# Total weight choosability of graphs

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**Abstract**

A graph $G = (V, E)$ is called $(k, k')$-total weight choosable if the following holds: For any total list assignment $L$ which assigns to each vertex $x$ a set $L(x)$ of $k$ real numbers, and assigns to each edge $e$ a set $L(e)$ of $k'$ real numbers, there is a mapping $f : V \cup E \to \mathbb{R}$ such that $f(y) \in L(y)$ for any $y \in V \cup E$ and for any two adjacent vertices $x, x'$, $\sum_{e \in E(x)} f(e) + f(x) \neq \sum_{e \in E(x')} f(e) + f(x')$. We conjecture that every graph is $(2, 2)$-total weight choosable and every graph without isolated edges is $(1, 3)$-total weight choosable. It follows from results in [7] that complete graphs, complete bipartite graphs, trees other than $K_2$ are $(1, 3)$-total weight choosable. Also a graph $G$ obtained from an arbitrary graph $H$ by subdividing each edge with at least three vertices is $(1, 3)$-total weight choosable. This paper proves that complete graphs, trees, generalized theta graphs are $(2, 2)$-total weight choosable. We also prove that for any graph $H$, a graph $G$ obtained from $H$ by subdividing each edge with at least two vertices is $(2, 2)$-total weight choosable as well as $(1, 3)$-total weight choosable.

## 1 Introduction

Suppose $G = (V, E)$ is a graph. For a vertex $x$ of $G$, $E(x)$ denotes the set of edges of $G$ incident to $x$ and $E^*(x) = E(x) \cup \{x\}$. An *edge weighting* of $G$ is a mapping that assigns to each edge $e$ of $G$ a real number $f(e)$. An edge-weighting $f$ of $G$ induces a vertex colouring $\phi_f$ of $G$ defined as $\phi_f(x) = \sum_{e \in E(x)} f(e)$ for every vertex $x$. An edge weighting $f$ is *proper* if the

[*]Department of Applied Mathematics, National Sun Yat-sen University. Grant number: NSC95-2115-M-110-004

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induced colouring $\phi_f$ is proper, i.e., for any edge $xx'$ of $G$, $\phi_f(x) \neq \phi_f(x')$. The real number assigned to an edge is called the *weight* of the edge. Instead of real numbers, one can use elements of some other field as weights. For simplicity, the weights are restricted to real numbers in this paper. The study of edge weighting is initiated by Karoński, Łuczak and Thomason [8]. They proposed the following conjecture in [8]:

**Conjecture 1.1.** *Every connected graphs $G \neq K_2$ has a proper edge weighting $f$ such that $f(e) \in \{1, 2, 3\}$ for every edge $e$.*

This conjecture is referred as the $1, 2, 3$-conjecture and has received some attention. It is shown in [8] if $S$ is a subset of $\mathbb{R}$ of size at least 183 and is independent over the field of rational numbers, then any connected graph $G \neq K_2$ has a proper edge weighting $f$ with $f(e) \in S$ for every edge $e$. Under the same restriction that $S$ being independent over rational numbers, the bound 183 is reduced to 4 in [1]. In [3], it is shown that any connected graph $G \neq K_2$ has a proper edge weighting $f$ with $f(e) \in \{1, 2, \ldots, 30\}$ for every edge $e$. The result was improved in [2], where it is shown that $f$ can be chosen so that $f(e) \in \{1, 2, \ldots, 16\}$ for every edge $e$.

In [7], Bartnicki, Grytczuk and Niwczykthe considered the choosability version of $1, 2, 3$-conjecture. A graph is said to be *$k$-edge-weight-choosable* if the following is true: For any list assignment $L$ which assigns to each edge $e$ a set $L(e)$ of $k$ real numbers, $G$ has a proper edge weighting $f$ such that $f(e) \in L(e)$ for each edge $e$.

**Conjecture 1.2.** *Every graph without isolated edges is* 3*-edge-weight-choosable.*

Conjecture 1.2 is stronger than Conjecture 1.1. Bartnicki, Grytczuk and Niwczyk [7] verified this conjecture for complete graphs, complete bipartite graphs and some other graphs. However, it is unknown if there is a constant $k$ such that every graph without isolated edges is $k$-edge-weight-choosable.

Przybyło and Woźniak [9, 10] studied weighting that involves both the edges and the vertices of $G$. Suppose $G = (V, E)$ is a graph. A mapping $f : V \cup E \to \mathbb{R}$ is called a *total weighting* of $G$. The *vertex-colouring $\phi_f$ of $G$ induced by $f$* is defined as $\phi_f(x) = \sum_{y \in E^*(x)} f(y)$. A total weighting is called proper if the induced vertex colouring $\phi_f$ is a proper colouring of $G$. Przybyło and Woźniak proposed the following conjecture and named it the $1, 2$-conjecture in [9]:

**Conjecture 1.3.** *Every simple graph $G$ has a proper total weighting $f$ such that $f(y) \in \{1, 2\}$ for all $y \in V \cup E$.*

Przybyło and Woźniak [9, 10] verified this conjecture for some special graphs, including complete graphs, 4-regular graphs and graphs $G$ with $\chi(G) \leq 3$. They also proved that every simple graph $G$ has a proper total weighting $f$ such that $f(y) \in \{1, 2, \ldots, 11\}$ for all $y \in V \cup E$.

In this paper, we consider the choosability version of total weighting. A *total list assignment* of $G$ is a mapping $L : V \cup E \to \mathcal{P}(\mathbb{R})$ which assigns to each element $y \in V \cup E$ a set $L(y)$ of real numbers as *permissible weights*. Given a total list assignment $L$, a total weighting $f$ is called an *$L$-total weighting* if for each $y \in V \cup E$, $f(y) \in L(y)$. We say $G$ is *$L$-total weightable* if there exists a $L$-total weighting $f$ of $G$. Given a pair $(k, k')$ of positive integers, a total list assignment $L$ is called a $(k, k')$-total list assignment if $|L(x)| = k$ for each vertex $x \in V$ and $|L(e)| = k'$ for each edge $e \in E$. We say $G$ is *$(k, k')$-total weight choosable* ($(k, k')$-choosable, for short) if for any $(k, k')$-total list assignment $L$, $G$ is $L$-total-weightable.

The notion of $L$-total weightability and the notion of $(k, k')$-total weight choosable is a common generalization of the above mentioned various edge weighting concepts. It also includes the ordinary vertex colouring and vertex choosability as its special cases. A total list assignment $L$ is called a *vertex list assignment* if $L(e) = \{0\}$ for each edge $e \in E$. If $L$ is a vertex list assignment for which $|L(x)| = k$ for each vertex $x$, then $L$ is a vertex $k$-list assignment. A graph $G$ is called *$k$-choosable* if for any $k$-vertex list assignment $L$, $G$ is $L$-total weightable. The *choosability $ch(G)$* of a graph $G$ (also called the choice number of $G$ or the list chromatic number of $G$ and denoted by $\chi_l(G)$) is the minimum integer $k$ such that $G$ is $k$-choosable.

