An efficient Wave Based Method for solving Helmholtz problems in three-dimensional bounded domains


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http://www.mech.kuleuven.be/mod/wbm/

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Overview

1. Introduction
2. The Wave Based Method
3. Numerical validation of the WBM for 3D acoustics
4. Conclusions
Overview

1 Introduction
   - Vibro-acoustic interactions
   - Numerical modelling approaches for mid-frequency vibro-acoustic problems

2 The Wave Based Method

3 Numerical validation of the WBM for 3D acoustics

4 Conclusions
Introduction

Vibro-acoustic interactions
Simulation of vibro-acoustic systems

Available approaches

**Deterministic element based methods**: FEM, BEM,...

- approximating shape functions
- fine discretisations

⇒ low-frequency
Simulation of vibro-acoustic systems

Available approaches

**Deterministic element based methods**: FEM, BEM,…

- approximating shape functions
- fine discretisations

\[ u(x) \]

\[ u_{fe}(x) \]

\[ 0 \]

\[ E_1, E_2, E_3, E_4, E_5, E_6, L_x \]

\[ \pm u_M \]
**Simulation of vibro-acoustic systems**

**Available approaches**

**Deterministic element based methods:** FEM, BEM,...

- approximating shape functions
- fine discretisations

⇒ low-frequency

\[
\begin{align*}
\mathbf{u}(x) &= u_0 + \sum_{i=1}^{n} E_i \mathbf{u}\left( E_i \right) \\
\mathbf{u}_{fe}(x) &= u_{fe}(E_1, E_2, E_3, E_4, E_5, E_6, L_x)
\end{align*}
\]
Simulation of vibro-acoustic systems

Available approaches

**Deterministic element based methods**: FEM, BEM,...

- approximating shape functions
- fine discretisations \(\Rightarrow\) low-frequency

\[ u(x) = u_M \]

\[ u_{fe}(x) \]
Simulation of vibro-acoustic systems

Available approaches

Statistical energy based methods: SEA, EFEM,...

- energy in SEA subsystems
- underlying assumptions

⇒ high-frequency
**Simulation of vibro-acoustic systems**

Current research

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**Deterministic element based methods**: FEM, BEM,...

- approximating shape functions
- fine discretisations $\Rightarrow$ low-frequency

**Statistical energy based methods**: SEA, EFEM,...

- energy in SEA subsystems
- underlying assumptions $\Rightarrow$ high-frequency

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FEM, BEM \(??\) SEA
Simulation of vibro-acoustic systems

Current research

**Deterministic element based methods**: FEM, BEM, ...
- approximating shape functions
- fine discretisations
  ⇒ low-frequency

**Statistical energy based methods**: SEA, EFEM, ...
- energy in SEA subsystems
- underlying assumptions
  ⇒ high-frequency

**Enhanced deterministic approaches**
- Process optimisation
- Reduction or elimination of numerical pollution effects
- Introduction of a priori information (Trefftz approaches)
Simulation of vibro-acoustic systems

Current research

**Deterministic element based methods**: FEM, BEM,...
- approximating shape functions
- fine discretisations  ⇒ low-frequency

**Statistical energy based methods**: SEA, EFEM,...
- energy in SEA subsystems
- underlying assumptions  ⇒ high-frequency

**Enhanced statistical approaches**
- Non-equipartition of modal energy
- Strong or indirect coupling
- SEA model parameter estimation

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FEM, BEM  SEA  frequency
Introduction

Vibro-acoustics

Numerical modelling

WBM

Numerical validation

Conclusions

Simulation of vibro-acoustic systems

Current research

**Deterministic element based methods**: FEM, BEM,...
- approximating shape functions
- fine discretisations
  \[\Rightarrow \text{low-frequency}\]

**Statistical energy based methods**: SEA, EFEM,...
- energy in SEA subsystems
- underlying assumptions
  \[\Rightarrow \text{high-frequency}\]

**Hybrid deterministic-statistical approaches**
- "Flexible" components: SEA subsystems
- "Stiff" components: FEM models
- Diffuse field reciprocity relation
Simulation of vibro-acoustic systems

