An efficient Wave Based Method for solving Helmholtz problems in three-dimensional bounded domains

B. Van Genechten, O. Atak, B. Bergen, E. Deckers,
S. Jonckheere, J.S. Lee, A. Maressa, K. Vergote,
B. Pluymers, D. Vandepitte and W. Desmet

Katholieke Universiteit Leuven
Departement Werktuigkunde

http://www.mech.kuleuven.be/mod/wbm/

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KATHOLIEKE UNIVERSITEIT
LEUVEN

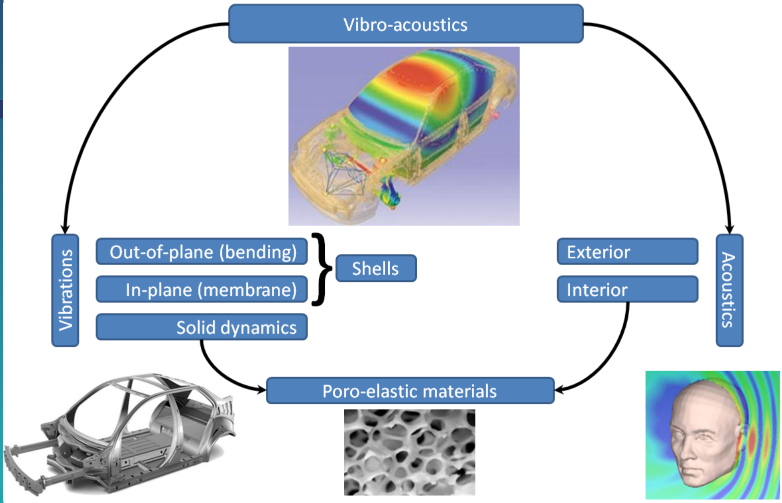
Overview

Overview

1. Introduction
   - Vibro-acoustic interactions
   - Numerical modelling approaches for mid-frequency vibro-acoustic problems
2. The Wave Based Method
3. Numerical validation of the WBM for 3D acoustics
4. Conclusions

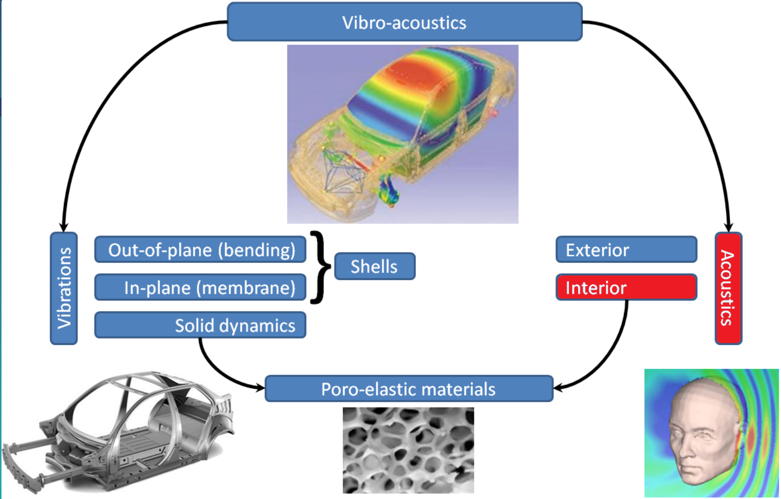
Introduction

Vibro-acoustic interactions



Introduction

Vibro-acoustic interactions



Simulation of vibro-acoustic systems

Available approaches

Deterministic element based methods: FEM, BEM,...

- approximating shape functions
- fine discretisations $\Rightarrow$ low-frequency

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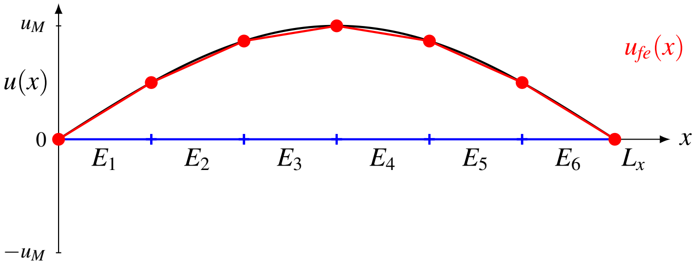