It follows from the definition that if a graph $G$ is $(k, 1)$-choosable, then $G$ is $k$-choosable (as one can let $L(e) = \{0\}$ for each edge $e$). The converse is also true. Assume $G$ is $k$-choosable, and $L$ is a $(k, 1)$-total list assignment with $L(e) = \{l(e)\}$ for each edge $e$. Let $L'$ be a $k$-vertex list assignment defined as $L'(x) = \{a + \sum_{e \in E(x)} l(e) : a \in L(x)\}$. Since $G$ is $k$-choosable, $G$ has an $L'$-colouring $f'$. Then $f(x) = f'(x) - \sum_{e \in E(x)} l(e)$ and $f(e) = l(e)$ is an $L$-total weighting of $G$.

It also follows from the definition that if a graph $G$ is $(1, k)$-choosable, then it is $k$-edge-weight-choosable. The converse is not true. For example, the path $P = (v_1, v_2, v_3, v_4)$ is 2-edge-weight-choosable, but it is not $(1, 2)$-

choosable. Let $L(v_1) = \{1\}$, $L(v_i) = \{0\}$ for $i = 2, 3, 4$, and let $L(e) = \{0, 1\}$ for each edge $e$. It can be verified that there is no $L$-total weighting. However, we shall show that those graphs shown in [7] to be $k$-edge-weight-choosable are also $(1, k)$-choosable. We propose two conjectures concerning total weight choosability of graphs.

**Conjecture 1.4.** *There are constants $k, k'$ such that every graph is $(k, k')$-choosable.*

**Conjecture 1.5.** *Every graph is $(2, 2)$-choosable. Every graph with no isolated edges is $(1, 3)$-choosable.*

Note that if a graph $G$ is $(k, k')$-choosable, then it is $(k+1, k')$-choosable and $(k, k'+1)$-choosable. If Conjecture 1.5 is true, then the integer pairs are best possible, as there are graphs which are not $(1, 2)$-choosable and for any integer $k$, there are graphs which are not $(k, 1)$-choosable.

Conjecture 1.5 is stronger than Conjectures 1.4, 1.2 and 1.3. Although $(1, 3)$-total weight choosability is stronger than the 3-edge weight choosability, we shall see that in some inductive proofs, the stronger version has its advantage. The argument in [7] actually shows that complete graphs, complete bipartite graphs and some other graphs are $(1, 3)$-choosable. This paper shows that complete graphs, trees, cycles, generalized theta graphs, and some other graphs are $(2, 2)$-choosable. Moreover, complete bipartite graphs $K_{2,n}$ are $(1, 2)$-choosable and $K_{3,n}$ are $(2, 2)$-choosable.

## 2 A polynomial associated with total weighting

Assume $G = (V, E)$ is a simple graph, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. In the study of edge weight choosability of graphs, Bartnicki, Grytczuk and Niwczyk [7] considered a polynomial associated to $G$. Fix an arbitrary orientation $D$ of $G$ and denote by $E(D)$ the set of directed edges of $D$. Each edge $e$ is associated with a variable $x_e$. Let

$$P(G) = \prod_{e=uv \in E(D)} \left( \sum_{e' \in E(v)} x_{e'} - \sum_{e' \in E(u)} x_{e'} \right).$$

It follows from the definition that an edge weight assignment $f : E \to \mathbb{R}$ is an edge weighting of $G$ if and only if $P(G) \neq 0$ when evaluated at $x_e = f(e)$ for $e \in E(G)$.

In this paper, we consider total weighting, i.e., mappings $f$ that assigns weights to edges as well as vertices of $G$. So we need a modification of the polynomial. Each vertex $v$ of $G$ is associated with a variable $x_v$, and each edge $e$ of $G$ is associated with a variable $x_e$. Denote by $X$ the set of variables $x_{v_1}, x_{v_2}, \ldots, x_{v_n}, x_{e_1}, x_{e_2}, \ldots, x_{e_m}$. For each vertex $v$ of $G$, let $Q_v = \sum_{e \in E(v)} x_e + x_v$. Fix an arbitrary orientation $D$ of $G$. Let

$$\tilde{P}(G) = \tilde{P}(x_{v_1}, x_{v_2}, \ldots, x_{v_n}, x_{e_1}, x_{e_2}, \ldots, x_{e_m}) = \prod_{(u,v) \in E(D)} (Q_v - Q_u).$$

Note that for each edge $(u, v)$, $Q_v - Q_u$ is a polynomial of degree 1 with variable set $X$. Thus $\tilde{P}(G)$ is a polynomial of degree $m = |E(G)|$ with variable set $X$. For different orientations $D$ of $G$, the polynomials defined may differ by a sign, which is irrelevant for our purpose. For convenience, we use $\tilde{P}(G)$ to denote any polynomial defined through an arbitrary orientation of $G$.

Suppose $f$ is a proper total weighting of $G$. Recall that $\phi_f$ is the colouring of $G$ induced by $f$. By definition, for each vertex $u$,

$$\phi_f(u) = Q_u(f(v_1), f(v_2), \ldots, f(v_n), f(e_1), f(e_2), \ldots, f(e_m))$$

is equal to the evaluation of the polynomial $Q_u$ with $x_v = f(v)$ for each $v \in V$ and $x_e = f(e)$ for each edge $e \in V$. Therefore $f$ is a proper total weighting if and only if

$$\tilde{P}(f(v_1), f(v_2), \ldots, f(v_n), f(e_1), f(e_2), \ldots, f(e_m)) \neq 0.$$

We shall need the following theorem, called the Combinatorial Nullstellensatz [5, 6], in our proofs.

**Theorem 2.1.** *Let $F$ be a field and let $p(x_1, x_2, \ldots, x_n)$ be a polynomial in $F[x_1, x_2, \ldots, x_n]$. Suppose the degree of $p$ is equal to $\sum_{j=1}^{n} t_j$ and the coefficient of $\prod_{j=1}^{n} x_j^{t_j}$ in the expansion of $p$ is nonzero. Then for any subsets $S_1, S_2, \ldots, S_n$ of $F$ with $|S_j| = t_j + 1$, there exist $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that*

$$p(s_1, s_2, \ldots, s_n) \neq 0.$$

It follows from Combinatorial Nullstellensatz that if $k : V \cup E \to \{0, 1, \ldots\}$ is a mapping such that

- $\sum_{v \in V} k(v) + \sum_{e \in E} k(e) = m$,

- the coefficient of the monomial $\prod_{v\in V} x_v^{k(v)} \prod_{e\in E} x_e^{k(e)}$ in the expansion of $\tilde{P}(G)$ is nonzero,

then for any total list assignment $L$ with $|L(v)| \geq k(v)+1$ for all $v \in V$ and $|L(e)| \geq k(e)+1$ for all $e \in E$, $G$ has a $L$-total weighting.