**Wave Based Method**

*Deterministic element based methods:* FEM, BEM,...
- approximating shape functions
- fine discretisations \(\Rightarrow\) low-frequency

*Statistical energy based methods:* SEA, EFEM,...
- energy in SEA subsystems
- underlying assumptions \(\Rightarrow\) high-frequency

*Deterministic Trefftz-based Wave Based Method*
- Basis functions = exact solutions of governing equations
- Enhanced numerical convergence properties
  \(\Rightarrow\) applicable to mid-frequency problems
Introduction

The Wave Based Method
- Problem definition
- Wave Based modelling approach

Numerical validation of the WBM for 3D acoustics

Conclusions
WBM modelling approach

Dynamic equations: Helmholtz problems

**Steady-state dynamic acoustic pressure** $p(\mathbf{r})$ in $V$:

Helmholtz equation

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = \mathcal{F}(\mathbf{r}), \; \mathbf{r} \in \Omega$$

At each point of $\Omega_v$: 1 boundary condition

$$\begin{cases} 
\mathbf{r} \in \Omega_v : \mathcal{L}_v(p(\mathbf{r})) = \bar{v}_n(\mathbf{r}) , \\
\mathbf{r} \in \Omega_z : \mathcal{L}_v(p(\mathbf{r})) = p(\mathbf{r})/\bar{Z}_n(\mathbf{r}) , \\
\mathbf{r} \in \Omega_p : p(\mathbf{r}) = \bar{p}(\mathbf{r}) . 
\end{cases}$$
WBM modelling approach
General WBM modelling process

1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions and source terms

WBM approximation

\[ p^{(\alpha)}(r) \simeq \sum_w p^{(\alpha)}_w \Phi^{(\alpha)}_w(r) + \hat{p}^{(\alpha)}_q(r) \]
WBM modelling approach

General WBM modelling process

1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions and source terms

2. Minimisation of the approximation errors using a Galerkin weighted residual formulation

\[
\int_{\Omega_v} \tilde{p}(\mathbf{r}) \, R_v(\mathbf{r}) \, d\Omega + \int_{\Omega_Z} \tilde{p}(\mathbf{r}) \, R_Z(\mathbf{r}) \, d\Omega \\
+ \int_{\Omega_p} -\mathcal{L}_v(\tilde{p}(\mathbf{r})) \, R_p(\mathbf{r}) \, d\Omega = 0
\]
WBM modelling approach
General WBM modelling process

1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions and source terms

2. Minimisation of the approximation errors using a Galerkin weighted residual formulation

3. Solution of the resulting system of equations

\[
\begin{bmatrix}
A^{1}_w & \cdots & C^{1,N}_{\Omega} \\
\vdots & \ddots & \vdots \\
C^{N,1}_w & \cdots & A^{N}_{\Omega}
\end{bmatrix}
\begin{Bmatrix}
p^{1}_w \\
\vdots \\
p^{N}_{\Omega}
\end{Bmatrix}
= \begin{Bmatrix}
F^{1}_w \\
\vdots \\
F^{N}_{\Omega}
\end{Bmatrix}
\]
WBM modelling approach

General WBM modelling process

1. Approximation of the field variables within each of the subdomains using physically meaningful wave functions and source terms
2. Minimisation of the approximation errors using a Galerkin weighted residual formulation
3. Solution of the resulting system of equations
4. Postprocessing, visualisation and interpretation of the response fields
The WB modelling approach has been successfully applied to:

**2D steady-state acoustic analysis**
- multi-domain interior acoustics
- acoustic scattering problems
- multiple scattering and inclusion problems

