

On Sparse Graphs with Given Colorings and Homomorphisms

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Abstract

We prove that for every graph H and positive integers k, l there exists a graph G with girth at least l such that for all graphs H' with at most k vertices there exists a homomorphism $G \rightarrow H'$ if and only if there exists a homomorphism $H \rightarrow H'$. This implies (for $H = K_k$) the classical result of Erdős and other generalizations (such as Sparse Incomparability Lemma). We refine the above statement to the 1-1 correspondence between the set of all homomorphisms $G \rightarrow H'$ and the set of all homomorphisms $H \rightarrow H'$. This in turn yields the existence of sparse uniquely H -colorable graphs and, perhaps surprisingly, provides a characterization of the graphs H for which the analog of Müller's theorem holds for H -colorings.

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1 Introduction

Our terminology is standard: Given a graph G , we denote by $\chi(G)$ its chromatic number, by $g(G)$ its girth. (Thus $g(G)$ is the shortest length of a cycle in G .) Given graphs G and H , a *homomorphism* from G to H is a vertex-mapping $f : V(G) \rightarrow V(H)$ such that $\{x, y\} \in E(G)$ implies $\{f(x), f(y)\} \in E(H)$. If there exists a homomorphism from G to H , then we say G is *homomorphic to* H , and write $G \rightarrow H$. It is easy to see that the question whether there exists a homomorphism from G to K_n (the complete graph with n vertices) is equivalent to the question whether G is n -colorable. Motivated by this, a homomorphism $G \rightarrow H$ is also called *H -coloring* of G .

The following are two main result of this paper:

Theorem 1.1 *For every graph H and for every positive integers k, l there exists a graph G with the following properties:*

- i. $g(G) > l$;*
- ii. For every graph H' with at most k vertices, there exists a homomorphism $g : G \rightarrow H'$ if and only if there exists a homomorphism $f : H \rightarrow H'$.*

For the statement of the second result we need the following notion:

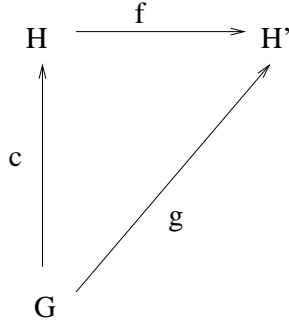
A graph H' is said to be *pointed for graph H* (or shortly *H -pointed*) if for any two homomorphisms $g, g' : H \rightarrow H'$ which satisfy $g(x) = g'(x)$ for all $x \neq x_0$ (for some fixed vertex $x_0 \in V(H)$) holds also $g(x_0) = g'(x_0)$.

Theorem 1.2 *For every graph H and every choice of positive integers k and l there exists a graph G together with a surjective homomorphism $c : G \rightarrow H$ with the following properties:*

- i. $g(G) > l$;*
- ii. For every graph H' with at most k vertices, there exists a homomorphism $g : G \rightarrow H'$ if and only if there exists a homomorphism $f : H \rightarrow H'$.*
- iii. For every H -pointed graph H' with at most k vertices and for every homomorphism $g : G \rightarrow H'$ there exists a unique homomorphism $f : H \rightarrow H'$ such that $g = f \circ c$.*

The conditions *ii.* and *iii.* may be expressed by the following diagram:

It is easy to give an example which shows that the statement *iii.* can not be true for all (i.e. not necessarily pointed) small graphs H : Given two homomorphisms $f', f'' : H \rightarrow H'$ satisfying $f'(x) = f''(x)$ for all $x \neq x_0$ and $f'(x_0) \neq f''(x_0)$.



The set $c^{-1}(x_0)$ may be splitted in two non-empty sets A and B and we can define $g : G \rightarrow H'$ by $f' \circ c(x)$ for all x with $c(x) \neq x_0$ and $g(x) = f' \circ c(x)$ for all $x \in A$ and $g(x) = f'' \circ c(x)$ for all $x \in B$. The homomorphism g cannot be written as $g = f \circ c$ for a homomorphism $f : H \rightarrow H'$. A graph H is called a *core* if any homomorphism $H \rightarrow H$ is an automorphism. Note that any core graph H is H -pointed and it follows that most graphs H on a large set are H -pointed.

