

試題及答案卷

學系：_____ 學號：_____ 姓名：_____

注意：禁止使用計算器。

問題	1: 10分	2: 10分	3: 10分	4: 10分	5: 10分	總分: 100分
得分						
問題	6: 10分	7: 10分	8: 10分	9: 10分	10: 10分	
得分						

1. 給定函數

$$f(x) = \begin{cases} \frac{\sin(x^2)}{x^2}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

(a) f 是否在0點連續? 為什麼?

答案: Yes

(b) f 是否在0點可微分? 為什麼?

答案: Yes

解答:

(a) Yes.

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{x^2 + o(x^2)}{x^2} \\ &= 1 = f(0) \end{aligned}$$

(b) Yes.

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{\frac{\sin h^2}{h^2} - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h \cos h^2 - 2h}{3h^2} \\ &= \lim_{h \rightarrow 0} \frac{2 \cos h^2 - 4h^2 \sin h^2 - 2}{6h} \\ &= \lim_{h \rightarrow 0} \frac{-2 \sin h^2 - 4h^2 \sin h^2}{6h} = 0 \quad \square \end{aligned}$$

2. 設 $y = f(x)$ 是一個可微分函數且滿足下列式子:

$$y = x^{x+y}$$

求 $y'(1)$ 。

答案: 2

解答: Note that if $x = 1$, then $y = 1$ since $y = 1^{1+y} = 1$. Taking derivative after taking \ln on both sides yields $\ln y = (x + y) \ln x$, $\frac{y'}{y} = (1 + y') \ln x + (x + y) \frac{1}{x}$. Hence $y' = \frac{\frac{xy+y^2}{x} + y \ln x}{1 - y \ln x}$ and $y'(1) = 2$. \square

3. 證明對所有實數 x , $f(x) = 3x^4 + 4x^3 + 1$ 恆大於等於 0。 答案: $f(x) \geq f(-1) = 0$

解答: $f'(x) = 12x^3 + 12x^2 = 12x^2(x + 1)$ and it attains its the minimal value 0 at $x = -1$. \square

4. 設 $g(x)$ 是一個定義在 $x > -1$ 上的函數, 且滿足下列式子:

$$g(3x^4 + 4x^3 + 1) = \ln(x + 2)$$

求 $g'(8)$ 之值。

答案: $1/72$

解答: $(dg/dy)(12x^3 + 12x^2) = \frac{1}{x+2}$, $g'(8) = \frac{\frac{1}{x+2}}{12x^3+12x^2} \Big|_{x=1} = 1/72$. \square

5. 寫出所有連續函數 f 滿足 $(f(x))^2 = x^2$, $x \in \mathbb{R}$, 並證明你已經寫出全部的這種連續函數。 答案: $\pm x$; $\pm |x|$

解答: There are 4 of them: $f_1(x) = x$; $f_2(x) = -x$; $f_3(x) = |x|$; $f_4(x) = -|x|$. $(f(x))^2 = x^2$, so $f(x) = \pm x$. If $f(x_1) > 0$ and $f(x_2) < 0$ for some $x_1 > 0, x_2 > 0$ then by intermediate value theorem for continuous function there exists a real number c between x_1 and x_2 (then $c > 0$) such that $f(c) = 0$, which contradicts to $(f(c))^2 = c^2$. Hence if $f(x_1) = x_1$ for some $x_1 > 0$ then $f(x) = x$ for all $x > 0$. \square

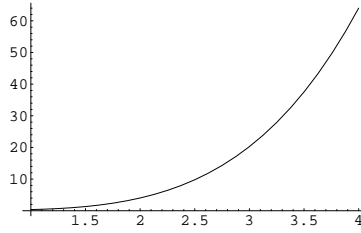
6. 試求 $\int_1^3 |x^2 - 4| dx$ 。 答案: 4

解答: 因為

$$|x^2 - 4| = \begin{cases} 4 - x^2, & \text{如果 } 1 \leq x \leq 2; \\ x^2 - 4, & \text{如果 } 2 \leq x \leq 3; \end{cases}$$

因此 $\int_1^3 |x^2 - 4| dx = \int_1^2 (4 - x^2) dx + \int_2^3 (x^2 - 4) dx = 4$ 。 \square

7. 求曲線 $y = \frac{x^4}{4} + \frac{1}{8x^2}$ 在 $x = 1$ 和 $x = 4$ 之間的長度。 答案: $\frac{8175}{128} = 63\frac{111}{128}$



解答: 令 $f(x) = \frac{x^4}{4} + \frac{1}{8x^2}$, $L = \int_1^4 \sqrt{1 + [f'(x)]^2} dx = \int_1^4 \sqrt{1 + (x^3 - \frac{1}{4x^3})^2} dx = \int_1^4 x^3 + \frac{1}{4x^3} dx = \frac{1}{4}x^4 - \frac{1}{8x^2} \Big|_1^4 = \frac{8175}{128} = 63\frac{111}{128}$ \square

8. 試求 $\int_0^{\pi/2} \sin^2 x \cos^4 x dx$ 。

答案: $\frac{\pi}{32}$

解答:

$$\begin{aligned} \int_0^{\pi/2} \sin^2 x \cos^4 x dx &= \int_0^{\pi/2} (\sin x \cos x)^2 \cos^2 x dx \\ &= \int_0^{\pi/2} \left(\frac{1}{2} \sin 2x\right)^2 \frac{1}{2} (1 + \cos 2x) dx \\ &= \frac{1}{8} \int_0^{\pi/2} \sin^2 2x dx + \frac{1}{8} \int_0^{\pi/2} \sin^2 2x \cos 2x dx \\ &= \frac{1}{16} \int_0^{\pi/2} (1 - \cos 4x) dx + \frac{1}{48} \sin^3 2x \Big|_0^{\pi/2} \\ &= \frac{1}{16} \left[x - \frac{\sin 4x}{4} \right] \Big|_0^{\pi/2} + 0 \\ &= \frac{\pi}{32} \end{aligned} \quad \square$$

9. 計算

答案: $\frac{\pi}{6}$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right]$$

解答: 我們有

$$\begin{aligned} s_n &= \frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \\ &= \frac{1}{n} \left[\frac{1}{\sqrt{4 - (\frac{1}{n})^2}} + \frac{1}{\sqrt{4 - (\frac{2}{n})^2}} + \cdots + \frac{1}{\sqrt{4 - (\frac{n}{n})^2}} \right] \end{aligned}$$

因此 s_n 是函數 $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) = \frac{1}{\sqrt{4-x^2}}$, 對於 $x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_n = \frac{n}{n} = 1$ 分割, 且中間點 $\xi_i = \frac{i}{n} \in [x_i, x_{i+1}]$ 的黎曼和。所以其解為

$$\lim_{n \rightarrow \infty} s_n = \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \arcsin \frac{x}{2} \Big|_0^1 = \frac{\pi}{6} \quad \square$$

10. 證明對任意的實數 x ，下列的級數和收斂並計算其和

答案: $\frac{1}{2}(\cos x + \cosh x)$

$$1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \frac{x^{12}}{12!} + \cdots$$

解答: 由以下泰勒展開式

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\end{aligned}$$

可以知道題目給定之級數和為 $\frac{1}{2}(\cos x + \cosh x)$ 的泰勒展開式。

□

~全卷完~