A quadrature rule for numerical integration based on Haar wavelets and hybrid functions

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Abstract

In this paper Haar wavelets and hybrid functions have been applied for numerical solution of double and triple integrals with variable limits of integration. This approach is the generalization and improvement of the method [31] where the numerical method is only applicable to the integrals with constant limits. Apart from generalization of the method [31], the new approach has two major advantages over the classical methods based on quadrature rule: (i) No need of finding optimum weights as the wavelet coefficients serve the purpose of optimal weights automatically (ii) Mesh points of the wavelets algorithm are used as nodal values instead of considering the n nodes as unknown roots of polynomial of degree n. The new method is more efficient. The novel method is compared with existing methods and applied to a number of benchmark problems. Accuracy of the method is measured in terms of absolute errors.

Keywords: Haar wavelets, Hybrid functions, Quadrature rule, Numerical method, Double integrals, Triple integrals.

1 Introduction

Numerical integration has several applications in science and engineering. A lot of work has been done in this area in terms quadrature rule of numerical integration. Quadrature rule is based on polynomial interpolation. Interpolating polynomials are used to find weights corresponding to nodes. Numerical quadrature bears some drawbacks, these include: (i) The use of large number of equally spaced nodes in the case of Newton-Cotes quadrature rule may cause erratic behavior with high degree polynomial interpolation (ii) Gaussian quadrature rule is also based on polynomial interpolation but the nodes as well as the weights are chosen to maximize the degree of accuracy of the resulting rule. Gaussian quadrature rule can be derived by the method of undetermined coefficients but the resulting equations for the $2n$ unknown nodes and weights are nonlinear. This procedure is quiet cumbersome for hand calculations and nodes and weights are tabulated in advance before evaluating integrals numerically. A number of polynomials based quadrature methods have been discussed in [13, 28, 19, 25, 12, 26, 11, 27] and the references therein. In order to overcome some of the difficulties listed above, we propose a new method based on Haar wavelets and hybrid functions to find numerical solutions of double and triple integrals. This work should be considered as a logical continuation of our previous work of

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[31]. In the earlier paper [31], Haar wavelets and hybrid functions are used to find numerical solution of definite integrals with constant limits and hence the methods could be used only to a limited number of numerical integration problems. In the present paper we extend the scope of applicability of the methods [31] to double and triple integrals with variable limits. In doing so we introduce a new approach which not only widens the area of applicability of those methods but also reduces the computational time and improves accuracy of the algorithm. The approach used in [31] was to approximate the function using two and three-dimensional Haar wavelets or two and three-dimensional hybrid functions in case of double integrals and triple integrals respectively. As we go to higher dimensions the number of coefficients increase exponentially and the computational cost of the method increase considerably. To avoid the rising computational cost, the present approach is based on considering one integral at a time and applying the Haar wavelet or hybrid function method for a single integral. After one integral has been solved the same method is applied for the evaluation of other integrals repeatedly in similar fashion.

Wavelets have been successfully used in the field of numerical approximations. Some of the wavelets applications are related to finding numerical solutions of integral equations and numerical integration [20, 31], ordinary differential equations [5, 30], partial differential equations [4] and fractional partial differential equations [33]. Various type of wavelets have been used in such applications, for example, Daubechies [32], Battle-Lemarie [34], B-spline [5], Chebyshev [1], Legendre [3, 29] and Haar wavelets [16, 6, 30, 31]. However because of their simplicity Haar wavelets received the attention of many researchers. Haar wavelets have been applied for numerical approximations by Hsiao and Chen [6], Hsiao [8], Hsiao and Wang [10], Lepik [16, 15, 14], Lepik and Tamme [18, 17], Maleknejad and Mirzaee [21], Babolian and Shshsawaran [2].

Hybrid functions have faster convergence than the Haar wavelets and can model discontinuities in better manner than Haar wavelets, Xing et. al. [35]. Another useful property of hybrid functions is a special product matrix and a related coefficient matrix with optimal order.The advantage of hybrid functions is that the order of block-pulse functions and Legendre polynomials are adjustable to obtain highly accurate numerical solutions than the piecewise constant orthogonal function for the solution of integral equations, Hsiao [9]. Recently, hybrid functions have been successfully used for the numerical solution of ordinary differential equations as well as integral equations, see [22, 24, 9, 23, 30, 31].

Wavelets have also been applied for numerical integration by Hashish et al [7] but their method is applicable to only those integrals that have constant limits of integration. This paper proposes a new method based on the simple Haar wavelets and hybrid functions. This approach has the following advantages

i Provides accurate solution in comparison with the exiting method,

ii Optimal weights are calculated using built in procedure in terms of wavelets or hybrid function coefficients. In the new approach we do not need to consult variety of tables for optimal weights.

iii No quadrature nodes are needed and the collocation points are used as nodal points

iv The new method calculates the integrals explicitly and it does not need solving a nonlinear system resulting form the unknown nodes and weights,

vi Simple and direct applicability with no need of other intermediate technique is required.