Theorem 1.1 is a generalization of the study of graphs with large chromatic number and large girth. The existence of such graphs is established by the following landmark result (and landmark proof):

Theorem 1.3 (Erdős [3]) *let k and l be positive integers. Then there exists a graph G with the following properties:*

- i. $\chi(G) > k$;*
- ii. $g(G) > l$.*

Theorem 1.1 has also the following corollary known as *Sparse Incomparability Lemma* [11], [13]:

Corollary 1.1 *For every pair H, H' of graphs such that H' is H -colorable and H fails to be H' -colorable there exists a graph G with the following properties:*

- i. $g(G) > l$*
- ii. G is H -colorable and G fails to be H' -colorable.*

(To obtain Corollary 1.1 we put $H = H$, $k = |V(H')|$ in Theorem 1.1.)

Theorem 1.2 may look like a technical extension of Theorem 1.1. However, it has several interesting corollaries which prove structural extensions of Erdős theorem.

We say a graph G is *uniquely H -colorable* if there is an onto homomorphism c from G to H , and any other homomorphism from G to H is the composition $\sigma \circ c$ of c with an automorphism σ of H . (Note that this implies that H is a core graph.)

The problem of the existence of uniquely k -colorable graphs with large girth has an interesting history: [10] settled triangle-free case (i.e. $l = 3$) and this was improved by Greenwell and Lovász [4] to graphs not containing short odd cycles. The general case was solved by Müller [8, 9]. Müller's proof is a constructive and uses a constructive proof of Theorem 1.3. A non-constructive proof has been published by Bollobás and Sauer [2]. Finally, Zhu [13] proved the following result which is a particular case (for $H = H'$) of our Theorem 1.2

Corollary 1.2 *For every core H and positive integer l there exists uniquely H -colorable graph G with girth $> l$.*

We prove here that there is such a graph G which is *strongly uniquely H -colorable* in the sense that any homomorphism $G \rightarrow H'$ to any small pointed graph H' is induced by a homomorphism $H \rightarrow H'$.

An explicit construction of this and of Corollary 1.2 is presently an open problem, cf. [12].

The existence of sparse uniquely k -colorable graphs was generalized in [8, 9] in another direction, where the following remarkable strengthening of the unique colorability was proved:

Theorem 1.4 (Müller [8, 9]) *Let k, l, t be positive integers, $k > 2$. Let A be a finite set and let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_t$ be distinct partitions of the set A each into at most k classes. Then there exists a k -chromatic graph $G = (V, E)$ such that the following holds:*

- i. $g(G) > l$;*
- ii. A is a subset of V ;*
- iii. up to a permutation of colors G has just t colorings $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_t$ by k -colors such that each of the coloring \mathcal{B}_i restricted to the set A coincides with the coloring \mathcal{A}_i , $i = 1, 2, \dots, t$.*

It seems that this result is little known (although it is included in [5]) and even the existence of an explicit construction of uniquely colorable graphs without short cycles was recently quoted as a problem [13]. In this paper we approach Müller's Theorem more generally and thus perhaps find a proper setting for it.

In Section 4 we show how Theorem 1.2 implies Theorem 1.4 and we completely characterize graphs H for which the analogy of Theorem 1.4 holds for H -colorings. This is related to the notion of *projective* graphs and *2-constructible* graphs which are studied in [6, 7].

The paper is organized as follows: In Section 2 we prove Theorem 1.2 (which in turn implies Theorem 1.1). As the proof is non-constructive we include in Section 3 a simple proof of a weaker statement with girth replaced by the odd girth (i.e. the length of the shortest odd cycle). In Section 4 we prove the analogy of Müller's theorem for H -colorings (this is stated as Corollary 4.1) and we characterize all such H (Theorem 4.1).

2 Proof of Theorem 1.2

Our proof uses probabilistic method and most of the calculations are fairly standard. But it is an indication of the proper setting of Theorem 1.2 that the proof is perhaps easier than the proofs of particular cases, [2, 13].