**2D steady-state structural dynamic analysis**
- plate membrane problems
- plate bending problems
- shell analysis

**2D steady-state vibro-acoustic analysis**
- interior vibro-acoustics
- exterior vibro-acoustics

**Current extension:**
- 3D steady-state interior acoustics
Acoustic pressure expansion: bounded subdomains

\[ p^{(\alpha)}(\mathbf{r}) \simeq \sum_{w=1}^{n_w^{(\alpha)}} p_w^{(\alpha)} \Phi_w^{(\alpha)}(\mathbf{r}) + \hat{p}_q^{(\alpha)}(\mathbf{r}) \]

\[ \Phi_w^{(\alpha)}(\mathbf{r}(x, y)) \]: Homogeneous solutions of Helmholtz eq.

\[ \Phi_w(\mathbf{r}(x, y, z)) = \begin{cases} 
\Phi_{wr}(x, y, z) = \cos(k_{xw} x) \cos(k_{yw} y) e^{-jk_{zw} z} \\
\Phi_{ws}(x, y, z) = \cos(k_{xw} x) e^{-jk_{yw} y} \cos(k_{zw} z) \\
\Phi_{wt}(x, y, z) = e^{-jk_{xw} x} \cos(k_{yw} y) \cos(k_{zw} z) 
\end{cases} \]

Trefftz approach: the wave number components should satisfy the associated dispersion relation:

\[ \begin{cases} 
(k_{xw})^2 + (k_{yw})^2 + (k_{zw})^2 = \lambda^2 \\
(k_{xw})^2 + (k_{yw})^2 + (k_{zw})^2 = \lambda^2 \\
(k_{xw})^2 + (k_{yw})^2 + (k_{zw})^2 = k^2 
\end{cases} \]
Acoustic pressure expansion: bounded subdomains

\[
p^{(\alpha)}(r) \simeq \sum_{w=1}^{n_w^{(\alpha)}} p_w^{(\alpha)} \Phi_w^{(\alpha)}(r) + \hat{p}_q^{(\alpha)}(r)
\]

\(\infty\) number of solutions \(\rightarrow\) selection of wave numbers:

\[
\begin{align*}
(k_{wx}, k_{wy}, k_{wz}) &= \left(\frac{w_1 \pi}{L_x}, \frac{w_2 \pi}{L_y}, \pm \sqrt{k^2 - (k_{wx})^2 - (k_{wy})^2}\right) \\
(k_{ws}, k_{ws}, k_{ws}) &= \left(\frac{w_3 \pi}{L_x}, \pm \sqrt{k^2 - (k_{ws})^2 - (k_{ws})^2}, \frac{w_4 \pi}{L_z}\right) \\
(k_{wt}, k_{wt}, k_{wt}) &= \left(\pm \sqrt{k^2 - (k_{wt})^2 - (k_{wt})^2}, \frac{w_5 \pi}{L_y}, \frac{w_6 \pi}{L_z}\right)
\end{align*}
\]

\(w_1, \ldots, w_6 = 0, 1, \ldots\)
WBM approximations: wave functions

\[ p^{(\alpha)}(\mathbf{r}) \simeq \sum_w p_w^{(\alpha)} \Phi_w^{(\alpha)}(\mathbf{r}) + \hat{p}_q^{(\alpha)}(\mathbf{r}) \]
WBM approximations: wave functions

\[ p^{(\alpha)}(r) \simeq \sum_w p_w^{(\alpha)} \Phi_w^{(\alpha)}(r) + \hat{p}_q^{(\alpha)}(r) \]
Acoustic pressure expansion: bounded subdomains

\[ p^{(\alpha)}(\mathbf{r}) \simeq \sum_{w=1}^{n_w^{(\alpha)}} p_w^{(\alpha)} \Phi_w^{(\alpha)}(\mathbf{r}) + \hat{p}_q^{(\alpha)}(\mathbf{r}) \]

\(\hat{p}_q^{(\alpha)}(x, y, z)\): Particular (free field) solution of Helmholtz eq.