To calculate the coefficient of the term $\prod_{v\in V} x_v^{k(v)} \prod_{e\in E} x_e^{k(e)}$, we need to calculate the permanent of some matrices.

For an $m \times m$ square matric $A = (a_{ij})$, the *permanent* of $A$ is defined by

$$\mathrm{per}(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)}$$

where $\sigma$ is taken over all the permutations of $\{1, 2, \ldots, m\}$.

Let $A$ be a matrix with $m$ rows and $s$ columns $A_1, A_2, \ldots, A_s$. Let $K = (k_1, k_2, \ldots, k_s)$ be a sequence of nonnegative integers with $k_1 + k_2 + \ldots + k_s = m$. Let $A(K)$ be the square matrix in which the $j$th column $A_j$ is repeated $k_j$ times (in case $k_j = 0$ then the column $A_j$ does not appear). The following lemma can be verified easily and is observed in [4].

**Lemma 2.2.** *Let $A$ be an $m\times s$ matrix. For the polynomial $P(x_1, x_2, \ldots, x_s) = \prod_{i=1}^{m} \sum_{j=1}^{s} a_{ij} x_j$, the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s}$ is equal to $\frac{\mathrm{per}(A(K))}{k_1! \cdots k_s!}$.*

For a graph $G$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = e_1, e_2, \ldots, e_m$ and with a fixed orientation, let $A_G$ be the $m \times (n+m)$ matrix with rows indexed by the edges of $G$ and columns indexed by vertices and edges of $G$ such that

$$\tilde{P}(G) = \prod_{e\in E} \left( \sum_{v\in V} a_{ev} x_v + \sum_{e'\in E} a_{ee'} x_{e'} \right).$$

For each vertex $v \in V$ (respectively, for each edge $e \in E$), we denote by $A_v$ (respectively, by $A_e$) the column of $A_G$ indexed by $v$ (respectively, indexed by $e$). So

$$A_G = [A_{v_1}, A_{v_2}, \ldots, A_{v_n}, A_{e_1}, A_{e_2}, \ldots, A_{e_m}].$$

The columns $A_v$ for $v \in V$ are called the vertex columns, and columns $A_e$ for $e \in E$ are called the edge columns. We denote by $B_G$ the submatrix of $A_G$ consisting the edge columns of $A_G$.

Given a matrix $A$, the *perrank* of $A$, denoted by perrank$(A)$, is the largest integer $k$ such that $A$ has a $k \times k$ submatrix with nonzero permanent. If $A$ is an $m \times s$-matrix, then perrank$(A) \leq m$. Let $A^{(k)} = [A, A, \ldots, A]$ be the matrix formed of $k$ copies of $A$. The *permanent index* of an $m \times s$ matrix $A$, denoted by $\text{pind}(A)$, is the minimum $k$ for which $\text{perrank}(A^{(k)}) = m$. If there is no such $k$, then let $\text{pind}(A) = \infty$.

The following lemma follows directly from Lemma 2.2 and Theorem 2.1.

**Lemma 2.3.** *Given two positive integers $k, k'$, let $A_G(k, k')$ be the matrix formed by the columns of $A_G$ in such a way that each vertex column appears $k$ times and each edge column appears $k'$ times. If the perrank of $A_G(k, k')$ is equal to $|E(G)|$, then $G$ is $(k + 1, k' + 1)$-choosable. In particular, if pind$(A_G) = 1$, then $G$ is $(2, 2)$-choosable. If pind$(B_G) = 1$, then $G$ is $(1, 2)$-choosable. If pind$(B_G) \leq 2$, then $G$ is $(1, 3)$-choosable.*

It was conjectured in [7] that for any graph with no isolated edges, $\text{pind}(B_G) \leq 2$. This conjecture implies that every graph with no isolated edges is $(1, 3)$-choosable. The following two conjectures are stronger than Conjecture 1.4 and the first half 1.5.

**Conjecture 2.4.** *There is a constant $k$ such that every graph $G$ has pind$(A_G) \leq k$.*

**Conjecture 2.5.** *Every graph $G$ has pind$(A_G) = 1$.*

## 3 Complete graphs

In this section, we prove that complete graphs $K_n$ are $(2, 2)$-choosable, by showing that $\text{pind}(A_{K_n}) = 1$.

For a complete graph $K_n$, the vertices are $v_1, v_2, \ldots, v_n$, and the edges are $e_{ij} = v_i v_j$ for $n \geq i > j \geq 1$. For $i > j$, we orient the edge $e_{ij}$ from $v_i$ to $v_j$. Let $\psi(k)$ be the function defined as $\psi(1) = 1$ and $\psi(k) = \psi(k-1)k!$ for $k \geq 2$.

**Lemma 3.1.** *Let $Q_{K_n}$ be obtained from $A_{K_n}$ by deleting the following columns*

$$A_{v_1}, A_{e_{21}}, A_{e_{32}}, \ldots, A_{e_{i(i-1)}}, \ldots, A_{e_{n(n-1)}}.$$

*Then $|per(Q_{K_n})| = \psi(n-1)$.*

As a consequence of this lemma and Lemmas 2.2 and 2.3, we have the following theorem, which is a main result of this paper.

**Theorem 3.2.** *Complete graphs are $(2, 2)$-choosable.*

We shall prove Lemma 3.1 by induction on $n \geq 2$. The case $n = 2$ is trivial. For $K_3$,

$$A_G = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{bmatrix}.$$

Remove the columns $A_{v_1}, A_{e_{21}}, A_{e_{32}}$ from $A_{K_3}$, we obtain the matrix

$$Q_{K_3} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}.$$

It is easy to verify that $|\mathrm{per}(Q_{K_3})| = 2 = \psi(2)$. (The case $n = 3$ need not be considered separately. We do it explicitly to illustrating the lemma). It remains to show that if $n \geq 4$ and Lemma 3.1 holds for $n - 1$, it also holds for $n$. Before the proof of Lemma 3.1, we shall prove some other lemmas.

**Lemma 3.3.** *If $e' = uv$ is an edge of $G$, then $A_{e'} = A_u + A_v$.*

*Proof.* It suffices to show that for any edge $e$, $a_{ee'} = a_{eu} + a_{ev}$. This follows from the following facts:

$$a_{eu} = \begin{cases} 1, \text{ if } u \text{ is the head of } e, \\ -1, \text{ if } u \text{ is the tail of } e, \\ 0, \text{ otherwise.} \end{cases}$$

$$a_{ee'} = \begin{cases} 1, \text{ if } e' \neq e \text{ is incident to the head of } e, \\ -1, \text{ if } e' \neq e \text{ is incident to the tail of } e, \\ 0, \text{ otherwise.} \end{cases}$$