Suppose that the graph H has a vertices and that the vertices are $\{1, 2, \dots, a\}$, and the edge set $E(H)$ has cardinality q . Let V_1, V_2, \dots, V_a be disjoint n -sets. Let G_0 be the graph with vertex set $V = V_1 \cup V_2 \cup \dots \cup V_a$, and $\{x, y\} \in E(G_0)$ if and only if $x \in V_i, y \in V_j$ and $(i, j) \in E(H)$. Then G_0 has qn^2 edges. Let \mathcal{G} be the set of all subgraphs G of G_0 with $m = \lfloor qn^{1+\epsilon} \rfloor$ edges, where $0 < \epsilon < 1/4l$. Put also $\delta = \min\{\epsilon l, 1/k\}$. Then $|\mathcal{G}| = \binom{qn^2}{m}$.

In the following, n is assumed to be sufficiently large. We consider \mathcal{G} as a probability space with each member occurring with the same probability $1/|\mathcal{G}|$. This is asymptotically the same thing as the random graph \mathbf{G} where we choose edges from the set $E(G_0)$ independently with the probability $n^{-1+\epsilon}$.

We shall make use of the fact that most graphs in \mathcal{G} have few short cycles which are pairwise vertex disjoint. On the other hand, the edges are "dense" in some sense. These results are stated as Claims 1-3:

Claim 1 The expected number of cycles of length $\leq l$ in a graph $G \in \mathcal{G}$ is bounded by $n^\delta n^{-\epsilon/2}$. The expected number of pairs of cycles of length $\leq l$ in a graph $G \in \mathcal{G}$ which intersect in at least one vertex is bounded by $n^{-1/2}$. Thus asymptotically almost all graphs from \mathcal{G} have at most n^δ cycles of length $\leq l$, and these cycles are all vertex disjoint.

A set $A \subset V$ is said to be *large* if there are $i, j, 1 \leq i < j \leq a, \{i, j\} \in E(H)$, such that $|A \cap V_i| \geq n/k$ and also $|A \cap V_j| \geq n/k$. We call an edge $\{i, j\}$ of H a *good edge* of A if $|A \cap V_i| \geq n/k$ and $|A \cap V_j| \geq n/k$. For a large set A denote by $|\mathbf{G}/A|$ the minimum number of edges of \mathbf{G} which lie in the set $\{\{x, y\}; x \in V_i, y \in V_j\}$ for a good edge $\{i, j\}$ of A .

Claim 2 The probability $Prob[A \text{ large implies } |\mathbf{G}/A| \geq n] = 1 - o(1)$.

Thus asymptotically almost all graphs from \mathcal{G} have the property that all good

edges (of H) of any large set induce at least n edges (of \mathbf{G}).

Claim 3 Almost all graphs from \mathcal{G} do not contain two non-empty sets $A \subset V_{i_0}, B \subset V_{j_0}, 1 \leq i_0 < j_0 \leq a$, with $\{i_0, j_0\} \in E(H)$, $|A| = n - (k-1)|B|$, $|B| \leq n/k$ such that the set $A \cup B$ contains at most $\min\{|B|, n^\delta\}$ edges and these edges form a matching (i.e. a set of mutually disjoint edges).

For an easy and standard proof of Claim 1 see e.g [2, 13]. We include proofs of Claims 2 and 3:

Proof. (of Claim 2)

We first estimate probability

$$\alpha = \text{Prob}[A \text{ large implies } |\mathbf{G}/A| \geq n].$$

We have

$$1 - \alpha \leq \sum_{A \text{ large}} \text{Prob}[|\mathbf{G}/A| < n] \leq 2^{an} \cdot \binom{qn^2}{n} \cdot (1-p)^{n^2/k^2-n}.$$

Now bounding very roughly

$$2^{an} \binom{qn^2}{n} \leq 2^{an} \binom{q^2 n^2}{n} \leq 2^{an} q^{2n} n^{2n} < e^{cn \ln n}$$

and

$$(1-p)^{n^2/k^2-n} \leq e^{-p(n^2/k^2-n)}$$

we obtain

$$1 - \alpha < e^{cn \ln n - c'n^{1+\epsilon}}$$

for some positive constants c and c' which are independent on n .

