.
\end{cases}
\] (7.1)

Derivation of the time-map formula \( T(\alpha) \).

\[
T(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{[F(\alpha) - F(u)]^{1/p}} du = \sqrt{\lambda} \quad \text{for} \quad 0 < \alpha < \beta. \quad (7.2)
\]

Multiplying (7.1) by \( u'(x) \) and integrating, we obtain

\[
\frac{(u'(x))^2}{2} + \lambda F(u(x)) = \text{constant}, \quad -1 < x < 1,
\] (7.3)

where \( F(u) = \int_0^u f(t) dt \).
Assume $x_0 \in (-1, 1)$ be a maximum point of $u(x)$ and $x \in (-1, 1)$. By (7.3), then we have

$$u'(x) = \sqrt{2\lambda \left[ F(u(x_0)) - F(u(x)) \right]}^{1/2} \text{sign}(x_0 - x), \quad \text{for } -1 < x < 1.$$
Assume $x_0 \in (-1, 1)$ be a maximum point of $u(x)$ and $x \in (-1, 1)$. By (7.3), then we have

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This implies that

$$x + 1 = \frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \left[ F(\|u\|_\infty) - F(w) \right]^{-1/2} dw, \text{ for } -1 < x \leq x_0$$

and

$$1 - x = \frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \left[ F(\|u\|_\infty) - F(w) \right]^{-1/2} dw, \text{ for } x_0 \leq x < 1.$$
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This implies that

$$x + 1 = \frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \left[ F(\|u\|_\infty) - F(w) \right]^{-1/2} dw, \text{ for } -1 < x \leq x_0$$

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Setting $x = x_0$, we have $x_0 = 0.$
Assume $x_0 \in (-1, 1)$ be a maximum point of $u(x)$ and $x \in (-1, 1)$. By (7.3), then we have

$$u'(x) = \sqrt{2\lambda} \left[ F(u(x_0)) - F(u(x)) \right]^{1/2} \text{sign}(x_0 - x), \quad \text{for} \ -1 < x < 1.$$ 

This implies that

$$x + 1 = \frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \left[ F(\|u\|_\infty) - F(w) \right]^{-1/2} dw, \quad \text{for} \ -1 < x \leq x_0$$

and

$$1 - x = \frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \left[ F(\|u\|_\infty) - F(w) \right]^{-1/2} dw, \quad \text{for} \ x_0 \leq x < 1.$$ 

Setting $x = x_0$, we have $x_0 = 0$. This implies that $u(x)$ has to be symmetric with respect to $x = 0$, and $u'(x) > 0$ for $-1 < x < 0$. 

Tzung-Shin Yeh (NUTN) A multiparameter diffusive logistic problem
Assume $x_0 \in (-1, 1)$ be a maximum point of $u(x)$ and $x \in (-1, 1)$. By (7.3), then we have

$$u'(x) = \sqrt{2\lambda} \left[ F(u(x_0)) - F(u(x)) \right]^{1/2} \text{sign}(x_0 - x), \text{ for } -1 < x < 1.$$ 

This implies that

$$x + 1 = \frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \left[ F(\|u\|_\infty) - F(w) \right]^{-1/2} dw, \text{ for } -1 < x \leq x_0$$

and

$$1 - x = \frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \left[ F(\|u\|_\infty) - F(w) \right]^{-1/2} dw, \text{ for } x_0 \leq x < 1.$$ 

Setting $x = x_0$, we have $x_0 = 0$. This implies that $u(x)$ has to be symmetric with respect to $x = 0$, and $u'(x) > 0$ for $-1 < x < 0$. Thus $\|u\|_\infty = u(0)$. 

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A multiparameter diffusive logistic problem
\[
\begin{aligned}
&\begin{cases}
u''(x) + \lambda f(u(x)) = 0, \quad -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{cases} \\
&\left(\frac{u'(x)}{2}\right)^2 + \lambda F(u(x)) = \text{constant}, \quad -1 < x < 1,
\end{aligned}
\] (7.1)

Let \( \alpha = u(0) \) and substitute \( x = 0 \) into (7.3), then

\[
\begin{aligned}
&\left(\frac{u'(x)}{2}\right)^2 + \lambda F(u(x)) = \left(\frac{u'(0)}{2}\right)^2 + \lambda F(u(0)) \\
&= \lambda F(\alpha).
\end{aligned}
\] (7.3)
\[
\begin{cases}
  u''(x) + \lambda f(u(x)) = 0, & -1 < x < 1, \\
  u(-1) = u(1) = 0.
\end{cases}
\] (7.1)

\[
\frac{(u'(x))^2}{2} + \lambda F(u(x)) = \text{constant}, & -1 < x < 1,
\] (7.3)

Let \( \alpha = u(0) \) and substitute \( x = 0 \) into (7.3), then

\[
\frac{(u'(x))^2}{2} + \lambda F(u(x)) = \frac{(u'(0))^2}{2} + \lambda F(u(0))
\]

\[
= \lambda F(\alpha).
\]

Hence

\[
\frac{du}{dx} = u'(x) = \sqrt{2\lambda} \left[ F(\alpha) - F(u) \right]^{1/2}, \quad -1 < x < 0.
\] (7.4)
\[
\begin{align*}
\begin{cases}
u''(x) + \lambda f(u(x)) = 0, & -1 < x < 1, \\
u(-1) = u(1) = 0.
\end{cases} & \quad (7.1)
\end{align*}
\]

\[
\frac{(u'(x))^2}{2} + \lambda F(u(x)) = \text{constant}, & \quad -1 < x < 1, \quad (7.3)
\]