Indeed, if $e$ is not incident to any of $u$ or $v$, then $a_{ee'} = 0, a_{eu} = a_{ev} = 0$. If $e \neq e'$ and $u$ or $v$ is the head of $e$, then $a_{ee'} = 1$ and $a_{eu} + a_{ev} = 1$. If $e \neq e'$ and $u$ or $v$ is the tail of $e$, then $a_{ee'} = -1$ and $a_{eu} + a_{ev} = -1$. If $e = e'$ then $a_{ee'} = 0$. Assume $e$ is oriented from $u$ to $v$. Then $a_{eu} = -1, a_{ev} = 1$, so $a_{eu} + a_{ev} = 0$. □

When we write the matrix $A_{K_n}$, we order the edges of $G$ as

$$e_{21}, e_{31}, e_{32}, e_{41}, e_{42}, e_{43}, \ldots, e_{n1}, e_{n2}, \ldots, e_{n(n-1)}.$$

Let $D$ be any square matrix obtained from $Q_{K_n}$ as follows: First delete the columns

$$A_{e_{n1}}, A_{e_{n2}}, \ldots, A_{e_{n(n-2)}}.$$

Then for some integer $0 \le s \le n-3$, add $s$ times the column $A_{v_n}$ and choose $n-2-s$ distinct columns from $A_{v_1}, A_{v_2}, \ldots, A_{v_{n-2}}$ and add them to $D$. Note that each of the columns $A_{v_2}, A_{v_3}, \ldots, A_{v_{n-2}}$ appeared in $Q_{K_n}$ once already. So if any of these columns is added to $D$, then it will appear in $D$ twice. However, $A_{v_1}$ is not contained in $Q_{K_n}$. So $A_{v_1}$ appears in $D$ at most once. Also observe that $A_{e_{n(n-1)}}$ is not contained in $Q_{K_n}$. So for any edge $e$ incident to $v_n$, $D$ does not contain the column $A_e$.

**Lemma 3.4.** *The matrix $D$ constructed above has zero permanent.*

*Proof.* By Lemma 2.2, the permanent of $D$ is the coefficient of the term

$$\prod_{v \in V} x_v^{k(v)} \prod_{e \in E} x_e^{k(e)}$$

where for each vertex $v$ (respectively, for each edge $e$), $k(v)$ (respectively, $k(e)$) is the number of times column $A_v$ (respectively, $A_e$) appears in $D$. By the construction of $D$, we know that

- $k(v_n) = s + 1$.
- $k(e) = 0$ for $e \in \{e_{21}, e_{32}, \ldots, e_{n(n-1)}, e_{n1}, e_{n2}, \ldots, e_{n(n-2)}\}$.
- $k(e) = 1$ for every other edge $e$.
- If $j \in \{2, 3, \ldots, n-2\}$ and $A_{v_j}$ is added to $D$ in the construction above (so $A_{v_j}$ appeared twice in $D$), then $k(v_j) = 2$.
- If $A_{v_1}$ is added to $D$ in the construction, then $k(v_1) = 1$. Otherwise $k(v_1) = 0$.
- $k(v) = 1$ for every other vertex $v$.

If $D$ has non-zero permanent, then by Theorem 2.1, $K_n$ has a $L$-total weighting for any total list assignment $L$ for which $|L(v)| = k(v)+1$ for each vertex $v$ and $|L(e)| = k(e)+1$ for each edge $e$. Let $L$ be the total list assignment defined as $L(v) = \{0, 1, \ldots, k(v)\}$ for each vertex $v$ and $L(e) = \{0, 1, \ldots, k(e)\}$ for each edge $e$.

Let $f$ be an $L$-total weighting of $K_n$. Consider the induced colouring $\phi_f$ of $K_n$. If $v = v_1$, or $v = v_{n-1}$, then $v$ is incident to two edges whose weights are 0: For $v_1$, the two edges are $e_{21}, e_{n1}$. For $v_{n-1}$, the two edges are $e_{(n-1)(n-2)}, e_{n(n-1)}$. Each other edge incident to $v$ has weight 0 or 1, and vertex $v$ itself has weight 0 or 1. Therefore $\phi_f(v) \in \{0, 1, \dots, n-2\}$. If $v = v_j$ for some $2 \le j \le n-2$, then $v$ is incident to three edges $v_{j(j-1)}, v_{(j+1)j}, v_{nj}$, of weight 0. The other edges have weights 0 or 1, and $v_j$ itself has weight $0, 1$ or 2. So again $\phi_f(v) \in \{0, 1, \dots, n-2\}$. All the edges incident to $v_n$ have weight 0, and $v_n$ itself has weight in $\{0, 1, \dots, s+1\}$. As $s \le n-3$, so $\phi_f(v_n) \in \{0, 1, \dots, n-2\}$. Therefore $\phi_f$ is not a proper colouring of $K_n$, in contrary to the assumption that $f$ is a $L$-total weighting of $K_n$. $\square$

The following lemma follows from the definition of permanent.

**Lemma 3.5.** *Assume $C = [A_1 + B_1, A_2, \dots, A_m]$ is a square matrix, where $A_i, B_1$ are column vectors of dimension $m$. Let $A = [A_1, A_2, \dots, A_m]$ and $B = [B_1, A_2, \dots, A_m]$. Then*

$$\text{per}(C) = \text{per}(A) + \text{per}(B).$$

Now we are ready to prove Lemma 3.1. For convenience, we move the column $A_{v_n}$ from its current position to the right most position (i.e., after the move, $A_{v_n}$ is the last column of $Q_{K_n}$. The order is not important, the moving is just for convenience). Let $m = n(n-1)/2$ and $m' = (n-1)(n-2)/2 = m - n + 1$. Then $Q_{K_n}$ is an $m \times m$ matrix, and after moving $A_{v_n}$ to the last column, the square submatrix of $Q_{K_n}$ induced by the first $m'$ row and first $m'$ columns is equal to $Q_{K_{n-1}}$. The last $n-1$ columns of $Q_{K_n}$ (after moving $A_{v_n}$ to the last column) are $A_{e_{n1}}, A_{e_{n2}}, \dots, A_{e_{n(n-2)}}, A_{v_n}$. By Lemma 3.3, $A_{e_{nj}} = A_{v_n} + A_{v_j}$. We write each of the $n-2$ columns $A_{e_{n1}}, A_{e_{n2}}, \dots, A_{e_{n(n-2)}}$ of $Q_{K_n}$ as the sum of two vertex columns.

By Lemma 3.5, $\text{per}(Q_{K_n})$ is the summation of a set of permanents $\text{per}(D)$, where each $D$ is obtained from $Q_{K_n}$ by choosing, for each of the $n-2$ columns that are of the form $A_{v_n} + A_{v_j}$, one of the two columns $A_{v_n}, A_{v_j}$. Altogether there are $2^{n-2}$ such $D$'s. One of the choice is that all the last $n-1$ columns are $A_{v_n}$. For this choice, the last $n-1$ columns has an all 1 square submatrix of order $(n-1) \times (n-1)$ at the bottom, and all the other entries are 0. Therefore for this choice, the permanent $D$ is equal to $\text{per}(Q_{K_{n-1}})(n-1)!$. For all the other choices, it follows from Lemma 3.4 that the permanent of $D$ is 0. Therefore $|\text{per}(Q_{K_n})| = \psi(n-1)$. This completes the proof of Lemma 3.1.

