Sparse H-colourable graphs of bounded maximum degree

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Jan, 2001

Abstract

Let F be a graph of order at most k. We prove that for any integer g there is a graph G of girth at least g and of maximum degree at most $5k^{13}$ such that G admits a surjective homomorphism c to F, and moreover, for any F-pointed graph H with at most k vertices, and for any homomorphism h from G to H there is a unique homomorphism f from F to H such that $h = f \circ c$. As a consequence, we prove that if H is a projective graph of order k, then for any finite family $\mathcal{F}$ of prescribed mappings from a set X to $V(H)$ (with $|\mathcal{F}| = t$), there is a graph G of arbitrary large girth and of maximum degree at most $5k^{26mt}$ (where $m = |X|$) such that $X \subseteq V(G)$ and up to an automorphism of H, there are exactly t homomorphisms from G to H, each of which is an extension of an $f \in \mathcal{F}$.

Keywords: Uniquely H-colourable graphs, girth, bounded maximum degree.

*Supported in part by the National Science Council under grant NSC89-2115-M-110-012.

1 Introduction

It was proved in 1959 [2] by Erdős that there exist graphs of arbitrary large girth and arbitrary large chromatic number. Since then, numerous generalizations of this landmark result have been published. Among these generalizations is the following result of Müller:

Theorem 1 *[4] Let k, g be positive integers, X a finite set, and $\mathcal{F}$ a finite family of k-colourings of X. Then there exists a graph $G = (V, E)$ with $X \subset V$ such that (i): G has girth $g(G) \geq g$; (ii): up to a permutation of colours, G has exactly $|\mathcal{F}|$ k-colourings, each of which is an extension of a member of $\mathcal{F}$.*

In particular, when $\mathcal{F}$ consists of a single colouring, the graph G is uniquely k-colourable, and hence has chromatic number k.

Suppose G and H are graphs. A *homomorphism* from G to H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. We say G is *H-colourable* if there exists a homomorphism from G to H. We say G is *uniquely H-colourable* if there is a surjective homomorphism h from G to H, and for any other homomorphism $f : G \to H$, there is an automorphism c of H such that $f = c \circ h$. It is easy to see that G is n-colourable if and only if G is K_n-colourable, and G is a uniquely n-colourable if and only if G is uniquely K_n-colourable. So H-colouring is a generalization of n-colouring. Nešetřil and Zhu [6] extracted the properties needed for Müller's result to hold, and defined the concept of G-pointed graphs. A graph H is said to be *G-pointed* if for any two distinct homomorphisms $g, g' : G \to H$ there are at least two vertices x, y such that $g(x) \neq g'(x)$ and $g(y) \neq g'(y)$. In other words, if two homomorphisms g, g' from G to H are identical on every vertex except one vertex x, then they must be identical on the vertex x. The following result was proved in [6].

Theorem 2 *[6] For every graph F and every choice of positive integers k and l there exists a graph G together with a surjective homomorphism $c : G \to F$ with the following properties:*

i. $g(G) > l$;

ii. For every graph H with at most k vertices, there exists a homomorphism $g : G \to H$ if and only if there exists a homomorphism $f : F \to H$.

iii. For every F-pointed graph H with at most k vertices and for every homomorphism $g : G \to H$ there exists a unique homomorphism $f : F \to H$ such that $g = f \circ c$.

For a graph $H = (V, E)$, the power H^k is a graph with vertex set V^k and in which $(x_1, x_2, \cdots, x_k)$ is adjacent to $(y_1, y_2, \cdots, y_k)$ if and only if x_i is adjacent to y_i for every $i \in \{1, 2, \cdots, k\}$. For each index i, the projection: $\pi_i : H^k \to H$ defined as $\pi_i(x_1, x_2, \cdots, x_k) = x_i$ is a homomorphism from H^k to H. A graph H is called *projective* if, up to an automorphism of H, the only homomorphisms from H^k to H are the projections. Theorem 2 implies the following:

Theorem 3 *[6] Suppose H is a projective graph, g is a positive integer and X is a finite set. Then for any finite family $\mathcal{F}$ of mappings from X to $V(H)$, there exists a graph $G = (V(G), E(G))$ with $X \subset V(G)$ such that (i): G has girth at least g; (ii): up to an automorphism of H, G has exactly $|\mathcal{F}|$ H-colourings, each of which is an extension of a member of $\mathcal{F}$.*