Thus we get $\text{Prob}[A \text{ large implies } |\mathbf{G}/A| \geq n] = 1 - o(1)$. ■

Proof. (of Claim 3)

This is similar, this time we give a counting version of the proof. We shall show that very few graphs in \mathcal{G} contain subgraphs induced by a pair of sets A, B , $|A| = n - (k-1)|B|$, $|B| \leq n/k$ with at most $\min\{|B|, n^\delta\}$ edges (even without matching condition). Towards this end for $b \leq n/k, s \leq \min\{b, n^\delta\}$ we denote by $P(b, s)$ the expected number of pairs A, B such that $A \subset V_i, B \subset V_j, \{i, j\} \in E(H)$, $|A| = n - (k-1)|B|$, $|B| = b$ and there are exactly s edges between A and B . Then

$$P(b, s) < 2q \binom{n}{n - (k-1)b} \binom{n}{b} \binom{(n - (k-1)b)b}{s} \binom{qn^2 - b(n - (k-1)b)}{m - s} \binom{qn^2}{m}^{-1}.$$

As $b \leq n/k$ we have

$$\binom{qn^2 - b(n - (k-1)b)}{m-s} \binom{qn^2}{m}^{-1} < \binom{qn^2 - bn/k}{m} \binom{qn^2}{m}^{-1} < e^{-\frac{bnm}{k}/qn^2} = e^{-bn^\epsilon/k}.$$

(Here we used the inequality $\binom{a-x}{b} \binom{a}{b}^{-1} \leq \left(\frac{a-x}{a}\right)^b < e^{-bx/a}$.)

Therefore

$$P(b, s) < 2qn^{(k-1)b} n^b (bn)^s e^{-\frac{bn^\epsilon}{k}} < 2q(bn)^s \exp\left(-\frac{bn^\epsilon}{k} + kb \ln n\right) < n^{2s} e^{-\frac{bn^\epsilon}{2k}}.$$

Let $P(b)$ be the sum of all $P(b, s)$ for which $s \leq \min\{b, n^\delta\}$.

If $1 \leq b < n^\delta$, then $s \leq b$, and hence

$$P(b) < \exp\left(-\frac{bn^\epsilon}{2k} + 2b \ln n\right) < e^{-\frac{bn^\epsilon}{3k}} < e^{-n^\epsilon/2}.$$

If $b \geq n^\delta$, then $s \leq n^\delta$, and hence

$$P(b) < \exp\left(-\frac{n^{(l+1)\epsilon}}{2k} + 2n^\delta \ln n\right) < e^{-\frac{n^{(l+1)\epsilon}}{3k}} < e^{-n^\delta} < e^{-n^\epsilon/2}.$$

Therefore

$$\sum_{1 \leq b \leq \frac{n}{2k}} P(b) < \frac{n}{2k} e^{-n^\epsilon/2} < e^{-n^\epsilon/3}.$$

This establishes proof of Claim 3. ■

Thus let G' be an instance of the graph from \mathcal{G} with all the properties claimed in Claims 1-3 for majority of graphs from \mathcal{G} .

Particularly, the graph G' contains at most n^δ cycles of length $\leq l$ and all these cycles are vertex disjoint.

Consequently, there exists a set M of edges of G' which forms a matching in G' of size at most n^δ such that the graph $G' - M = (V(G'), E(G') - M)$ has no cycles of length $\leq l$. Put $G = G' - M = (V, E)$. We prove that the graph G is our desired graph which satisfies Theorem 1.2. It is clear that G has girth $> l$

Define the mapping $c : V \rightarrow V(H)$ by $c(x) = i$ iff $x \in V_i, i = 1, \dots, a$. Clearly c is a surjective homomorphism $G \rightarrow H$.

Let H' be a fixed graph with at most k vertices and let $g : G \rightarrow H'$ be a homomorphism.

We define a mapping $\phi : V(H) \rightarrow V(H')$ as follows: For each $i \in V(H)$, there exists a vertex $x \in V(H')$ such that $|V_i \cap g^{-1}(x)| \geq n/k$ by the pigeonhole principle. We let $\phi(i)$ be any (fixed) x with $|V_i \cap g^{-1}(x)| \geq n/k$. (If there are more than one x which satisfy the condition, we arbitrarily choose one.)

We prove that ϕ is a homomorphism $H \rightarrow H'$. Thus let $\{i, j\}$ be an edge of H . Put $A_i = g^{-1}(\phi(i))$ and $A_j = g^{-1}(\phi(j))$. As $|A_i| \geq n/k$ and $|A_j| \geq n/k$ there exists an edge e of G with its ends in $A_i \cup A_j$ (this follows from Claim 2). Thus also $\phi(i)$ and $\phi(j)$ form an edge of H' . This proves part *ii.* of the Theorem 1.1. (At this point we also concluded proof of Theorem 1.1.)