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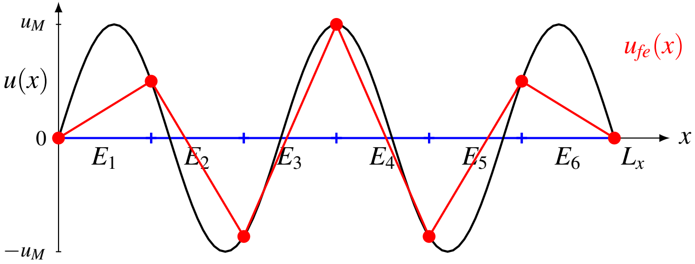

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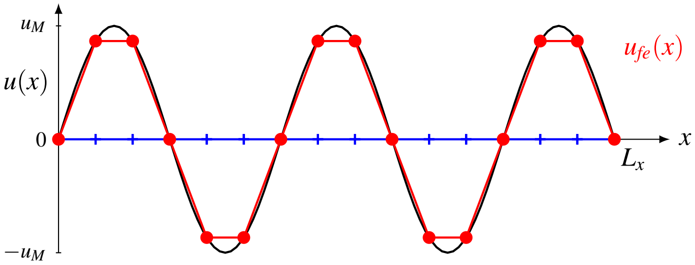
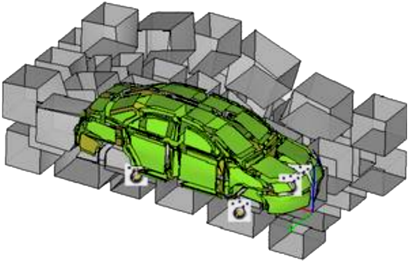


Simulation of vibro-acoustic systems

Available approaches

***Statistical energy based methods*: SEA, EFEM,...**

- energy in SEA subsystems
- underlying assumptions $\Rightarrow$ high-frequency



Simulation of vibro-acoustic systems

Current research

Deterministic element based methods: FEM, BEM,...

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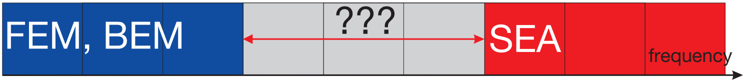

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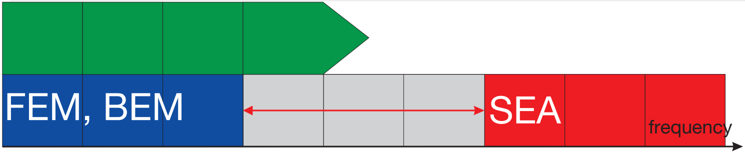
Enhanced deterministic approaches

- Process optimisation
- Reduction or elimination of numerical pollution effects
- Introduction of a priori information (Trefftz approaches)



Simulation of vibro-acoustic systems

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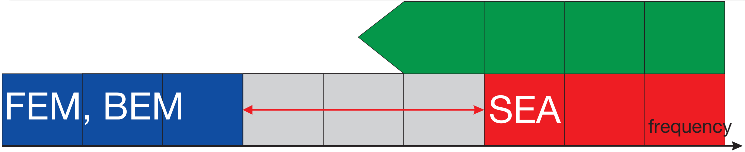
Enhanced statistical approaches

- Non-equipartition of modal energy
- Strong or indirect coupling
- SEA model parameter estimation



Simulation of vibro-acoustic systems

Current research

Deterministic element based methods: FEM, BEM,...

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- fine discretisations ⇒ <u>low-frequency</u>

Statistical energy based methods: SEA, EFEM,...

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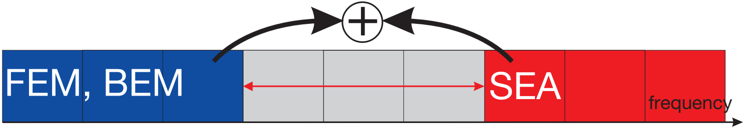
Hybrid deterministic-statistical approaches

- "Flexible" components: SEA subsystems
- "Stiff" components: FEM models
- Diffuse field reciprocity relation



Simulation of vibro-acoustic systems

Wave Based Method

Deterministic element based methods: FEM, BEM,...

- approximating shape functions
- fine discretisations $\Rightarrow$ <u>low-frequency</u>

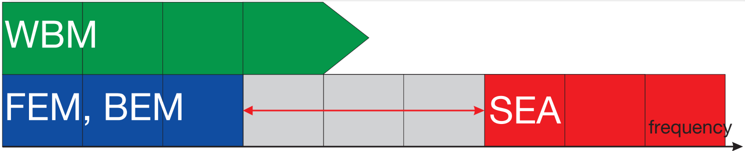
Statistical energy based methods: SEA, EFEM,...