As the complete graphs K_n are projective, Theorem 3 is a generalization of Theorem 1. If $k \geq 2d$ are positive integers such that $(k, d) = 1$, then the *circular complete graph* $K_{k/d}$ is the graph with vertex set $\{0, 1, \cdots, k-1\}$ in which $i \sim j$ if and only if $d \leq |i - j| \leq k - d$. A graph G is said to be (k, d)-colourable if G admits a homomorphism to $K_{k/d}$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the infimum of those ratios k/d for which G is (k, d)-colourable. It is known [7, 8] that for every graph G, $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$. As the graphs $K_{k/d}$ are projective [3, 5, 6], Theorem 3 applies to the circular chromatic number of graphs as well.

Recently, it is proved in [1] that there exist uniquely k-colourable graphs G of large girth with maximum degree $\Delta(G) \leq 5k^{13}$. We note that uniquely k-colourable graphs of large girth were constructed in [4]. However, for the graphs constructed in [4], the maximum degree goes to infinity along with the number of vertices.

In this note, we show that the same result is true for H-colouring problems. We shall prove the following:

Theorem 4 *Let k and l be positive integers. For every graph F on at most k vertices there exists a graph G together with a surjective homomorphism $c : G \to F$ with the following properties:*

i. $g(G) > l$ and $\Delta(G) \leq 5k^{13}$;

ii. For every graph H with at most k vertices, there exists a homomorphism $g : G \to H$ if and only if there exists a homomorphism $f : F \to H$.

iii. For every F-pointed graph H with at most k vertices and for every homomorphism $g : G \to H$ there exists a unique homomorphism $f : F \to H$ such that $g = f \circ c$.

The following result is a consequence of Theorem 4:

Corollary 5 *Suppose H is a projective graph on k vertices, g is a positive integer, X is a set of size $m > 0$, and $\mathcal{F}$ is a family of mappings from X to $V(H)$ with $|\mathcal{F}| = t > 0$. Then there exists a graph $G = (V, E)$ with $X \subset V$ such that (i): G has girth at least g and maximum degree $\Delta(G) \leq 5k^{26mt}$; (ii): up to an automorphism of H, G has exactly t H-colourings, each of which is an extension of a member of $\mathcal{F}$.*

2 Proof of Theorem 4

The proof is by the probabilistic method. The construction of the random graph and the calculations are similar to that of [1]. Suppose F is a graph with k vertices, $V(F) = [k] = \{1, \ldots, k\}$. Given a positive integer n, let $G(n, F) = F[\overline{K}_n]$ be the lexicographic product of F and $\overline{K}_n$. In other words, $G(n, F)$ has vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$, where $|V_i| = n$ and $x \in V_i$ is adjacent to $y \in V_j$ if and only if ij is an edge of F.

Lemma 6 *If n is sufficiently large, then there exists a subgraph G of $G(n, F)$ with the following properties:*

1. *For any edge ij of F and any $U \subset V_i, W \subset V_j$, of size $|U| = \lceil \frac{k^{-3}n}{2} \rceil$ and $|W| = \lceil \frac{(k-1)n}{k} \rceil$, there are at least $\frac{k^7 n}{4}$ edges between U and W in G.*

2. *For any edge ij of F and any $U \subset V_i, W \subset V_j$, of size $|U| = |W| = \lceil \frac{n}{40k} \rceil$, there is at least one edge between U and W in G.*

3. *For any edge ij of F and any $U \subset V_i, W \subset V_j$, of size $k \leq |W| = k|U| \leq \frac{n}{40}$, there are less than $\frac{|U|k^{10}}{2}$ edges between U and W in G.*

4. *Let $g := \lfloor \frac{1}{11} \frac{logn}{logk} \rfloor$ and define $C := \{v : v$ is a vertex contained in a cycle in G of length at most $g-1\}$. Then $|C| \leq \frac{n}{4k}$.*

5. *Let $Y := \{v : v$ has degree in G larger than $5k^{13}\}$. Then $|Y| \leq \frac{n}{4k} - 1$.*

To prove Lemma 6, we construct a random subgraph G of $G(n, F)$ as follows: Let $p = k^{10}n^{-1}$. Put each edge of $G(n, F)$ into G independently with probability p. We shall prove that with a positive probability, a random graph G so constructed has all the properties listed in Lemma 6. The calculation of the probability are standard and similar to the proof in [1].