Now we prove *iii.* Thus from now on let H' be a H -pointed graph (with at most k vertices).

It follows that we can further assume that $|V_i \cap g^{-1}(j)| < n/k$ for all $j \neq i$. For otherwise if there are i, j such that $j \neq i$ with $|V_i \cap g^{-1}(j)| \geq n/k$ then we can define another mapping $\phi' : V(H) \rightarrow V(H')$ which agrees with ϕ on every vertex $u \neq j$, and $\phi'(i) = j$. As above it is easy to check that ϕ' is a homomorphism $H \rightarrow H'$. However this is a contradiction as H' is a H -pointed graph.

Thus the homomorphism ϕ is uniquely determined by the homomorphism g . The set $V_i \cap g^{-1}(\phi(i))$ will be denoted by W_i , $i = 1, \dots, a$.

It remains to prove that $\phi \circ c = g$.

Assume to the contrary that $\phi \circ c \neq g$.

Thus there exist distinct vertices x, x_0 of H' and $i_0 \in V(H)$ such that $\phi(i_0) = x_0 \neq x$ while the set $W = g^{-1}(x) \cap V_{i_0}$ is non-empty.

Choose $i_0 \in V(H)$ such that $t = |g^{-1}(\phi(i_0) \cap V_{i_0})| = |W_{i_0}|$ is minimal among $|W_i|, i \in V(H)$.

Then $t \geq n/k$ and also $t < n$ (as we are assuming that $g \neq \phi \circ c$). Clearly $V_{i_0} - g^{-1}(\phi(i_0)) = \bigcup \{g^{-1}(\phi(i)) \cap V_{i_0}; i \in V(H), \phi(i) \neq \phi(i_0)\}$. By the pigeonhole principle, there exists $j_0 \neq i_0, \phi(j_0) \neq \phi(i_0)$ such that $|g^{-1}(\phi(j_0)) \cap V_{i_0}| \geq (n-t)/(k-1)$.

Put $B = g^{-1}(\phi(j_0)) \cap V_{i_0}$ and $|B| = b$. Thus $b \geq (n-t)/(k-1)$ and $t \geq n-b(k-1)$.

We distinguish two cases:

Case 1 $\{i_0, j_0\} \in E(H)$

In this case the sets $A = W_{j_0} = g^{-1}(\phi(j_0)) \cap V_{j_0}$ and $B = g^{-1}(\phi(j_0)) \cap V_{i_0}$ are in the situation of Claim 3 and there exists an edge e of G with its end-vertices in the set $A \cup B$ which is a contradiction with g being a homomorphism.

Case 2 $\{i_0, j_0\} \notin E(H)$

First we claim that there exists $p_0 \in V(H)$ such that $\{p_0, i_0\} \in E(H)$, $\{p_0, j_0\} \notin E(H)$. For if every vertex p of H which is incident with i_0 is also incident with j_0 then we can define the mapping $\phi' : V(H) \rightarrow V(H')$ by $\phi'(i) = i$ for all $i \neq i_0$ and $\phi'(i_0) = j_0$. It is $\phi \neq \phi'$ and ϕ' is clearly a homomorphism which contradicts the fact that the graph H' is H -pointed.

Secondly, we claim that there exists $p_0 \in V(H)$ and $v \in V(H')$ such that $\phi(p_0) = v$, $\{p_0, i_0\} \in E(H)$, $\{p_0, j_0\} \notin E(H)$, $\{v, \phi(i_0)\} \in E(H')$, $\{v, \phi(j_0)\} \notin E(H')$.

This is analogous to the first part of case 2: Suppose that every $p_0 \in V(H)$ with $\{p_0, i_0\} \in E(H)$, $\{p_0, j_0\} \notin E(H)$ satisfies both $\{\phi(p_0), \phi(i_0)\} \in E(H')$, $\{\phi(p_0), \phi(j_0)\} \in E(H')$. Then we can again define the mapping $\phi' : V(H) \rightarrow V(H')$ by $\phi'(i) = i$ for all $i \neq i_0$ and $\phi'(i_0) = j_0$ which again contradicts the fact that the graph H' is H -pointed.