**Corollary 3.6.** *Suppose $G$ is obtained from $K_n$ by deleting a set $X$ of edges that can be extended to a Hamilton path of $K_n$. Then $pind(A_G) = 1$.*

*Proof.* Since $X$ is a set of edges that can be extended to a Hamilton path of $K_n$, we may assume, without loss of generality, that $X \subseteq \{e_{21}, e_{32}, \ldots, e_{n(n-1)}\}$. Observe that $A_G$ is obtained from $A_{K_n}$ by removing those columns and row corresponding to edges in $X$. Let $D$ be the matrix formed by those columns of $A_G$ corresponding to the columns of $Q_{K_n}$. Then $D$ can be obtained from $Q_{K_n}$ by removing the rows corresponding to edges in $X$. Assume $|X| = s$, and let $m = \frac{n(n-1)}{2} - s$ be the number of edges of $G$. Denote by $Q_X$ the $s$ row matrix formed by the $s$ rows of $Q_{K_n}$ corresponding to the edges in $X$. The permanent of $Q_{K_n}$ can be calculated as the sum $\sum \text{per}(S)\text{per}(R)$, where $S$ ranges over all matrices formed by $s$ columns of $Q_X$, and $R$ is the complement matrix of $S$ (i.e., $R$ is obtained from $Q_{K_n}$ by deleting the columns and row of $S$. Since $\text{per}(Q_{K_n})$ is nonzero, one of the matrices $R$ has nonzero permanent. That $R$ is a required $m \times m$ square submatrix of $A_G$. So $\text{pind}(A_G) = 1$. $\square$

# 4 Bipartite graphs

Suppose $G = (X \cup Y, E)$ is a bipartite graph. In writing the polynomial $A_G$, we assume that the edges of $G$ are oriented from $Y$ to $X$. We denote by $R_e$ the row of $B_G$ corresponding to edge $e$. The entry at row $R_e$ and column $A_{e'}$ is denoted by $a_{e,e'}$. So $a_{e\tilde{e}} = 1$ if $e, \tilde{e}$ have a common end vertex in $X$, $a_{e\tilde{e}} = -1$ if $e, \tilde{e}$ have a common end vertex in $Y$, and $a_{e\tilde{e}} = 0$ otherwise.

The matrix $B_G$ defines a bipartite graph $H$ with bipartite sets $S, T$ as follows: $S = \{s_e : e \in E(G)\}$ and $T = \{t_e : e \in E(G)\}$. If $a_{e\tilde{e}} \neq 0$ then there is an edge between $s_e$ and $t_{\tilde{e}}$. For a perfect matching $M$ of $H$, let $\sigma(M) = \prod_{s_e t_{\tilde{e}} \in M} a_{e\tilde{e}}$. By definition of permanent, $\text{per}(B_G) = \sum \sigma(M)$, where the summation ranges over all perfect matchings of $H$.

Identify $s_e$ with $t_e$ in $H$, we obtain the line graph $L(G)$ of $G$ (parallel edges are replaced by a single edge). We call a spanning subgraph $F$ of $L(G)$ a semi-2-factor of $L(G)$ if each component of $F$ is either a cycle or an edge. If each cycle in $F$ is assigned a direction to form a directed cycle, then $F$ is called a directed semi-2-factor. A perfect matching $M$ in $H$

corresponds to a directed semi-2-factor in $L(G)$. As each cycle in a semi-2-factor can be assigned two direction, if $F$ is semi-2-factor with $q$ cycles then it corresponds to $2^q$ perfect matchings in $H$. For each semi-2-factor $F$ of $L(G)$, let $m(F)$ be the number of cycles in $F$ and let $\sigma(F) = \prod_{e\tilde{e} \in F} a_{e\tilde{e}}$. Then $\text{per}(B_G) = \sum \sigma(F) 2^{m(F)}$.

If $G = K_{n,m}$ is a complete bipartite graph, then $L(G)$ is the Cartesian product $K_n \Box K_m$ of $K_n$ and $K_m$. We draw the graph $K_n \Box K_m$ in the shape of rectangle, with $m$ horizontal copies of $K_n$, and $n$ vertical copies of $K_m$. So each edge of $K_n \Box K_m$ is either a horizontal edge or a vertical edge. For a semi-2-factor $F$ of $L(G) = K_n \Box K_m$, $\sigma(F)$ is equal to 1 or $-1$, depending on the number of horizontal edges is even or odd.

**Lemma 4.1.** *For any positive integer $m$, per$(B_{K_{2,m}}) \neq 0$. As a consequence, $K_{2,m}$ is $(1,2)$-choosable.*

*Proof.* If $G = K_{2,m}$, the $m$ horizontal edge of $K_2 \Box K_m$ form a cut of $K_2 \Box K_m$. By the discussion above, $\text{per}(B_G) = \sum \sigma(F) 2^{m(F)}$. Since each cycle of a semi 2-factor contains an even number of edges of any cut, so $F$ contains an even number of horizontal edges of $K_2 \Box K_m$. Therefore $\sigma(F) = 1$ for all semi 2-factor $F$ of $L(K_{2,m})$, and hence *per*$(B_{K_{2,m}}) \neq 0$. □

**Lemma 4.2.** *Assume $G$ is a graph, $x_1, x_2, \ldots, x_k$ is a sequence of vertices of $G$ and $y_i \in N_G[x_i]$ for $i = 1, 2, \ldots, k$. Assume the following hold:*

- *For $i = 2, 3, \ldots, k$, $N_G[x_i] \cap (\cup_{j=1}^{i-1} N_G[x_j]) \subseteq \{y_1, y_2, \ldots, y_i\}$.*
- *For any $1 \leq i \leq k$, if a vertex $z$ is contained in $t$ of the sets $N_G[x_1], N_G[x_2], \ldots, N_G[x_i]$, then there are at least $t-1$ indices $j \leq i$ such that $y_j = z$.*
- *$B_{G-\{x_1, x_2, \ldots, x_k\}}$ has nonzero permanent.*

*Then pind$(A_G) = 1$.*

*Proof.* For $i = 0, 1, \ldots, k$, let $G_i = G - \{x_{i+1}, x_{i+2}, \ldots, x_k\}$. We shall prove by induction on $i$ that there is a square matrix $B_i$ with nonzero permanent formed by some edge columns of $A_{G_i}$ and some vertex columns (each edge column and each vertex column appears in $B_i$ at most once), and the vertex columns are contained in the set $\{A_v : v \in \cup_{j=1}^{i} N_G[x_i]\}$. Moreover, if $z$ is contained in $t$ of the sets $N_G[x_i]$ and there are at least $t$ indices $j \leq i$ with $y_j = z$, then $A_z$ is not a column of $B_i$.