First we prove that if n is sufficiently large, then with probability at least $9/10$, a random graph G constructed above has property (3).

Let U and W be as in the statement of property (3) and let $q := |W| = k|U|$. For any set S of $\frac{qk^9}{2}$ edges between U and W, the probability that $S \subset E(G)$ is $p^{\frac{qk^9}{2}}$. There are $\binom{\frac{q^2}{k}}{\frac{qk^9}{2}}$ choices of the set S. So the probability that there are at least $\frac{|U|k^{10}}{2} = \frac{qk^9}{2}$ edges between U and W is at most:

$$\binom{\frac{q^2}{k}}{\frac{qk^9}{2}} p^{\frac{qk^9}{2}} \leq (\frac{2eq^2}{qk^{10}}.\frac{k^{10}}{n})^{\frac{qk^9}{2}} = (\frac{2eq}{n})^{\frac{qk^9}{2}}.$$

Summing over all possible choices for U and W, we conclude that the probability that G does not fulfill the statement of property 3 is at most:

$$\begin{aligned} \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} k^2 \binom{n}{q} \binom{n}{\frac{q}{k}} (\frac{2eq}{n})^{\frac{qk^9}{2}} &\leq \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} (\frac{ekn}{q})^{2q} (\frac{q}{n})^{\frac{qk^9}{4}} (\frac{4e^2q}{n})^{\frac{qk^9}{4}} \\ &\leq \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} (ek)^{2q} (\frac{4e^2q}{n})^{\frac{qk^9}{4}}, \end{aligned}$$

where we used the fact that $k^9 \geq 8$. Furthermore, by assuming that $k \geq 3$ (as the case that $k = 2$ is trivial) and using the fact that $(ek)^{\frac{8}{k^9}} \leq (e3)^{\frac{8}{3^9}} \leq \frac{9}{e^2}$, we can bound the last sum by:

$$\sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} (\frac{(ek)^{\frac{8}{k^9}} 4e^2 q}{n})^{\frac{qk^9}{4}} \leq \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} (\frac{36q}{n})^{2q}.$$

Because $f(q) = (\frac{36q}{n})^{2q}$ is convex in q, $f(q)$ is bounded by $max\{f(1), f(\lfloor \frac{n}{40} \rfloor)\}$ in the interval $[1, \lfloor \frac{n}{40} \rfloor]$. Therefore the sum is at most:

$$\frac{n}{40} max\{(\frac{36}{n})^2, (\frac{36}{40})^{\frac{n}{20}}\}.$$

It is obvious that $\frac{n}{40} max\{(\frac{36}{n})^2, (\frac{36}{40})^{\frac{n}{20}}\}$ approaches 0 as n goes to infinity. Therefore, if n is large enough, the maximum is at most $\frac{1}{10}$. So with probability at least 9/10, G has property (3).

Similar argument shows that, with probability at least 9/10, G has each of the properties (1), (2), (4), (5). (See [1] for detailed calculations for the proofs of similar properties.) Therefore with probability at least 1/2, a random graph G constructed above has all the properties (1)-(5). This completes the proof of Lemma 6.

Given a graph G with properties listed in Lemma 6, we now describe a method (given in [1]) which constructs a graph G^* satisfying the condition of Theorem 4, by deleting some vertices from G. Let $C := \{v : v$ is a vertex contained in a cycle in G of length at most $g-1\}$. Let $Y := \{v : v$ has degree in G larger than $5k^{13}\}$. Let $s := max\{|V_i \cap (C \cup Y)| : i \in [k]\}$. First of all we remove from each V_i exactly s vertices in such a way that all vertices from $C \cup Y$ are removed. Call the remaining sets W_i. The resulting graph G' now has girth at least g, and has maximum degree at most $5k^{13}$. We further modify the graph G' by the following WHILE-loop:

WHILE there is a vertex $x \in W_i$ such that x has less than $\frac{k^{10}}{2}$ neighbors in W_j and ij is an edge of F, then
DO the following: delete x from W_i, and remove for every $r \neq i$ an arbitrary vertex from W_r.

Note that in the WHILE-loop, each time we search in the new graph (not in the original graph G) for the "bad" vertex x. So a vertex x may be good at the beginning, and become bad later. However, it is proved in [1] that the WHILE-loop will eventually stop.