- energy in SEA subsystems
- underlying assumptions $\Rightarrow$ <u>high-frequency</u>

Deterministic Trefftz-based Wave Based Method

- Basis functions = exact solutions of governing equations
- Enhanced numerical convergence properties

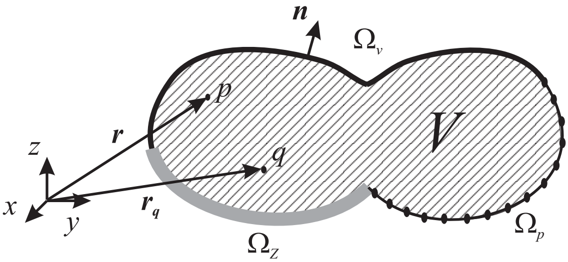
$\Rightarrow$ <u>applicable to mid-frequency problems</u>



Overview

WBM modelling approach

Dynamic equations: Helmholtz problems



- Steady-state dynamic acoustic pressure $p(\mathbf{r})$ in V: Helmholtz equation

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = \mathcal{F}(\mathbf{r}), \mathbf{r} \in \Omega$$

- At each point of $\Omega_\bullet$: 1 boundary condition

$$\begin{cases} \mathbf{r} \in \Omega_v : \mathcal{L}_v(p(\mathbf{r})) = \bar{v}_n(\mathbf{r}) \ , \\ \mathbf{r} \in \Omega_Z : \mathcal{L}_v(p(\mathbf{r})) = p(\mathbf{r})/\bar{Z}_n(\mathbf{r}) \ , \\ \mathbf{r} \in \Omega_p : p(\mathbf{r}) = \bar{p}(\mathbf{r}) \ . \end{cases}$$

WBM modelling approach

General WBM modelling process

1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions en source terms

WBM approximation

$$p^{(\alpha)}(\mathbf{r}) \simeq \sum_{w} p_w^{(\alpha)} \Phi_w^{(\alpha)}(\mathbf{r}) + \hat{p}_q^{(\alpha)}(\mathbf{r})$$

WBM modelling approach

General WBM modelling process

1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions en source terms
2. Minimisation of the approximation errors using a Galerkin weighted residual formulation

$$\int_{\Omega_v} \widetilde{p}(\mathbf{r})\ R_v(\mathbf{r}) d\Omega + \int_{\Omega_Z} \widetilde{p}(\mathbf{r})\ R_Z(\mathbf{r}) \mathrm{d}\Omega$$

$$+ \int_{\Omega_p} -\mathcal{L}_v(\widetilde{p}(\mathbf{r}))\ R_p(\mathbf{r}) d\Omega = 0$$

WBM modelling approach

General WBM modelling process

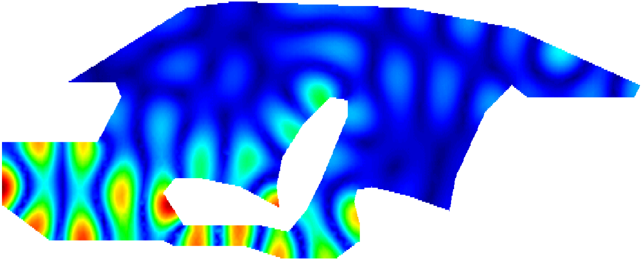
1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions en source terms
2. Minimisation of the approximation errors using a Galerkin weighted residual formulation
3. Solution of the resulting system of equations

$$\begin{bmatrix} \boldsymbol{A}_w^1 & \cdots & \boldsymbol{C}_w^{1,N_\Omega} \\ \vdots & \ddots & \vdots \\ \boldsymbol{C}_w^{N_\Omega,1} & \cdots & \boldsymbol{A}_w^{N_\Omega} \end{bmatrix} \begin{Bmatrix} \boldsymbol{p}_w^1 \\ \vdots \\ \boldsymbol{p}_w^{N_\Omega} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}_w^1 \\ \vdots \\ \boldsymbol{F}_w^{N_\Omega} \end{Bmatrix}$$

WBM modelling approach

General WBM modelling process

1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions en source terms
2. Minimisation of the approximation errors using a Galerkin weighted residual formulation
3. Solution of the resulting system of equations
4. Postprocessing, visualisation and interpretation of the response fields



WBM modelling approach

Applications areas

The WB modelling approach has been successfully applied to

2D steady-state acoustic analysis

- multi-domain interior acoustics
- acoustic scattering problems
- multiple scattering and inclusion problems