If $i = 0$, then this follows from our assumption that $B_{G_0}$ has nonzero permanent. Assume $i \geq 1$ and the statement holds for $i-1$. The graph $G_i$, which is obtained from $G_{i-1}$ by adding vertex $x_i$ and edges incident to $x_i$ in $G_i$. Assume the edges in $G_i$ incident to $x_i$ are $e_1 = x_i u_1, e_2 = x_i u_2, \ldots, e_s = x_i u_s$.

Since the statement holds for $i-1$, there is a square matrix $B_{i-1}$ formed by some edge columns and vertex columns of $A_{G_{i-1}}$. Moreover, the assumption implies that the vertex columns $A_{u_1}, A_{u_2}, \ldots, A_{u_s}$ are not contained in $B_{i-1}$, except that in case $y_i = u_j$, then $A_{u_j}$ may be contained in $B_{i-1}$. The vertex columns in $B_{i-1}$ do not contain the columns corresponding to the neighbours of $x_i$ in $G_{i-1}$. Let $D$ be the submatrix of $A_{G_i}$ formed by the columns corresponding to those columns in $B_{i-1}$. The matrix $D$ can also be viewed as obtained from $B_{i-1}$ by adding $s$ rows (at the bottom) corresponding to those edges in $G_i$ incident to $x_i$.

By Lemma 3.5, for $j = 1, 2, \ldots, s$, $A_{e_j} - A_{u_j} = A_{x_i}$. Let $D'$ be the matrix obtained from $D$ by adding $s$ copies of the columns $A_{x_i}$. Since $A_{x_i}$ has 1's in the last $s$ rows and 0's in the other rows, we conclude that $per(D') = s! per(B_{i-1}) \neq 0$. If $y_i = x_i$, then we replace the $s$ copies of $A_{x_i}$ by the columns $A_{e_j} - A_{u_j}$ for $j = 1, 2, \ldots, s$. If $y_i = u_t$, then we replace $s-1$ copies of $A_{x_i}$ by the columns $A_{e_j} - A_{u_j}$ for $j = 1, 2, \ldots, s, j \neq t$. Let $Z = \{e_j : j = 1, 2, \ldots, s\} \cup \{u_j : j = 1, 2, \ldots, s\} - \{y_i\}$. By Lemma 3.5, there is a matrix $B_i$ with nonzero permanent which is obtained from $D$ by adding $s$ distinct columns from $\{A_y : y \in Z\}$. This completes the proof of the lemma. □

**Corollary 4.3.** *If $G$ is a graph for which $B_G$ has nonzero permanent, and $G'$ is obtained from $G$ by adding a set $\{x_1, x_2, \ldots, x_k\}$ of vertices and some edges so that $N(x_i) \cap N(x_j) = \emptyset$ if $i \neq j$, and for each $i \geq 2$, there is at most one index $j < i$ such that $x_i$ is adjacent to $x_j$, then pind$(A_{G'}) = 1$.*

*Proof.* Arbitrarily choose $y_1$. For $i \geq 2$, if there is an index $j$ such that $x_i$ is adjacent to $x_j$, then let $y_i = x_j$. Otherwise, arbitrarily choose $y_i$. Then the conclusion follows from Lemma 4.2. □

Since $B_{K_{2,m}}$ has nonzero permanent, any graph $G$ obtained from $K_{2,m}$ by adding one vertex and some edges has pind$(A_G) = 1$. In particular, we have the following corollary.

**Corollary 4.4.** *For any positive integer $m$, pind$(A_{K_{3,m}}) = 1$. Hence $K_{3,m}$ is $(2,2)$-choosable.*

The question whether every complete bipartite graph is $(2,2)$-choosable remains open.

# 5 Construction of $(2,2)$-choosable graphs

This section proves that some other graphs, including trees, cycles, generalized theta graphs are $(2,2)$-choosable.

**Lemma 5.1.** *Suppose $G$ is obtained from $G'$ by adding one vertex $v$ and one edge $e^* = uv$, where $u$ is a vertex of $G'$. If pind$(A_{G'}) = 1$, then pind$(A_G) = 1$. If $G'$ is $(2,2)$-choosable, then $G$ is $(2,2)$-choosable.*

*Proof.* Assume $A_{G'}$ is an $m \times (m+n)$ matrix. Then $A_G$ is obtained from $A_{G'}$ by adding one row and two columns. Let $B'$ be an $m \times m$ submatrix of $A_{G'}$ with nonzero permanent. Let $B$ be the submtrix of $A_G$ obtained from $B'$ by adding the last row (the row corresponding to the edge $e^*$) and the column $A_v$. Note that the column $A_v$ has 1 in the last row and 0 in other rows. Therefore $per(B) = per(B') \neq 0$.

Assume $G'$ is $(2,2)$-choosable and $L$ is a $(2,2)$-total-list assignment of $G$. Let $L'$ be the $(2,2)$-total list assignment of $G'$ defined as $L'(y) = L(y)$ if $y \in V(G') \cup E(G')$ and $y \neq u$. Arbitrarily choose $w \in L(e^*)$ and let $L'(u) = w + L(u)$. By assumption, $G'$ has a proper $L'$-total weighting $f$. By let $f(e^*) = w$ and by choosing $f(v) \in L(v)$ so that $\phi_f(v) \neq \phi_f(u)$, we obtain a proper $L$-total weighting of $G$. □

A *thread* in a graph $G$ is a path $P = (v_1, v_2, \ldots, v_k)$ in $G$ such that $d_G(v_i) = 2$ for $i = 2, 3, \ldots, k-1$. The vertices $v_1, v_k$ need not be distinct. If we need to specify the two end vertices of a thread, then we say $P$ is a $v_1$-$v_k$-thread. By deleting the thread $P = (v_1, v_2, \ldots, v_k)$ from $G$, we mean delete the vertices $v_2, v_3, \ldots, v_{k-1}$ (and hence edges incident to them).

**Lemma 5.2.** *Assume $G$ is a graph and $P_1 = (x, u, y), P_2 = (x, v, y)$ are two $x$-$y$-threads. Let $G'$ be obtained from $G$ by deleting the two threads $P_1, P_2$. If pind$(A_{G'}) = 1$, then pind$(A_G) = 1$.*

*Proof.* Assume $A_{G'}$ is an $m \times (m+n)$ matrix. Let $B'$ be an $m \times m$ submatrix of $A_{G'}$ with per$(B') \neq 0$. Let $e_1 = xu, e_2 = xv, e_3 = yu, e_4 = yv$.

Orient the edges $e_1, e_2, e_3, e_4$ from $u, v$ to $x, y$. Let $B''$ be obtained from $B'$ by adding the 4 rows corresponding to added edges and the following 4 columns: $A_u, A_v, A_{e_1} - A_{e_2}, A_{e_3} - A_{e_4}$. The added 4 columns have zero entries everywhere except those entries at the added 4 rows. Let $C$ be the $4 \times 4$ submatrix of $B''$ consisting of the added rows and columns. Then $\mathrm{per}(B'') = \mathrm{per}(B') \times \mathrm{per}(C)$.