Lemma 7 *[1] The WHILE-loop is executed at most $\frac{n}{2k}$ times.*

Proof. Assume to the contrary that the WHILE-loop is executed more than $\frac{n}{2k}$ times. Let W_i', $i \in [k]$, be the remaining parts of the sets W_i after $\lceil \frac{n}{2k} \rceil$ executions of the WHILE-loop. Then $|W_i'| \geq \frac{(k-1)n}{k}$ as $s \leq \frac{n}{2k} - 1$.

For each execution of the WHILE-loop, an ordered pair (i, j) is selected so that ij is an edge of F and W_i contains a vertex which has less than $\frac{k^{10}}{2}$ neighbors in W_j'. Since there are only k^2 ordered pairs, there exists an ordered pair (i, j) which was selected more than $\lceil \frac{n}{2k^3} \rceil$ times. Therefore, there exists a set $U \subset V_i$ of size $\lceil \frac{n}{2k^3} \rceil$ such that all vertices from U have less than $\frac{k^{10}}{2}$ neighbors in W_j', and ij is an edge of F. Let W be a subset of W_j' of size $\lceil \frac{(k-1)n}{k} \rceil$. Then there are less than $\frac{k^7 n}{4}$ edges between U and W, contradicting property 1. ∎

After the WHILE-loop is completed, we obtain a subgraph G^* of $G(n, F)$, whose k partite sets are $W_1', W_2', \cdots, W_k'$, with $|W_1'| = |W_2'| = \cdots = |W_k'| \geq \frac{(k-1)n}{k}$. With V_i replaced by W_i', the graph G^* has properties (1)-(5). Moreover, G^* has girth at least g, maximum degree at most $5k^{13}$, and for each edge ij of F, each vertex x of W_i' has at least $\frac{k^{10}}{2}$ neighbors in W_j'. Now we shall prove that G^* has the properties of Theorem 4.

It follows from the construction that G^* admits a surjective homomorphism to F, with W_i' mapped to i. Assume that H has at most k vertices and there exists a homomorphism $g : G^* \to H$. We need to prove that there exists a homomorphism $f : F \to H$. For each $i \in V(F)$, let $v \in V(H)$ be a vertex such that $|g^{-1}(v) \cap W_i'| \geq \frac{(k-1)n}{k^2}$. Let $f(i) = v$. We claim that f is a homomorphism from F to H. Otherwise, there is an edge ij of F for which $f(i)f(j) = uv$ is not an edge of H. This implies that in G^* there is no edge between $g^{-1}(u) \cap W_i'$ and $g^{-1}(v) \cap W_j'$. However, $g^{-1}(u) \cap W_i'$ and $g^{-1}(v) \cap W_j'$ are subsets of W_i' and W_j' of size at least $\frac{(k-1)n}{k^2} > \frac{n}{40k}$. This is in contrary to property (2).

Finally, assume that H is an F-pointed graph with at most k vertices. We need to prove that for every homomorphism $g : G \to H$ there exists a unique homomorphism $f : F \to H$ such that $g = f \circ c$, where c is the homomorphism from G to F with $c(W_i') = i$. Let f be the homomorphism from F to H defined as in the previous paragraph. We shall prove that $g = f \circ c$. Assume to the contrary that $g \neq f \circ c$. So for some i, $W_i' - g^{-1}(f(i)) \neq \emptyset$.

First we observe that for any i, j such that $f(i) \neq f(j)$, $|W_i' \cap g^{-1}(f(j))| < \frac{n}{40k}$. Otherwise, let f' be the mapping from $V(F)$ to $V(H)$ be defined as $f'(a) = f(a)$ if $a \neq i$, and $f'(i) = f(j)$. The argument in the previous paragraph shows that f' is also a homomorphism from F to H, contrary to the assumption that H is F-pointed.

Let i_0 be an index such that $|W_{i_0}' - g^{-1}(f(i_0))| = q$ is maximum among $|W_i' - g^{-1}(f(i))|$. Let j_0 be an index such that $f(i_0) \neq f(j_0)$ and $|W_{i_0}' \cap g^{-1}(f(j_0))| = q'$ is maximum among $|W_{i_0}' \cap g^{-1}(f(j))|$ where $f(i_0) \neq f(j)$. Then $q' \geq q/k$. By the previous paragraph, $q' < \frac{n}{40k}$.