Thus let p_0, v satisfy $\{p_0, i_0\} \in E(H)$, $\{p_0, j_0\} \notin E(H)$, $\phi(j_0) \neq \phi(i_0)$, $\phi(p_0) = v$, $\{v, \phi(i_0)\} \in E(H')$, $\{v, \phi(j_0)\} \notin E(H')$.

In this situation the sets $A = W_{p_0} = g^{-1}(v) \cap V_{p_0}$ and $B = g^{-1}(\phi(j_0)) \cap V_{i_0}$ satisfy $|A| \geq t = |g^{-1}(\phi(i_0)) \cap V_{i_0}|$ (by the choice of i_0 and $|B| = b$, $b \geq (n-t)/(k-1)$). Thus there exists a set $A \subset W_{p_0}$ such that $|A| = n - |B|(k-1)$ such that the set $A \cup B$ is an independent set in G (as $g(B) = \phi(j_0)$ and $g(A) = v$ and $\{\phi(j_0), v\} \notin E(H')$). However this contradicts Claim 3.

Thus $g = \phi \circ c$. This finishes the proof of Theorem 1.2.

3 Odd Girth Construction

For a non-bipartite graph G denote by $oddg(G)$ the *odd girth* of G , i.e. the shortest length of an odd cycle in G . Here we prove the analogy of Theorem 1.1 and 1.2 for graphs of a given odd girth by giving an explicit construction. The reason why we prove this weaker statement is that for general graphs H such an explicit construction is not known (as stated above; in a companion paper [12] we give the constructive proof of Theorem 1.1 for graphs G_k^d). We shall use products of graphs:

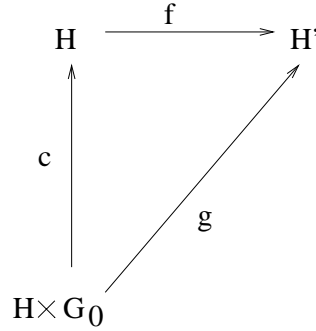
By a *product* we mean here *categorical product* (sometimes called *direct product*) defined as follows: Suppose G and H are simple finite graphs. The categorical product $G \times H$ of G and H has vertex set $V(G \times H) = \{(x, y) : x \in V(G) \text{ and } y \in V(H)\}$ and edge set $E(G \times H) = \{\{(x, y), (x', y')\} : \{x, x'\} \in E(G) \text{ and } \{y, y'\} \in E(H)\}$. The mapping $\pi : V(G \times H) \rightarrow V(G)$ defined by $\pi(x, y) = x$ is called a *projection*. It is an easy exercise to verify that $oddg(G \times H) = \max\{oddg(G), oddg(H)\}$.

Theorem 3.1 For every graph H and every choice of positive integers k and l there exists a graph G_0 such that the graph $G = H \times G_0$ together with the projection $c : H \times G_0 \rightarrow H$ has the following properties:

i. $\text{oddg}(G) > l$;

ii. For every graph H' with at most k vertices, there exists a homomorphism $g : G \rightarrow H'$ if and only if there exists a homomorphism $f : H \rightarrow H'$.

iii. For every H -pointed graph H' with at most k vertices and for every homomorphism $g : G \rightarrow H'$ there exists a unique homomorphism $f : H \rightarrow H'$ such that $g = f \circ c$.



Proof.

Let G_0 be a connected graph with odd girth $> l$ and chromatic number $> k^{|V(H)|}$ (a construction is provided e.g. by iterated shift graphs). Put $G = H \times G_0$. It has $\text{oddg}(G) > l$. Thus let H' be a graph with at most k vertices and let $g : G \rightarrow H'$ be a homomorphism.

For every $y \in V(G_0)$ define the mapping $f_y : V(H) \rightarrow V(H')$ by $f_y(x) = g((x, y))$. Note that the mapping f_y need not be a homomorphism $H \rightarrow H'$ but as $\chi(G_0) > k^{|V(H)|}$ there exists an edge $\{y, y'\} \in E(G_0)$ such that $f_y = f_{y'}$. However in this case the mapping $f = f_y = f_{y'}$ is a homomorphism $H \rightarrow H'$: given $\{x, x'\} \in E(H)$ we have $\{(x, y), (x', y')\} \in E(H \times G_0)$ and thus $\{g(x, y), g(x', y')\} = \{f_y(x), f_{y'}(x')\} = \{f_y(x), f_y(x')\} = \{f(x), f(x')\} \in E(H)$.