Since

$$C = \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix},$$

which has nonzero permanent, we conclude that $\mathrm{per}(B'') \neq 0$. By Lemma 3.3, $\mathrm{per}(B'')$ is a linear combination of the permanents of matrices obtained from $B''$ by replacing the row $A_{e_1} - A_{e_2}$ with either $A_{e_1}$ or $A_{e_2}$, and replacing the row $A_{e_3} - A_{e_4}$ with either $A_{e_3}$ or $A_{e_4}$. Since the linear combination is nonzero, there exists $i \in \{1, 2\}$ and $j \in \{3, 4\}$ such that the matrix obtained from $B''$ by replacing $A_{e_1} - A_{e_2}$ with $A_{e_i}$ and replacing $A_{e_3} - A_{e_4}$ with $A_{e_j}$ has nonzero permanent. Such a matrix $B$ is a submatrix of $A_G$. So $\mathrm{pind}(A_G) = 1$. □

It is unknown if the following is true: If $G'$ is obtained from $G$ by deleting a pair of $x$-$y$-threads of length 2, and $G'$ is $(2, 2)$-choosable, then $G$ is also $(2, 2)$-choosable.

Given a total list assignment $L$ of $G = (V, E)$, to find a proper $L$-total weighting $f$ of $G$, we need to determine $f(y)$ for all $y \in V \cup E$. In the following, we shall determine the weighting on edges first. Assume $f(e)$ are determined for all edges $e$. For each vertex $v$, let $L_f(v) = \{w + \sum_{e \in E(v)} f(e) : w \in L(v)\}$. Then $L_f$ is a vertex list assignment. It is easy to see that $f$ can be extended to a $L$-total weighting of $G$ (by determining $f(x)$ for each vertex $x \in V(G)$) if and only if there is an $L_f$-colouring of $G$.

Suppose $s$ is a real number and $A$ is a set of real numbers. Then the set $s + A$ is defined as $s + A = \{s + a : a \in A\}$. If $A, B$ are sets of real numbers, then $A + B = \{a + b : a \in A, b \in B\}$.

**Lemma 5.3.** *Suppose $G'$ is obtained from a graph $G$ by deleting a thread $P = (v_0, v_1, v_2, v_3)$ of length 3. If $G'$ is $(2, 2)$-choosable, then $G$ is $(2, 2)$-choosable.*

*Proof.* Suppose $L$ is a $(2,2)$-total-weight-list assignment of $G$. Let $e_0 = v_0v_1, e_1 = v_1v_2, e_2 = v_2v_3$. We need to find an $L$-total weighting of $G$. Let $e_0 = v_0v_1, e_1 = v_1v_2, e_2 = v_2v_3$. Choose $w_0 \in L(e_0), w_2 \in L(e_2)$ so that $w_0 + L(v_1) \neq w_2 + L(v_2)$. (This can be done as there are two different choices for each of $w_0, w_2$).

Let $L'$ be the $(2,2)$-total-list assignment of $G'$ defined as follows: $L'(y) = L(y)$ if $y \neq v_0, v_3$, and let $L'(v_0) = w_0 + L(v_0)$ and $L'(v_3) = w_2 + L(v_3)$ (in case $v_0 = v_3$, then let $L'(v_0) = w_0 + w_2 + L(v_0)$). By assumption, $G'$ has a proper $L'$-total-weighting $f$. Let $\phi_f$ be the induced colouring of $G'$.

Assume $L(e_1) = \{a, b\}$ and $a < b$. Let $f_1$ be the extension of $f$ to the edges $e_0, e_1, e_2$ so that $f(e_0) = w_0, f(e_2) = w_2$ and $f(e_1) = a$. Let $f_2$ be the extension of $f$ to the edges $e_0, e_1, e_2$ so that $f(e_0) = w_0, f(e_2) = w_2$ and $f(e_1) = b$. We shall show that one of $f_1, f_2$ can be extended to a proper $L$-total weighting of $G$. This is equivalent to say that either $\phi_f$ can be extended to an $L_{f_1}$-colouring of $G$ or an $L_{f_2}$-colouring of $G$.

For $i = 1, 2$, assume $L_{f_1}(v_i) = \{a_i, b_i\}$, where $a_i < b_i$. Let $t = b - a > 0$. Then $L_{f_2}(v_i) = \{a_i + t, b_i + t\}$. If $\phi_f(v_0) \notin L_{f_i}(v_1)$ for some $i = 1, 2$, then $\phi_f$ can be easily extended to an $L_{f_i}$-colouring of $G$: first choose $\phi_f(v_2) \in L_{f_i}(v_2)$ so that $\phi_f(v_2) \neq \phi_f(v_3)$ (as $|L_{f_i}(v_2)| = 2$, there is such a choice). Then choose $\phi_f(v_1) \in L_{f_i}(v_1)$ so that $\phi_f(v_1) \neq \phi_f(v_2)$. Since $\phi_f(v_0) \notin L_{f_i}(v_1)$, we automatically have $\phi_f(v_0) \neq \phi_f(v_1)$. Thus we assume that $\phi_f(v_0) \in L_{f_i}(v_1)$ for $i = 1, 2$. Similarly we assume that $\phi_f(v_3) \in L_{f_i}(v_2)$ for $i = 1, 2$. Thus $\phi_f(v_0) \in \{a_1, b_1\} \cap \{a_1 + t, b_1 + t\}$ and $\phi_f(v_3) \in \{a_2, b_2\} \cap \{a_2 + t, b_2 + t\}$. Since $a_i < b_i$, it must be the case that $\phi_f(v_0) = a_1 + t = b_1$ and $\phi_f(v_3) = a_2 + t = b_2$. Since $\{a_1, b_1\} \neq \{a_2, b_2\}$, we conclude that $a_1 \neq a_2$. Now $\phi_f$ can be extended to an $L_{f_1}$-colouring of $G$ by letting $\phi_f(v_1) = a_1$ and $\phi_f(v_2) = a_2$. □

It is unknown if the following is true: Assume $G'$ is obtained from $G$ by deleting a thread of length 3. If $\text{pind}(A_{G'}) = 1$, then $\text{pind}(A_G) = 1$. However, if $G'$ is obtained from $G$ by deleting a thread of length at least 4 and $\text{pind}(A_{G'}) = 1$, then it can be proved that $\text{pind}(A_G) = 1$.

**Corollary 5.4.** *Assume $G'$ is obtained from $G$ by repeatedly deleting pairs of $x$-$y$-threads of length 2, threads of lengths at least 3, and leaves. If pind$(A_{G'}) = 1$, then $G$ is $(2,2)$-choosable. In particular, if $G$ has only one cycle, or $G$ is has at most two vertices of degree greater than 2, or $G$ is obtained from another graph $H$ by subdividing each edge with two vertices, then $G$ is $(2,2)$-choosable.*

## 6 $(1,3)$-choosable graphs

By Lemma 2.3, if $\text{pind}(B_G) \le k$, then $G$ is $(1, k+1)$-choosable. It was shown in [7] that if $\text{pind}(B_{G'}) \le 2$, then $\text{pind}(B_G) \le 2$, if one of the following holds:

- $G$ is obtained from $G'$ by adding two vertices $x, y$, and $N_G(x) - \{y\} = N_G(y) - \{x\}$.
- $G$ is obtained from $G'$ by adding two vertices $x, y$ and two edges $xy, yz$, where $z \in V(G')$.
- $G$ is obtained from $G'$ by adding three vertices $x, y, z$ and edges $ux, xy, yz, zv$, where $u, v$ are (not necessarily distinct) vertices of $G'$.