2D steady-state structural dynamic analysis

- plate membrane problems
- plate bending problems
- shell analysis

2D steady-state vibro-acoustic analysis

- interior vibro-acoustics
- exterior vibro-acoustics

Current extension:

- 3D steady-state interior acoustics

WBM modelling approach

WBM field variable approximations

Acoustic pressure expansion: bounded subdomains

$$p^{(\alpha)}(\mathbf{r}) \simeq \sum_{w=1}^{n_w^{(\alpha)}} p_w^{(\alpha)} \Phi_w^{(\alpha)}(\mathbf{r}) + \hat{p}_q^{(\alpha)}(\mathbf{r})$$

$\Phi_w^{(\alpha)}(\mathbf{r}(x,y))$: *Homogeneous* solutions of Helmholtz eq.

$$\Phi_w(\mathbf{r}(x,y,z)) = \begin{cases} \Phi_{w_r}(x,y,z) = \cos(k_{xw_r}x)\ \cos(k_{yw_r}y)\ e^{-jk_{zw_r}z} \\ \Phi_{w_s}(x,y,z) = \cos(k_{xw_s}x)\ e^{-jk_{yw_s}y}\ \cos(k_{zw_s}z) \\ \Phi_{w_t}(x,y,z) = e^{-jk_{xw_t}x}\ \cos(k_{yw_t}y)\ \cos(k_{zw_t}z) \end{cases}$$

Trefftz approach: the wave number components should satisfy the associated dispersion relation:

$$\begin{cases} (k_{xw_r})^2 + (k_{yw_r})^2 + (k_{zw_r})^2 = \\ (k_{xw_s})^2 + (k_{yw_s})^2 + (k_{zw_s})^2 = \\ (k_{xw_t})^2 + (k_{yw_t})^2 + (k_{zw_t})^2 = k^2 \end{cases}$$

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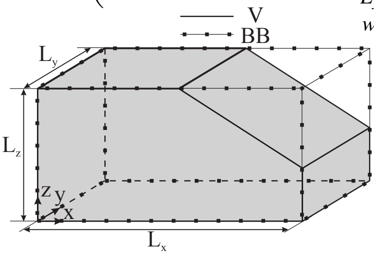
∞ number of solutions $\rightarrow$ selection of wave numbers:

$$\begin{cases} (k_{xw_r}, k_{yw_r}, k_{zw_r}) = \left(\dfrac{w_1\pi}{L_x}, \dfrac{w_2\pi}{L_y}, \pm\sqrt{k^2 - (k_{xw_r})^2 - (k_{yw_r})^2}\right) \\ (k_{xw_s}, k_{yw_s}, k_{zw_s}) = \left(\dfrac{w_3\pi}{L_x}, \pm\sqrt{k^2 - (k_{xw_s})^2 - (k_{zw_s})^2}, \dfrac{w_4\pi}{L_z}\right) \\ (k_{xw_t}, k_{yw_t}, k_{zw_t}) = \left(\pm\sqrt{k^2 - (k_{yw_t})^2 - (k_{zw_t})^2}, \dfrac{w_5\pi}{L_y}, \dfrac{w_6\pi}{L_z}\right) \end{cases}$$

$$w_1, \ldots, w_6 = 0, 1, \ldots$$



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WBM approximations: wave functions

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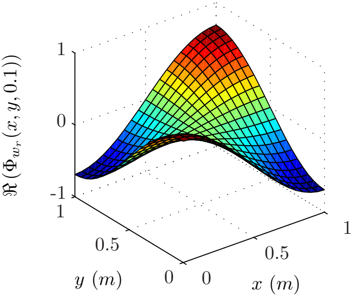

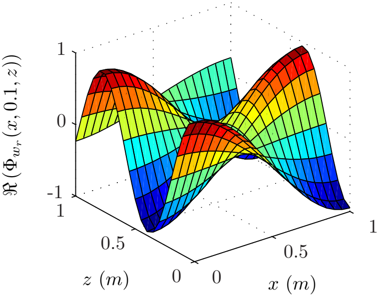

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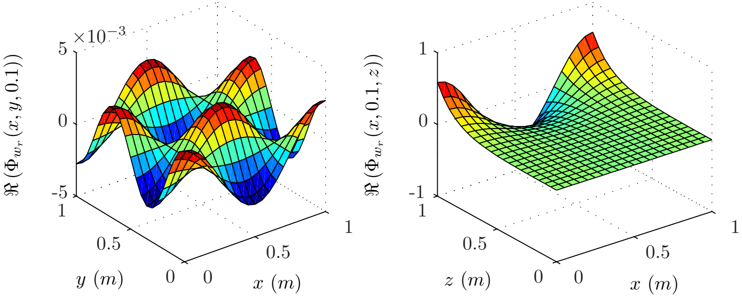

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$\hat{p}_q^{(\alpha)}(x,y,z)$: *Particular* (free field) solution of Helmholtz eq.