This proves *ii*.

So suppose that in addition the graph H' is a H -pointed core. Under this assumption the validity of *iii*. follows readily from the following (which generalizes [4, 11, 15]):

Claim If f_y (defined above) is a homomorphism $H \rightarrow H'$ for some $y \in V(G_0)$ and that $\{y, z\} \in E(G_0)$, then $f_z = f_y$.

Assume to the contrary that $f_z(x_0) \neq f_y(x_0)$ for a vertex $x_0 \in V(H)$. Define

mapping $g : V(H) \rightarrow V(H')$ as $g(x) = f_y(x)$ for $x \neq x_0$ and $g(x_0) = f_z(x_0)$. Then g is a homomorphism $H \rightarrow H'$. (It suffices to check edges of H incident with x_0 : If $\{x, x_0\} \in E(H)$ then $\{(x, y), (x_0, z)\} \in E(H \times G_0)$ and thus $\{f_y(x), f_z(x_0)\} = \{g(x), g(x_0)\} \in E(H')$.) However this is in contrary with the fact that H' is H -pointed. ■

4 Unique extensions, constructible sets and projectivity

In this section we show that uniquely colorable graphs and, more generally, graphs which satisfy Müller's theorem can be obtain as a corollary of Theorem 1.2 for $H = K_k^t = K_k \times \dots \times K_k$ (t -times), $H' = K_k$. We shall need the following property of complete graphs (which is established in [9]):

The only homomorphisms $f : K_k^t \rightarrow K_k$ satisfying $f(x, x, \dots, x) = x$ are projections.

In a recent paper by B. Larose and C. Tardif [6, 7], this property was studied (in relationship to Hedetniemi's product conjecture) and called projectivity: A graph H is said to be t -projective if every homomorphism $f : H^t \rightarrow H$ which satisfies $f(x, x, \dots, x) = x$ for every $x \in V(H)$ is a projection. A graph is *projective* if it is t -projective for every t (i.e. for projective graphs, up to an automorphism, the only homomorphisms $H^t \rightarrow H$ are projections). Note that every projective graph is necessarily a core.

Thus the above mentioned result of Müller can be stated by saying that complete graphs are projective. Larose and Tardif proved some sufficient conditions for a graph to be projective. It is easy to derive from these conditions that many classes of graphs are projective, including Kneser graphs, graphs G_k^d , etc.. Our companion paper [12] contains a short proof of the projectivity of complete graphs and graphs G_k^d .

The notion of projective graphs and Theorem 1.2 leads to the following (which extends Müller's theorem to non-complete graphs):

Corollary 4.1 *Let H be projective graph with k vertices, l a positive integer. Let A be a finite set and let f_1, f_2, \dots, f_l be distinct mappings $A \rightarrow V(H)$. Then there exists a graph $G = (V, E)$ such that the following holds:*

- i. A is a subset of V ;*
- ii. For every $i = 1, 2, \dots, l$ there exists unique homomorphism $g_i : G \rightarrow H$ such that g_i restricted to the set A coincides with the mapping f_i ;*

iii. For every homomorphism $f : G \rightarrow H$ there exists $i, 1 \leq i \leq t$ and an automorphism h of H such that $h \circ f_i = f$;

iv. G has girth $> l$.

