國立中山大學、國立成功大學合辦94學年度基礎學科
微積分競試試題與詳解

1.試求 $\lim\limits_{x\to\infty}(\frac{x^2-4}{x^2-1})^{x^2+1}$.

Ans: Consider $\ln((\frac{x^2-4}{x^2-1})^{x^2+1}) = (x^2+1)(\ln(x^2-4)-\ln(x^2-1))$

$$\lim_{x\to\infty}(x^2+1)(\ln(x^2-4)-\ln(x^2-1)) = \lim_{x\to\infty}\frac{\ln(x^2-4)-\ln(x^2-1)}{\frac{1}{x^2+1}}$$

$$\lim_{x\to\infty}\frac{\frac{2x}{x^2-4}-\frac{2x}{x^2-1}}{\frac{-2x}{(x^2+1)^2}} = \lim_{x\to\infty}\frac{(x^2+1)^2(x^2-1-x^2+4)}{-(x^2-4)(x^2-1)} = -3$$

Since ln is a continuous function, $\lim\limits_{x\to a}\ln(x) = \ln(\lim\limits_{x\to a}x)$, and by the L'Hospital's rule $\lim\limits_{x\to\infty}(\frac{x^2-4}{x^2-1})^{x^2+1} = e^{-3}$.

2.試求(a) $\int_0^1 x^5 e^{x^3}\,dx$.

(b) $\int_0^{\frac{\pi}{4}} \frac{d\theta}{(\tan^2\theta+4\tan\theta+3)\cos^2\theta}$

(c) $\int \frac{8x^2+4x-11}{(x+3)(x-1)^2}dx$

Ans: (a) Let $u = x^3$. Then $\int_0^1 x^5 e^{x^3}dx = \int_0^1 u\cdot\frac{1}{3}\cdot e^u\,du$

$$=\frac{1}{3}\left[ue^u|_0^1 - \int_0^1 e^u du\right] = \frac{1}{3}$$

(b) Let $u = \tan\theta \Rightarrow du = \sec^2\theta d\theta$

原式 $= \int_0^1 \frac{du}{u^2+4u+3} = \int_0^1 \frac{1}{2}\left[\frac{1}{u+1}-\frac{1}{u+3}\right]du = \frac{1}{2}\left[\ln(u+1)-\ln(u+3)\right]|_0^1$

$= \frac{1}{2}[\ln 3 - \ln 2]$

(c) $\frac{8x^2+4x-11}{(x+3)(x-1)^2} = \frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2} \Rightarrow A = \frac{49}{16}, B = \frac{79}{16}, C = \frac{1}{4}$

原式 $= \int \frac{\frac{49}{16}}{x+3}+\frac{\frac{79}{16}}{x-1}+\frac{\frac{1}{4}}{(x-1)^2}dx = \frac{49}{16}\ln|x+3|+\frac{79}{16}\ln|x-1|-\frac{1}{4}\frac{1}{x-1}+\mathbf{C}$

3.試求(a) $\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{2t-t^2}}dt$ (b) $\lim\limits_{x\to 1}\frac{1}{x-1}(\int_{\frac{1}{2}}^x \frac{1}{\sqrt{2t-t^2}}dt - \frac{\pi}{6})$

Ans: (a) Letting $1-t = \sin\theta$, we have $dt = -\cos\theta d\theta$ and for $\frac{1}{2}\le t\le 1$, $0\le\sin\theta\le\frac{1}{2} \Rightarrow \cos\theta\ge 0$ and $\sqrt{1-(1-t)^2} = \cos\theta$.

Thus, $\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1-(1-t)^2}}dt = \int_{\frac{\pi}{6}}^0 \frac{-\cos\theta d\theta}{\cos\theta} = -\theta|_{\frac{\pi}{6}}^0 = \frac{\pi}{6}$

(b) Let $f(x) = \int_{\frac{1}{2}}^x \frac{1}{\sqrt{2t-t^2}}dt$. Then $\lim\limits_{x\to 1}\frac{1}{x-1}(\int_{\frac{1}{2}}^x \frac{1}{\sqrt{2t-t^2}}dt - \frac{\pi}{6}) = \lim\limits_{x\to 1}\frac{f(x)-f(1)}{x-1}$.

Since $\sqrt{2t-t^2} > 0$ for $0 < t < 2$, $f(x)$ is differentiable at $x = 1$, and the

Fundamental Theorem of Calculus implies that
$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}. = f'(1) = \left.\frac{1}{\sqrt{2t-t^2}}\right|_{t=1} = 1$

4. 請判定瑕積分$\int_0^\infty e^{x-2e^x}dx$是否存在? 如果此瑕積分存在, 試求其值.