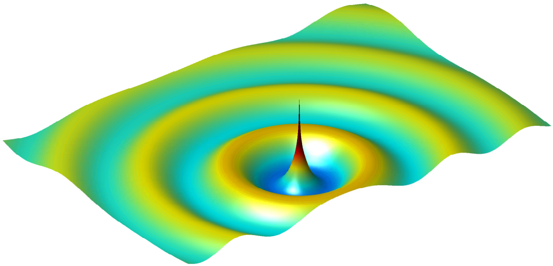
- Acoustic point source:

$$\widehat{p}_q(x,y,z) = \frac{j\rho_0\omega Q}{4\pi}\frac{e^{-jkr_q}}{r_q}$$

with Q the source strength $Q = \int_V q\,\mathrm{d}V$ and with r_q the distance to the source point.

WBM approximation: source terms

$$p^{(\alpha)}(\mathbf{r}) \simeq \sum_{w} p_w^{(\alpha)} \Phi_w^{(\alpha)}(\mathbf{r}) + \hat{p}_q^{(\alpha)}(\mathbf{r})$$



WBM modelling approach

WBM model properties

Application of a Galerkin weighted residual formulation yields

$$\begin{bmatrix} A_w^1 & \cdots & C_w^{1,N_\Omega} \\ \vdots & \ddots & \vdots \\ C_w^{N_\Omega,1} & \cdots & A_w^{N_\Omega} \end{bmatrix} \begin{Bmatrix} p_w^1 \\ \vdots \\ p_w^{N_\Omega} \end{Bmatrix} = \begin{Bmatrix} F_w^1 \\ \vdots \\ F_w^{N_\Omega} \end{Bmatrix}$$

Properties:

+ Small number of degrees of freedom
+ No accuracy decrease for derived variables
+ High convergence rate

- Fully populated matrices with complex coefficients
- Frequency-dependent matrices
- More complex numerical integrations
- Bad numerical conditioning
- Convergence only guaranteed for convex (sub)domains

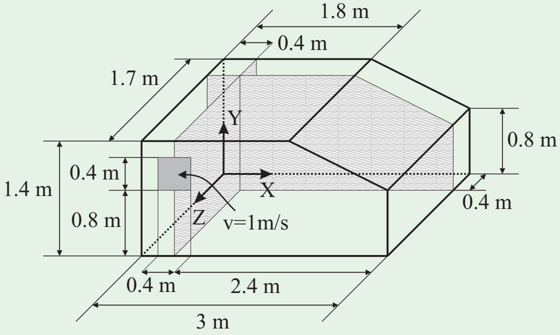
$\rightarrow$ applicable to practical mid-frequency applications of moderate geometrical complexity

Overview

Numerical validation

Convex acoustic cavity

Problem definition



- Acoustic fluid: Air

$$c = 340\sqrt{1+j\eta}\; m/s,\; \rho_0 = 1.225/(1+j\eta)\; kg/m^3,\; \eta = \{0\%.1\%\}$$

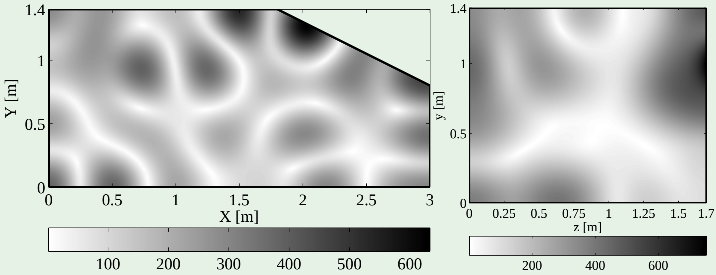
- Boundary conditions: acoustically rigid ($\bar{v}_n(\mathbf{r}) = 0$)
- Excitation: discontinuous unit normal velocity on patch in front

Numerical validation

Convex acoustic cavity

Acoustic variables at $450Hz$, damped case

- Acoustic pressure amplitude [Pa]