Proof. Consider the graph $H_0 = H^t \times K_N$ where K_N is the complete graph with N vertices, $N > \max\{k^{kt}, |A|\}$. We apply Theorem 1.2 for the graphs H_0 (in place of H) and H (in place of H'). Thus there exists a graph G and a surjective homomorphism $c : G \rightarrow H_0$ such that for any homomorphism $g : G \rightarrow H$ there exists a homomorphism $f : H_0 \rightarrow H$ such that $g = f \circ c$. Now, up to automorphisms of H , all the homomorphisms $H_0 \rightarrow H$ are induced by t projections $\pi_1, \pi_2, \dots, \pi_t : H^t \rightarrow H$. In the other words, every homomorphism $f : H_0 \rightarrow H$ for which $f(x, \dots, x, a) = x$ is of the form $f(x, a) = \pi_i(x)$ for every vertex (x, a) of H_0 and some $i, 1 \leq i \leq t$. Hence, up to an automorphism of H , there are exactly t homomorphisms from G to H : $\pi_i \circ c, i = 1, 2, \dots, t$. (Here we use $N > k^{kt}$, see Section 3 where this has been explained in a greater detail.) Now consider mappings f_1, f_2, \dots, f_t together with an injective mapping $f_0 : A \rightarrow V(K_N)$. Then the corresponding mapping $\phi = (f_1, f_2, \dots, f_t, f_0) : A \rightarrow V(H_0)$ is injective. Thus we can identify A with its image of $\phi(A)$ and similarly we can assume that A is a subset of G (by replacing some vertices of the graph G by the set $\phi(A)$). Call the resulting graph again G . Then each of the t homomorphisms $\pi_i \circ c$ is an extension of the mapping f_i . Clearly all homomorphisms $f : G \rightarrow H$ coincide on $A = \phi(A)$ with one of the maps $f_i, i = 1, 2, \dots, t$. ■

One can ask to what extend is the projectivity a necessary condition for a validity of Corollaries 4.1.

Recent work of Tardif and Larose allows us to characterize all graphs H for which an analogy of Müller's Theorem is valid. This is non-trivial and it is based on the following notions which are introduced in [6, 7] and goes back to [1]:

A set C of vertices of a graph H is said to be *constructible* if there exists a graph G , vertices x_0, x_1, \dots, x_n of G and vertices y_1, \dots, y_n of H such that C is the set of all $g(x_0)$ where g is any homomorphism from G to H such that $g(x_i) = y_i$ for all $i = 1, \dots, n$.

Larose and Tardif [6] proved a remarkable result which states that the graph H is projective if and only if every subset of its vertex set is constructible and this is equivalent to that every two element subset of $V(H)$ is constructible.

It is interesting to note that every graph H which is H -pointed is 1-constructible. Thus 1-constructibility is close to the validity of Theorem 1.2. Perhaps surprisingly, the constructibility of 2-element sets is equivalent with the validity of Müller's theorem. We state this as:

Theorem 4.1 For a core graph H , the following statements are equivalent:

I. For any choice of a finite set A and distinct mappings $f_1, f_2, \dots, f_t : A \rightarrow V(H)$, for any positive integer l , there exists a graph $G = (V, E)$ such that the following holds:

i. A is a subset of V ;

ii. For every $i = 1, 2, \dots, t$ there exists unique homomorphism $g_i : G \rightarrow H$ such that g_i restricted to the set A coincides with the mapping f_i ;

iii. For every homomorphism $f : G \rightarrow H$ there exists $i, 1 \leq i \leq t$ and a homomorphism $h : H \rightarrow H$ such that $h \circ f_i = f$;

iv. G has girth $> l$.

II. The graph H is projective;

III. Every 2-subset of $V(H)$ is constructible.

Proof. The equivalence of **II.** and **III.** was established in [6], **II.** implies **I.** by Corollary 4.1. We prove **I.** implies **III.**:

Assume H has vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let $\{v_i, v_j\}$ be a 2-element subset of V . We need to show that $\{v_i, v_j\}$ is constructible.

Consider *i.* for $A = V^2$ with projections $f_1 = \pi_1$ and $f_2 = \pi_2$. Let G be a graph satisfying **I.** Let $x_0 = (v_i, v_j)$ and for $i = 1, 2, \dots, n$, let $x_i = (v_i, v_i)$ and $y_i = v_i$, $i = 1, \dots, n$. Then for any homomorphism $f : G \rightarrow H$ with $f(x_i) = y_i$ for $i = 1, 2, \dots, n$ holds $f(x_0) \in \{v_i, v_j\}$. Furthermore, there is a homomorphism f from G to H , namely the extension of f_1 to G , which satisfies $f(x_i) = y_i$ for $i = 1, 2, \dots, n$ and $f(x_0) = v_i$; and there is a homomorphism f from G to H , namely the extension of f_2 to G , which satisfies $f(x_i) = y_i$ for $i = 1, 2, \dots, n$ and $f(x_0) = v_j$. Thus $\{v_i, v_j\}$ is a constructible set. ■

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