First we show that there is an index j such that j is adjacent to i_0 in F but $f(j)$ is not adjacent to $f(j_0)$ in H (the vertex j could be j_0 itself). For otherwise, the mapping $f' : V(F) \to V(H)$ defined as $f'(a) = f(a)$ for $a \neq i_0$ and $f'(i_0) = f(j_0)$ would be a homomorphism from F to H, contrary to the assumption that H is F-pointed.

Let $U = W_{i_0}' \cap g^{-1}(f(j_0))$ and $W = W_j' - g^{-1}(f(j))$. By the choice of i_0, we have $|W| \leq |W_{i_0}' - g^{-1}(f(i_0))| \leq k|U|$.

By property (3), there are less than $\frac{|U|k^{10}}{2}$ edges between U and W in G^*. However, each vertex of U is adjacent to at least $\frac{k^{10}}{2}$ neighbors in W_j'. So there is an edge, say xy, between U and $W_j' \cap g^{-1}(f(j))$. However $g(x) = f(j_0)$ and $g(y) = f(j)$ and $f(j_0)$ is not adjacent to $f(j)$, contrary to the assumption that g is a homomorphism. This completes the proof of Theorem 4.

To prove Corollary 5, assume that $\mathcal{F} = \{f_1, f_2, \cdots, f_t\}$. Let $F = H^t \times K_N$ where K_N is the complete graph with N vertices, $N = max\{k^t, m\}$. Then $|F| \leq k^{2mt}$. We apply Theorem 4 for the graphs F and H. Thus there exists a graph G and a surjective homomorphism $c : G \to F$ such that for any homomorphism $g : G \to H$ there exists a homomorphism $f : F \to H$ such that $g = f \circ c$.

Since $N \geq |H^t| = k^t$, there is a homomorphism $\phi : H^t \to K_N$. For any homomorphism $f : F \to H$, the restriction of f to $H^t \times \phi(H^t)$ defines a homomorphism f' from $H^t \to H$ as follows: for $x \in H^t$, let $f'(x) = f(x, \phi(x))$. As H is projective, we conclude that, up to an automorphism of H, all the homomorphisms $F \to H$ are induced by the t projections $\pi_1, \pi_2, \cdots, \pi_t : H^t \to H$. In other words, every homomorphism $f : F \to H$ for which $f(x, \cdots, x, a) = x$ is of the form $f(x, a) = \pi_i(x)$ for every vertex (x, a) of F and some $i, 1 \leq i \leq t$, here $x \in H^t$. Hence, up to an automorphism of H, there are exactly t homomorphisms from G to H: $\pi_i \circ c$, $i = 1, 2, \cdots, t$. Now consider mappings $f_1, f_2, \cdots, f_t$ together with an injective mapping $f_0 : X \to V(K_N)$. Then the corresponding mapping $\phi = (f_1, f_2, \cdots, f_t, f_0) : X \to V(F)$ is injective. Thus we can identify X with its image $\phi(X)$ and also replace some vertices of the graph G with elements of $\phi(X)$. Call the resulting graph again G. Then each of the t homomorphisms $\pi_i \circ c$ is an extension of the mapping f_i. Clearly all homomorphisms $f : G \to H$ coincide on $X = \phi(X)$ with one of the maps f_i, $i = 1, 2, \cdots, t$.

A graph H is a *core* if H does not admit a homomorphism to any of its proper subgraphs.

Corollary 8 *If H is a core on k vertices, then for any integer g, there is a uniquely H-colourable graph G of girth at least g and with maximum degree at most $5k^{13}$.*

This corollary follows easily from Theorem 4 and the fact that if H is a core then it is H-pointed.

The question that for which graph G we have $\chi_c(G) = \chi(G)$ has been studied extensively in the literature. For all such graphs G constructed before, if G is χ_c-critical (i.e., deleting any edge will decrease its circular chromatic number), then its maximum degree increases to infinity along with the size of G. As uniquely $K_{k/d}$-colourable graphs have circular chromatic number k/d [8], it follows that there are graph G of arbitrary large girth and of bounded maximum degree which have circular chromatic number k/d. As any such graph contains a χ_c-critical subgraph, it follows that for any k/d, we can find a χ_c-critical graph G with $\chi_c(G) = k/d$ with maximum degree bounded by a constant which is independent of the size of G.

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