2 Numerical Examples

Example 1.

$$\int_0^1 \int_0^y e^{|x+y-1|}\, \mathrm{d}x\, \mathrm{d}y = -2 + e$$

This example has also been solved numerically by Rathod et al [27] using finite element method. Their best result is accurate up to 7 decimal places while we obtained results accurate up to 10 decimal places which clearly indicates superiority of our method over their method. Moreover, their method can only be applied to integrals over triangular regions while our methods can be applied to a variety of regions including triangular regions. Absolute errors are shown in Table 1.

Example 2.

$$\int_0^{\frac{\pi}{4}} \int_0^{\sin y} \frac{1}{\sqrt{(1-x^2)}}\, \mathrm{d}x\, \mathrm{d}y = \frac{\pi^2}{32}$$

Absolute errors are shown in Table 2.

Example 3.

$$\int\int_R (x+y)^{-\frac{1}{2}}\, \mathrm{d}x\, \mathrm{d}y = \frac{2}{3}\left(2 - 7\sqrt{3} - 15\sqrt{5} + 20\sqrt{6}\right) \approx 3.549613026789713,$$

where R is a quadrilateral region connecting the points $(-1,2), (2,1), (3,3)$ and $(1,4)$. This example has been solved numerically by Islam and Hossain [11] using finite element method for quadrilateral regions only. Their best result is accurate up to 14 decimal places while our hybrid functions method gave result accurate to 15 decimal places.

In order to solve this problem using present methods we divide the region R into three subregions so that the given integral can be written as sum of three integrals as

$$\begin{aligned}\int\int_R (x+y)^{-\frac{1}{2}}\, \mathrm{d}x\, \mathrm{d}y &= \int_1^2 \int_{(5-3y)}^{\frac{1}{2}(y+3)} (x+y)^{-\frac{1}{2}}\, \mathrm{d}x\, \mathrm{d}y \\ &+ \int_2^3 \int_{(y-3)}^{\frac{1}{2}(y+3)} (x+y)^{-\frac{1}{2}}\, \mathrm{d}x\, \mathrm{d}y + \int_3^4 \int_{y-3}^{(9-2y)} (x+y)^{-\frac{1}{2}}\, \mathrm{d}x\, \mathrm{d}y \end{aligned} \quad (1)$$

Absolute errors are shown in Table 3.

Example 4.

$$\int_0^{\pi} \int_0^{z} \int_0^{zy} \frac{1}{y} sin\left(\frac{x}{y}\right)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \frac{1}{2}(4+\pi^2)$$

Absolute errors are shown in Table 4.

Table 1: Absolute Errors of Example 1

Haar	Absolute Errors	Hybrid	Absolute Errors
$M = 4, N = 5$	$1.6425E - 03$	$m = 3, n = 25$	$9.0521E - 07$
$M = 5, N = 10$	$4.9512E - 04$	$m = 4, n = 50$	$1.4728E - 08$
$M = 10, N = 15$	$1.9801E - 04$	$m = 6, n = 100$	$1.3179E - 09$
$M = 16, N = 16$	$1.1857E - 04$	$m = 8, n = 100$	$7.9406E - 10$

Table 2: Absolute Errors of Example 2

Haar	Absolute Errors	Hybrid	Absolute Errors
$M = 3, N = 3$	$1.7115E - 04$	$m = 3, n = 10$	$2.5489E - 08$
$M = 4, N = 3$	$9.6650E - 05$	$m = 5, n = 10$	$4.0726E - 11$
$M = 5, N = 4$	$6.2718E - 05$	$m = 7, n = 10$	$1.0419E - 13$
$M = 6, N = 5$	$4.3844E - 05$	$m = 9, n = 10$	$3.3307E - 16$

Table 3: Absolute Errors of Example 3

Haar	Absolute Errors	Hybrid	Absolute Errors
$M = N = 2$	$1.1497E - 02$	$m = 3, n = 20$	$7.0503E - 08$
$M = N = 4$	$2.9874E - 03$	$m = 4, n = 20$	$3.6491E - 08$
$M = N = 8$	$7.5501E - 04$	$m = 5, n = 20$	$5.1215E - 11$
$M = N = 16$	$1.8929E - 04$	$m = 6, n = 20$	$3.0780E - 11$
$M = N = 32$	$4.7355E - 05$	$m = 7, n = 20$	$6.2172E - 14$
$M = N = 64$	$1.1841E - 05$	$m = 8, n = 20$	$3.8192E - 14$
$M = N = 128$	$2.9604E - 06$	$m = 9, n = 20$	$3.9968E - 15$

Table 4: Absolute Errors of Example 4

Haar	Absolute Errors	Hybrid	Absolute Errors
$M = N = P = 8$	$3.5959E - 03$	$m = 3, n = 5$	$6.4408E - 05$
$M = N = P = 16$	$9.0291E - 04$	$m = 4, n = 10$	$2.0353E - 06$
$M = N = P = 32$	$2.2597E - 04$	$m = 5, n = 10$	$2.7290E - 09$
$M = N = P = 64$	$5.6508E - 05$	$m = 6, n = 20$	$2.5509E - 11$

3 Conclusion

In this paper Haar wavelets and hybrid functions were applied for numerical integration of double and triple integrals with variable limits. Comparison between the two methods shows that hybrid functions method gives better results than Haar wavelets method. The hybrid functions method was also compared with finite element method and its advantages can easily be observed which are that hybrid functions method is easy to implement, does not need to modify for different regions and gives more accurate results.

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