Let \(\alpha = u(0)\) and substitute \(x = 0\) into (7.3), then

\[
\frac{(u'(x))^2}{2} + \lambda F(u(x)) = \frac{(u'(0))^2}{2} + \lambda F(u(0)) = \lambda F(\alpha).
\]

Hence

\[
\frac{du}{dx} = u'(x) = \sqrt{2\lambda} \left[ F(\alpha) - F(u) \right]^{1/2}, & \quad -1 < x < 0. \quad (7.4)
\]

Now integrating (7.4) on \([-1, 0]\), we obtain

\[
\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_{0}^{\alpha} \left[ F(\alpha) - F(u) \right]^{-1/2} du = T(\alpha) \quad \text{for} \quad 0 < \alpha < \beta. \quad (7.2)
\]
5. Holling type I-IV functions

\[ \begin{cases} 
    u''(x) + \lambda \left[ ru \left( 1 - \frac{u}{q} \right) - p(u) \right] = 0, \quad -1 < x < 1, \\
    u(-1) = u(1) = 0. 
\end{cases} \tag{\ast} \]

Holling type-I function

\[ p(u) = p_1(u) = \begin{cases} 
    u, & 0 \leq u < k/a, \\
    k, & u \geq k/a, 
\end{cases} \]

where \( k, a > 0 \).

Holling type-II function

\[ p(u) = p_2(u) = \frac{u}{1+u}. \]

Holling type-IV function

\[ p(u) = p_4(u) = \frac{u}{1+u^2}. \]
Classified bifurcation curves of \((*)\) with \(p(u) = \frac{u}{1 + u^2}\), drawn on the first quadrant of \((q,r)\)-parameter plane.
6. Dynamical behavior of the reaction-diffusion equation in one space variable with
\[(q, r) \in R^*_1 \equiv \{(q, r) : (q, r) \in R_1 \text{ and } r \leq \eta_1 q\}\]

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u'' + \lambda f(u), \quad -1 < x < 1, \quad t > 0, \\
u(-1, t) &= u(1, t) = 0, \quad t > 0, \\
u(x, 0) &= u_0(x) \geq 0, \quad -1 < x < 1.
\end{align*}
\]

(12.1)

where

\[
\begin{align*}
f(u) &= ru \left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}.
\end{align*}
\]

(12.2)
References

Let
\[ \tilde{C} = \{ u \in C([-1, 1]) : u(x) \geq 0, -1 < x < 1; u(-1) = u(1) = 0 \}. \]

Then
1. If \( 0 < \lambda \leq \hat{\lambda} \), then \( \lim_{t \to \infty} u(x, t) = 0 \) (which is the trivial steady state solution) uniformly for \(-1 \leq x \leq 1\), for all nontrivial initial conditions \( u_0 \in \tilde{C} \).
2. If \( \hat{\lambda} < \lambda < \lambda_* \), then \( \lim_{t \to \infty} u(x, t) = u_1(\lambda, x) \) (which is the unique positive steady state solution) uniformly for \(-1 \leq x \leq 1\), for all nontrivial initial conditions \( u_0 \in \tilde{C} \).
3. If $\lambda = \lambda_*$, then there exists a co-dimension one manifold $M \subset \tilde{C}$ such that $\tilde{C} \setminus M$ has exactly two connected components $X_1$ and $X_2$, such that

(i) if $u_0 \in X_1 \cup M$, then $\lim_{t \to \infty} u(x, t) = u_2(\lambda_*, x)$ which is the unique maximal positive steady state solution;

(ii) if $u_0 \in X_2$, then $\lim_{t \to \infty} u(x, t) = u_1(\lambda_*, x)$ which is the unique minimal positive steady state solution.
4. If $\lambda_* < \lambda < \lambda^*$, then there exists a co-dimension one manifold $M \subset \tilde{C}$ such that $\tilde{C} \setminus M$ has exactly two connected components $X_1$ and $X_2$, such that
(ii) if $u_0 \in X_1$, then $\lim_{t \to \infty} u(x, t) = u_3(\lambda, x)$ which is the unique maximal positive steady state solution;
(iii) if $u_0 \in X_2$, then $\lim_{t \to \infty} u(x, t) = u_1(\lambda, x)$ which is the unique minimal positive steady state solution;
(iii) if $u_0 \in M$, then $\lim_{t \to \infty} u(x, t) = u_2(\lambda, x)$ which is the unique non-maximal, non-minimal positive steady state solution.
5. If $\lambda = \lambda^*$, then there exists a co-dimension one manifold $M \subset \tilde{C}$ such that $\tilde{C} \setminus M$ has exactly two connected components $X_1$ and $X_2$, such that

(i) if $u_0 \in X_1$, then $\lim_{t \to \infty} u(x, t) = u_3(\lambda^*, x)$ which is the unique maximal positive steady state solution;

(ii) if $u_0 \in X_2 \cup M$, then $\lim_{t \to \infty} u(x, t) = u_2(\lambda^*, x)$ which is the unique minimal positive steady state solution.
6. If $\lambda > \lambda^*$, then $\lim_{t \to \infty} u(x, t) = u_3(\lambda, x)$ (which is the unique positive steady state solution) uniformly for $-1 \leq x \leq 1$, for all nontrivial initial populations $u_0(x)$. 