Rigidity conditions accurately taken into account

Numerical validation

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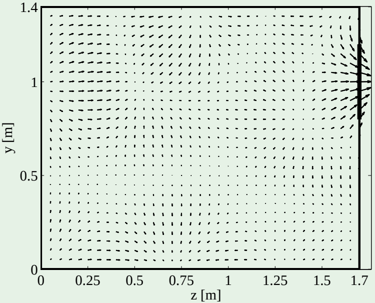
Acoustic variables at $450Hz$, damped case

- Acoustic velocity vector field [m/s]



Discontinuous velocity excitation conditions accurately taken into account

Numerical validation

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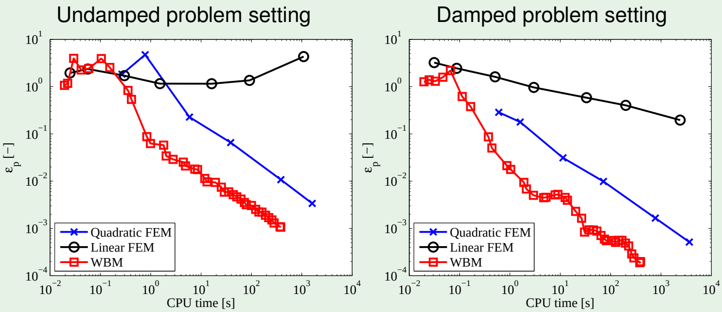
Computational efficiency comparison at $450Hz$

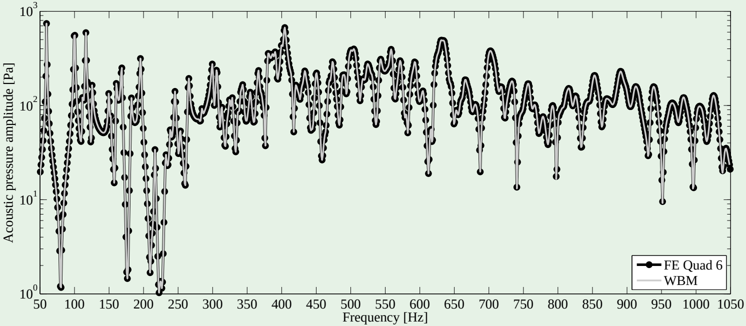


$$\epsilon_p = \frac{1}{108}\sum_{j=1}^{108}\epsilon_j = \frac{1}{108}\sum_{j=1}^{108}\left|\frac{\hat{p}(\mathbf{r_j}) - p_{ref}(\mathbf{r_j})}{p_{ref}(\mathbf{r_j})}\right|$$

Numerical validation

Convex acoustic cavity

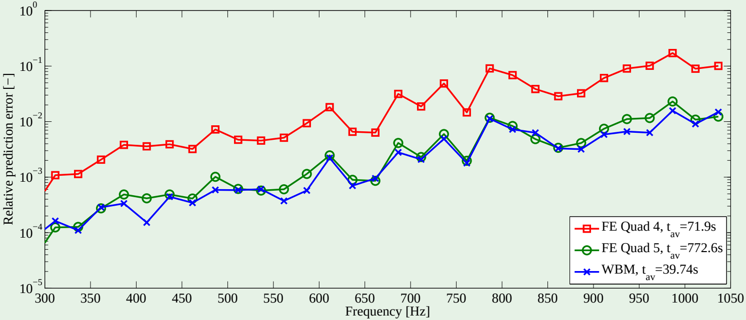
Frequency response comparison damped problem setting



Numerical validation

Convex acoustic cavity

Relative pressure amplitude prediction errors damped problem setting



- For a similar calculation time, the WBM is an order of magnitude more accurate than the quadratic FEM
- WBM reaches a similar prediction accuracy more than 10 times faster than the quadratic FEM.

Overview

Conclusions

Wave Based Method for steady-state dynamics

+ globally defined exact solutions = wave functions
+ low, mid (and high) frequency applications

Extension to 3D acoustics

- Derivation of wave function sets
- Weighted residual formulation

⇒ Efficient modelling of 3D problems of moderate geometrical complexity

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Next steps

Extension of the modelling approach to:

- 3D multi-domain acoustic problems
- 3D acoustic (multiple) scattering problems
- 3D acoustic multiple inclusion problems

Thank you for your attention!

Contact details:

dr. ir. Bert Van Genechten

K.U.Leuven, dept. of Mechanical Engineering
Celestijnenlaan 300B - box 2420
3001 Heverlee. Belgium

e-mail: Bert.VanGenechten@mech.kuleuven.be
`http://www.mech.kuleuven.be/mod/wbm/`