- Acoustic point source:

\[ \hat{p}_q(x, y, z) = \frac{j\rho_0 \omega Q}{4\pi} \frac{e^{-jkr_q}}{r_q} \]

with \(Q\) the source strength \(Q = \int_V q \, dV\) and with \(r_q\) the distance to the source point.
WBM approximation: source terms

\[ p^{(\alpha)}(r) \simeq \sum_w p_w^{(\alpha)} \Phi_w^{(\alpha)}(r) + \hat{p}_q^{(\alpha)}(r) \]
Application of a Galerkin weighted residual formulation yields

\[
\begin{bmatrix}
A_{w}^{1} & \cdots & C_{w}^{1,N\Omega}

\vdots & \ddots & \vdots

C_{w}^{N\Omega,1} & \cdots & A_{w}^{N\Omega}
\end{bmatrix}
\begin{bmatrix}
p_{w}^{1}

\vdots

p_{w}^{N\Omega}
\end{bmatrix}
= \begin{bmatrix}
F_{w}^{1}

\vdots

F_{w}^{N\Omega}
\end{bmatrix}
\]

Properties:

+ Small number of degrees of freedom
+ No accuracy decrease for derived variables
+ High convergence rate

- Fully populated matrices with complex coefficients
- Frequency-dependent matrices
- More complex numerical integrations
- Bad numerical conditioning
- Convergence only guaranteed for convex (sub)domains

→ applicable to practical mid-frequency applications of moderate geometrical complexity
Overview

1. Introduction

2. The Wave Based Method

3. Numerical validation of the WBM for 3D acoustics
   - Problem setting
   - Spatial predictions of acoustic quantities
   - Computational efficiency comparison

4. Conclusions
Numerical validation
Convex acoustic cavity

Problem definition

- Acoustic fluid: Air
  \[ c = 340\sqrt{1 + j\eta} \text{ m/s}, \quad \rho_0 = \frac{1.225}{(1 + j\eta)} \text{ kg/m}^3, \quad \eta = \{0\%, 1\%\} \]
- Boundary conditions: acoustically rigid \((\vec{v}_n(r) = 0)\)
- Excitation: discontinuous unit normal velocity on patch in front

Acoustic fluid: Air

\[ c = 340\sqrt{1 + j\eta} \text{ m/s}, \quad \rho_0 = \frac{1.225}{(1 + j\eta)} \text{ kg/m}^3, \quad \eta = \{0\%, 1\%\} \]
Numerical validation
Convex acoustic cavity

Acoustic variables at 450 Hz, damped case

- Acoustic pressure amplitude [Pa]

Rigidity conditions accurately taken into account
Acoustic variables at 450\(Hz\), damped case

- Acoustic velocity vector field [m/s]

Discontinuous velocity excitation conditions accurately taken into account
Numerical validation
Convex acoustic cavity

Computational efficiency comparison at 450Hz

\[ \varepsilon_p = \frac{1}{108} \sum_{j=1}^{108} \varepsilon_j = \frac{1}{108} \sum_{j=1}^{108} \left| \frac{\hat{p}(r_j) - p_{\text{ref}}(r_j)}{p_{\text{ref}}(r_j)} \right| \]

undamped problem setting

Damped problem setting
Numerical validation
Convex acoustic cavity

Frequency response comparison damped problem setting

![Frequency response comparison graph](image-url)
For a similar calculation time, the WBM is an order of magnitude more accurate than the quadratic FEM.

WBM reaches a similar prediction accuracy more than 10 times faster than the quadratic FEM.
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Conclusions

Wave Based Method for steady-state dynamics
+ globally defined exact solutions = wave functions
+ low, mid (and high) frequency applications

Extension to 3D acoustics
- Derivation of wave function sets
- Weighted residual formulation
⇒ Efficient modelling of 3D problems of moderate geometrical complexity
Conclusions

Wave Based Method for steady-state dynamics
+ globally defined exact solutions = wave functions
+ low, mid (and high) frequency applications

Extension to 3D acoustics
- Derivation of wave function sets
- Weighted residual formulation
⇒ Efficient modelling of 3D problems of moderate geometrical complexity

Next steps
Extension of the modelling approach to:
- 3D multi-domain acoustic problems
- 3D acoustic (multiple) scattering problems
- 3D acoustic multiple inclusion problems
Thank you for your attention!

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