It follows from this result that complete graphs, complete bipartite graphs and trees (other than $K_2$) have $\text{pind}(B_G) \le 2$ (and hence are $(1,3)$-choosable).

The following simple result shows that there are some advantages by considering $(1,3)$-total weight choosability instead of 3-edge weight choosability.

**Lemma 6.1.** *Suppose $G'$ is obtained from $G$ by deleting a thread of length 3. If $G'$ is $(1,3)$-choosable, then $G$ is $(1,3)$-choosable.*

*Proof.* Assume $P = (v_0, v_1, v_2, v_3)$ is a thread in $G$ and $G'$ is obtained from $G$ by deleting this thread. Let $e_i = v_i v_{i+1}$ for $i = 0, 1, 2$ be the added edges. Let $L$ be a $(1,3)$-total list assignment of $G$. We shall find an $L$-total weighting of $G$. For each vertex $v$ of $G$, let $f(v)$ be the unique permissible weight of $v$, i.e., $L(v) = \{f(v)\}$.

Arbitrarily choose a weight $f(e_0) \in L(e_0)$ and a weight $f(e_2) \in L(e_2)$ so that $f(e_0) + f(v_1) \ne f(e_2) + f(v_2)$ (such a choice is certainly possible).

Let $\tilde{L}$ be the $(1,3)$-total list assignment of $G'$ defined as follows: $\tilde{L}(y) = L(y)$ if $y \in E(G) \cup V(G)$ and $y \ne v_0, v_3$. Let $\tilde{L}(v_0) = \{f(v_0) + f(e_0)\}$ and $\tilde{L}(v_3) = \{f(v_3) + f(e_2)\}$. If $v_0 = v_3$, then $\tilde{L}(v_0) = \{f(v_0) + f(e_0) + f(e_2)\}$. Since $G'$ is $(1,3)$-choosable, there is an $\tilde{L}$-weighting $f$ of $G'$. Assume $\phi_f(v_0) = \alpha$ and $\phi_f(v_3) = \beta$.

To extend $f$ to a proper total weighting of $G$, it remains to determine $f(e_1)$ (as each vertex has only one permissible weight, so we do not need to choose the weight for vertices). For the extend $f$ to be a weighting of $G$, it suffices to choose $f(e_1)$ so that $\phi_f(v_1) \ne \alpha$ and $\phi_f(v_2) \ne \beta$.

Now the edge $e_1$ has three permissible weights, say $L(e_1) = \{a, b, c\}$ where $a < b < c$. By letting $f(e_1) = a, b, c$, respectively, $\phi_f(v_1)$ takes values $\alpha_1 < \alpha_2 < \alpha_3$, respectively, and $\phi_f(v_2)$ takes values $\beta_1 < \beta_2 < \beta_3$, respectively. At most one of the three choices makes $\phi_f(v_1) = \alpha$ and at most one of the three choices makes $\phi_f(v_2) = \beta$. Therefore there is a choice of $f(e_1)$ so that $\phi_f(v_1) \neq \alpha$ and $\phi_f(v_2) \neq \beta$. □

The argument above can be easily modified to show that if $G'$ is $(1,3)$-choosable and $G$ is obtained from $G'$ by adding a path of length at least 3 connecting two vertices of $G'$, then $G$ is $(1,3)$-choosable.

A *hanging edge* of a graph $G$ is an edge $uv$ of $G$ such that $d_G(v) = 1$ and $d_G(u) = 2$ or 3. A *pair of leaves* is a pair of vertices $u, v$ such that $d_G(u) = d_G(v) = 1$ and $u, v$ have the same neighbour.

**Lemma 6.2.** *Assume $G'$ is obtained from $G$ by deleting the vertices of a hanging edge. If pind$(B_{G'}) \leq 2$, then pind$(B_G) \leq 2$.*

*Proof.* Assume $e = uv$ is a hanging edge of $G$ (with $d_G(v) = 1$ and $d_G(u) = 2$ or 3) and $G' = G - \{u, v\}$. Let $B'$ be a square formed by columns of $B_{G'}$ with $\text{per}(B') \neq 0$. If $d_G(u) = 2$, then let $w$ be the other neighbour of $u$ and let $e' = uw$. Let $B$ be the square matrix formed by columns of $B_G$ which is obtained from $B'$ by adding the two rows corresponding to the added edges $e', e$, and two columns $A_e$ and $A_{e'}$. It can be easily verified that $|\text{per}(B)| = |\text{per}(B')| \neq 0$. If $d_G(u) = 3$, then let $w, w'$ be the two other neighbours of $u$ and let $e_1 = uw, e_2 = uw'$. Let $B$ be obtained from $B'$ by adding three rows corresponding to the three added edges, and three columns $A_e, A_e, A_{e_1}$. It can be easily checked that $|\text{per}(B)| = 2|\text{per}(B')|$. □

It follows from the above mentioned results in [7] that if $G'$ is obtained from $G$ by deleting a pair of leaves or by deleting a pair of $x$-$y$-threads of length 2, and $\text{pind}(B_{G'}) \leq 2$, then $\text{pind}(B_G) \leq 2$. Thus we have the following corollary.

**Corollary 6.3.** *Assume $G'$ is obtained from $G$ by repeatedly deleting pairs of $x$-$y$-threads of length 2, threads of lengths at least 3, vertices of hanging edges and pairs of leaves. If pind$(B_{G'}) \leq 2$, then $G$ is $(1,3)$-choosable. In particular, if $G$ has only one cycle, or $G$ has at most two vertices of degree greater than 2, or $G$ is obtained from another graph $H$ by subdividing each edge with two vertices, then $G$ is $(1,3)$-choosable.*

*Proof.* If $G$ has only one cycle, or has at most two vertices of degree greater than 2, or is obtained from another graph $H$ by subdividing each edge with two vertices, then by repeatedly deleting pairs of $x$-$y$-threads of length 2, threads of lengths at least 3, vertices of hanging edges and pairs of leaves, we can get a graph $G'$ which is either a tree with at least two edges, or a $K_3$, or a $K_3$ plus a leaf vertex, or a $C_4$, or a $K_4$ minus one edge. Each of these graphs $G'$ has $\text{pind}(B_G) \le 2$. Hence $G$ is $(1, 3)$-choosable. $\square$

## Acknowledgement

We thank Daqing Yang for writing a program for us to calculate the permanent matrices.

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