Ans: Consider $\int_0^t \frac{e^x}{e^{2e^x}}dx$. (Let $u = e^x$)

$$= \int_1^{e^t} \frac{du}{e^{2u}} = \int_1^{e^t} -\frac{1}{2}(-2e^{-2u})du$$
$$= -\frac{1}{2}e^{-2u}\Big|_1^{e^t} = -\frac{1}{2}(e^{-2e^t} - e^{-2})$$

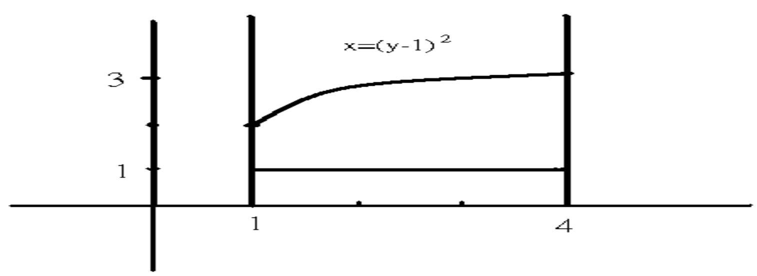
$$\lim_{t\to\infty} -\frac{1}{2}(e^{-2e^t} - e^{-2}) = \lim_{t\to\infty}\frac{-1}{2}\left(\frac{1}{e^{2e^t}} - \frac{1}{e^2}\right) = \frac{1}{2e^2}$$

5.令$R = \{(x,y) \in \mathbb{R}^2 | 1 \le x \le 4, 1 < y \le 1 + x^{\frac{1}{2}}\}$. 試求:

(a) R 對$x = 1$ 旋轉所得之旋轉體體積.

(b) R 對$x = 0$ 旋轉所得之旋轉體體積.

Ans:



(a) $Vol = \int_1^4 2\pi(x-1)[1+\sqrt{x}-1]dx$
$$= \int_1^4 2\pi(x^{\frac{3}{2}} - x^{\frac{1}{2}})dx$$

$= 2\pi(\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}})\Big|_1^4$

$= 2\pi(\frac{62}{5} - \frac{14}{3})$

另解: $Vol = \pi \cdot 3^2 + \int_2^3 \pi\{3^2 - [(y-1)^2 - 1]^2\}dy$

$= \pi \cdot 3^2 \cdot 2 - \int_2^3 \pi[(y-1)^2 - 1]^2 dy$

(b) $Vol = \int_1^4 2\pi x[1 + \sqrt{x} - 1]dx$

$= \int_1^4 2\pi x^{\frac{3}{2}} dx$

$= \frac{4\pi}{5}x^{\frac{5}{2}}\Big|_1^4 = \frac{124\pi}{5}$

另解: $Vol = \pi(4^2 - 1^2) + \int_2^3 \pi[4^2 - (y-1)^4]dy$

$= \pi \cdot 4^2 \cdot 2 - \pi \cdot 1^2 \cdot 1 - \int_2^3 \pi[(y-1)^2]^2 dy$

6.考慮曲線$y = x^{\frac{1}{2}}$對$y = 0$ 旋轉. 試求當$1 \le x \le 4$ 時所得之旋轉體表面積.

Ans: $Area = \int_1^4 2\pi \cdot x^{\frac{1}{2}} \cdot \sqrt{1 + (\frac{1}{2}x^{-\frac{1}{2}})^2}\,dx = \int_1^4 2\pi \cdot x^{\frac{1}{2}} \cdot \sqrt{1 + \frac{1}{4}x^{-1}}\,dx$

$= 2\pi \int_1^4 \sqrt{x + \frac{1}{4}}\,dx = 2\pi(x + \frac{1}{4})^{\frac{3}{2}} \cdot \frac{2}{3}\Big|_1^4$

$= \frac{4\pi}{3}(\frac{17^{\frac{3}{2}}}{8} - \frac{5^{\frac{3}{2}}}{8}) = \frac{\pi}{6}(17^{\frac{3}{2}} - 5^{\frac{3}{2}})$

7. 設$f(x) = \sqrt{x^3 + x + 6}$,求$(f^{-1})'(4)$,也就是求$f$的反函數在4的導數.

Ans: $4 = \sqrt{x^3 + x + 6} \Rightarrow x = 2$ (另一根$x^2 + 2x + 5$虛根)

$\therefore (f^{-1})'(4) = \frac{1}{f'(2)} = \frac{1}{\frac{13}{8}} = \frac{8}{13}$

8.設$f(x) = \begin{cases} 3x + x^2 \sin\frac{1}{x} & \text{當}x \neq 0 \\ 0 & \text{當}x = 0 \end{cases}$

請判定$f'(x)$存在的位置.當$f'(x)$存在時,試求其公式.

Ans: When $x \neq 0, f'(x) = 3 + 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x} \cdot (-\frac{1}{x^2})$

$= 3 + 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

At $x = 0, \lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\frac{3x+x^2\sin\frac{1}{x}}{x}$

$= \lim\limits_{x\to 0}(3 + x\sin\frac{1}{x})$

And, since $\left|\sin\frac{1}{x}\right| \le 1$, $0 \le \left|x\sin\frac{1}{x}\right| \le |x|$

$\Rightarrow \lim\limits_{x\to 0}0 \le \lim\limits_{x\to 0}\left|x\sin\frac{1}{x}\right| \le \lim\limits_{x\to 0}|x| = 0$

$\Rightarrow \lim\limits_{x\to 0}\left|x\sin\frac{1}{x}\right| = 0$

$\Rightarrow \lim\limits_{x\to 0}x\sin\frac{1}{x} = 0$

$\Rightarrow f'(0) = 3$

9.令$f(x)=\frac{x^3}{x^2-1}$. 請討論$f(x)$ 遞增, 遞減, 凹凸, 漸近線, 極值之位置, 並畫出$y=f(x)$的圖.

Ans: 漸近線; $x=1$, $x=-1$, $y=x$.

local maximum: $(-\sqrt{3},-\frac{3\sqrt{3}}{2})$.

local minimum: $(\sqrt{3},\frac{3\sqrt{3}}{2})$.

point of reflection: $(0,0)$.

圖形: