# Bipartite subgraphs of triangle-free subcubic graphs

Xuding Zhu *

**Abstract**

Suppose $G$ is a graph with $n$ vertices and $m$ edges. Let $n'$ be the maximum number of vertices in an induced bipartite subgraph of $G$ and let $m'$ be the maximum number of edges in a spanning bipartite subgraph of $G$. Then $b(G) = m'/m$ is called the bipartite density of $G$, and $b^*(G) = n'/n$ is called the bipartite ratio of $G$. This paper proves that every 2-connected triangle-free subcubic graph, apart from seven exceptions, has $b(G) \geq 17/21$. Every 2-connected triangle-free subcubic graph other than the Petersen graph and the dodecahedron has $b^*(G) \geq 5/7$. The bounds that $b^*(G) \geq 5/7$ and $b(G) \geq 17/21$ are tight in the sense that there are infinitely many 2-connected triangle-free cubic graphs $G$ for which $b(G) = 17/21$ and $b^*(G) = 5/7$. On the other hand, if $G$ is not cubic (i.e., $G$ have vertices of degree at most 2), then the strict inequalities $b^*(G) > 5/7$ and $b(G) > 17/21$ hold, with a few exceptions. Nevertheless, the bounds are still sharp in the sense that for any $\epsilon > 0$, there are infinitely many 2-connected subcubic graphs $G$ with minimum degree 2 such that $b^*(G) < 5/7 + \epsilon$ and $b(G) < 17/21 + \epsilon$. The bound that $b(G) \geq 17/21$ is a common improvement of an earlier result of Bondy and Locke and a recent result of Xu and Yu: Bondy and Locke proved that every triangle-free cubic graph other than the Petersen graph and the dodecahedron has $b(G) > 4/5$. Xu and Yu confirmed a conjecture of Bondy and Locke and proved that every 2-connected triangle free subcubic graph with minimum degree 2 apart from five exceptions has $b(G) > 4/5$. The bound $b^*(G) \geq 5/7$ is a strengthening of a well-known result (first proved by Fajtlowicz and by Staton, and with a few new proofs found later) which says that any triangle-free subcubic graph $G$ has independence ratio at least 5/14. The proofs imply a linear time algorithm that finds an induced bipartite subgraph $H$ of $G$ with $|V(H)|/|V(G)| \geq 5/7$ and a spanning bipartite subgraph $H'$ of $G$ with $|E(H')|/|E(G)| \geq 17/21$.

*Keywords:* triangle-free; subcubic; independence ratio; bipartite density; bipartite ratio.
*AMS 2000 Subject Classifications:* 05C35; 05C75

## 1 Introduction

The problem of determining the maximum number of edges contained in a spanning bipartite subgraph of a given graph $G$ is called the Max-Cut problem. It has applications in VLSI, and has been studied extensively in the literature. Given a graph $G$ and an integer $m$, the problem to determine if $G$ has a bipartite subgraph $H$ with $m$ edges is NP-complete even when restricted to triangle-free cubic graphs [16]. A natural question is to find lower bounds for the number of

*Department of Applied Mathematics, National Sun Yat-sen University, and National Center for Theoretical Sciences, Taiwan. Grant number: NSC95-2115-M-110-013-MY3.

edges in a maximum bipartite subgraph of $G$. The *bipartite density* $b(G)$ of $G$ is defined as

$$b(G) = \max\{|E(B)|/|E(G)| : B \text{ is a bipartite subgraph of } G\}.$$

Erdős [4] proved that if $G$ is $2m$-colourable then $b(G) \geq \frac{m}{2m-1}$. Staton [13] and Locke [11] proved that if $G$ is cubic and $G \neq K_4$, then $b(G) \geq \frac{7}{9}$. Hopkins and Staton [9] proved that if $G$ is cubic and triangle-free then $b(G) \geq \frac{4}{5}$. Bondy and Locke [3] give a polynomial time algorithm that, for a given triangle-free cubic graph $G$, finds a bipartite subgraph $H$ of $G$ with at least $4|E(G)|/5$ edges. They further proved the following result.

**Theorem 1.1** *[3] If $G$ is a triangle-free cubic graph, then the strict inequality $b(G) > \frac{4}{5}$ holds, provided that $G$ is not the Petersen graph and not the dodecahedron.*

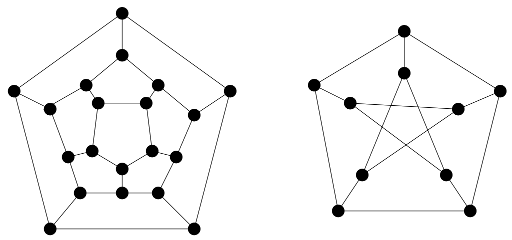

Figure 1: The dodecahedron and the Petersen graph.

A graph $G$ is called *subcubic* if the maximum degree of $G$ is at most 3. The inequality $b(G) \geq \frac{4}{5}$ applies to subcubic graphs as well. However, there are triangle-free subcubic graphs other than the Petersen graph and the dodecahedron for which the strict inequality $b(G) > \frac{4}{5}$ does not hold. Graphs $F_1, F_2, F_3, F_4, F_5$ in Figure 2 were found by Bondy and Locke. They all have bipartite density $\frac{4}{5}$. Bondy and Locke [3] conjectured that these five graphs are the only exceptions. Namely, they conjectured that if $G$ is a triangle-free subcubic graph, then $b(G) > \frac{4}{5}$, provided that $G$ is not the Petersen graph, not the dodecahedron, and $G \neq F_i$ for $i = 1, 2, 3, 4, 5$. Since Bondy and Locke have settled the case for cubic graphs, to prove this conjecture, it suffices to show it is true for connected triangle-free subcubic graphs with minimum degree 2. Xu and Yu [15] have recently settled the conjecture by proving the following result.

**Theorem 1.2** *[15] If $G$ is a connected triangle-free subcubic graph with minimum degree 2. Then $b(G) > \frac{4}{5}$, provided that $G \notin \{F_i : 1 \leq i \leq 5\}$.*

A simple proof of Theorem 1.2 is given in [17].

In this paper, we present a common improvement of Theorem 1.1 and Theorem 1.2. We may restrict ourselves to 2-connected subcubic graphs, for otherwise, $G$ has an cut-edge $e$. If $G_1, G_2$ are the two components of $G - e$, then a maximum spanning bipartite subgraph of $G$ is obtained from the union of maximum spanning bipartite subgraphs of $G_1$ and $G_2$ by adding the edge $e$.

**Theorem 1.3** *Suppose $G$ is a 2-connected triangle-free subcubic graph. Then $b(G) \geq \frac{17}{21}$, provided that $G$ is not the Petersen graph, not the dodecahedron and $G \neq F_i$ for some $1 \leq i \leq 5$. Moreover, if $G$ has minimum degree 2, then $b(G)$ is strictly larger than $\frac{17}{21}$, provided that $G \neq F_i$ for $i = 1, 2, 3, 4, 5, 8$, where $F_8$ is depicted in Figure 4.*

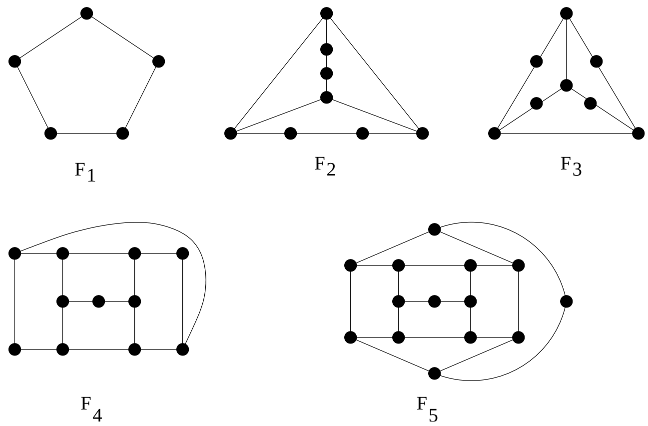


Figure 2: Triangle-free subcubic graphs with bipartite density 4/5.

For a positive integer $k$, let $\alpha_k(G)$ denote the maximum number of vertices contained in an induced $k$-colourable subgraph of $G$. The parameter $\alpha_1(G)$ is called the *independence number* of $G$ (and is usually denoted by $\alpha(G)$). The ratio $i(G) = \alpha_1(G)/|V(G)|$ is called the *independence ratio* of $G$. The independence ratio of triangle-free subcubic graphs $G$ has attracted considerable attention. By Brooks' Theorem, $G$ is 3-colourable, and hence $i(G) \geq 1/3$. Albertson, Bollobás and Tucker [1] proved that $i(G)$ is strictly larger than 1/3. Fajtlowicz [5] and Staton [14] proved that $i(G) \geq 5/14$. This bound is sharp as the generalized Petersen graph $P(7, 2)$ has 14 vertices and independence number 5. A shorter proof of the result was found by Jones [10]. Griggs and Murphy [6] designed a linear-time algorithm to find an independent set in $G$ of size at least $5(|V(G)| - k)/14$, where $k$ is the number of components of $G$ that are 3-regular. Heckman and Thomas [8] gave an even shorter proof of the inequality $i(G) \geq 5/14$ and gave a linear-time algorithm to find an independent set in $G$ of size $5|V(G)|/14$. Heckman and Thomas [8] conjectured that $G$ has fractional chromatic number at most 14/5. In other words, the conjecture says that there is a multi-set $\mathcal{K}$ of independent sets of $G$ of average size at least $5|V(G)|/14$ that evenly covers the vertices of $G$ (i.e., each vertex is contained in the same number of independent sets in $\mathcal{K}$). The conjecture is open, and the best known result in this direction is that $G$ has fractional chromatic number at most $3 - \frac{3}{64}$, which was proved by Hatami and Zhu [7].

We define the *bipartite ratio* $b^*(G)$ of $G$ as

$$b^*(G) = \alpha_2(G)/|V(G)|.$$

In this paper, we are interested in lower bounds for $b^*(G)$ for triangle-free subcubic graphs $G$. It is obvious that for any graph $G$, $b^*(G) \leq 2i(G)$. We shall prove the following result.

**Theorem 1.4** *If $G$ is a 2-connected triangle-free subcubic graph, then $b^*(G) \geq 5/7$, provided that $G$ is not the Petersen graph and not the dodecahedron. Moreover, if $G$ has minimum degree 2, then $b^*(G)$ is strictly larger than 5/7, provided that $G \neq F_5$.*

The bound is sharp in the sense that there are infinitely many triangle-free 2-connected cubic graphs $G$ with $b^*(G) = 5/7$.

The proofs in this paper imply a linear time algorithm that finds, for any triangle-free subcubic graph $G$ other than the few exceptions, an induced bipartite subgraph $H$ of $G$ with $|V(H)|/|V(G)| \geq 5/7$, and a spanning bipartite subgraph $H$ of $G$ with $|E(H)|/|E(G)| \geq 17/21$.

## 2 A technical result

Both Theorem 1.3 and Theorem 1.4 are consequences of a more technical result. Suppose $G$ is a triangle-free subcubic graph and $H$ is a maximum induced bipartite subgraph of $G$. Intuitively, the more vertices $G$ has, the more vertices $H$ has. However, the contribution of each vertex of $G$ to the number of vertices of $H$ is different. A vertex $x$ of degree $i$ is called an *$i$-vertex*. It is obvious that each 0-vertex and each 1-vertex of $G$ contributes 1 vertex to $H$ (if $x$ is a 0-vertex or a 1-vertex of $G$, then we must have $x \in V(H)$ and $H - x$ is a maximum induced bipartite subgraph of $G - x$). It turns out that in general, each 2-vertex of $G$ contributes at least $\frac{6}{7}$ vertices to $H$ and each 3-vertex of $G$ contributes at least $\frac{5}{7}$ vertices to $H$. Let $n_i(G)$ (abbreviated as $n_i$ if the graph $G$ is clear from the context) be the number of $i$-vertices of $G$. Let

$$\sigma(G) \quad = \quad (5n_3 + 6n_2 + 7n_1 + 7n_0)/7.$$

Our main result in this paper says that in general, $\alpha_2(G) \geq \sigma(G)$. However, this general rule has a few exceptions. An error term needs to be added to the inequality $\alpha_2(G) \geq \sigma(G)$. Here is our technical result:

**Theorem 2.1** *If $G$ is a triangle-free subcubic graph and each connected component of $G$ has a vertex of degree at most 2, then $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.*

The parameter $\epsilon(G)$ is the error term, which we have not defined yet. To define this error term, we need to construct a few families of graphs.

First of all, let

$$\mathcal{G}_1 = \{F_i : 1 \leq i \leq 5\}.$$

Starting from $\mathcal{G}_1$, we construct three other classes of graphs through some graph operations. Figure 3 below defines eleven graph operations.

In the figures, an unfilled circle indicates a vertex of $G$ of degree at most 2. A filled circle indicates an arbitrary vertex of $G$. The filled squares are added vertices. A broken line is a deleted edge of $G$. A solid line indicates an edge. A solid line with two backslashes on it indicates that it is a non-edge (i.e., the two vertices at the end of this line are not adjacent). Those solid lines incident to added vertices are added edges, and the other solid lines are original edges of $G$. All the edges incident to added vertices are shown in the figures. However, for original vertices of $G$, not all edges incident to them are shown in the figures.

For example, $G \circ_1 (x, y, z)$ is obtained from $G$ by adding vertices $a, b, c$ and adding edges $ax, ab, bz, bc, cy$, where $x, y, z$ have degree at most 2 in $G$. It is possible that there are some edges connecting vertices $x, y, z$. The graph $G \circ_2 (u, v, x, y)$ is obtained from $G$ by adding vertices $a, b, c$ and edges $au, ab, av, bx, bc, cy$ and deleting the edge $xy$, where $u, v$ are two nonadjacent 2-vertices of $G$. It is possible that there are edges between $\{x, y\}$ and $\{u, v\}$. On the other hand, $xy$ must

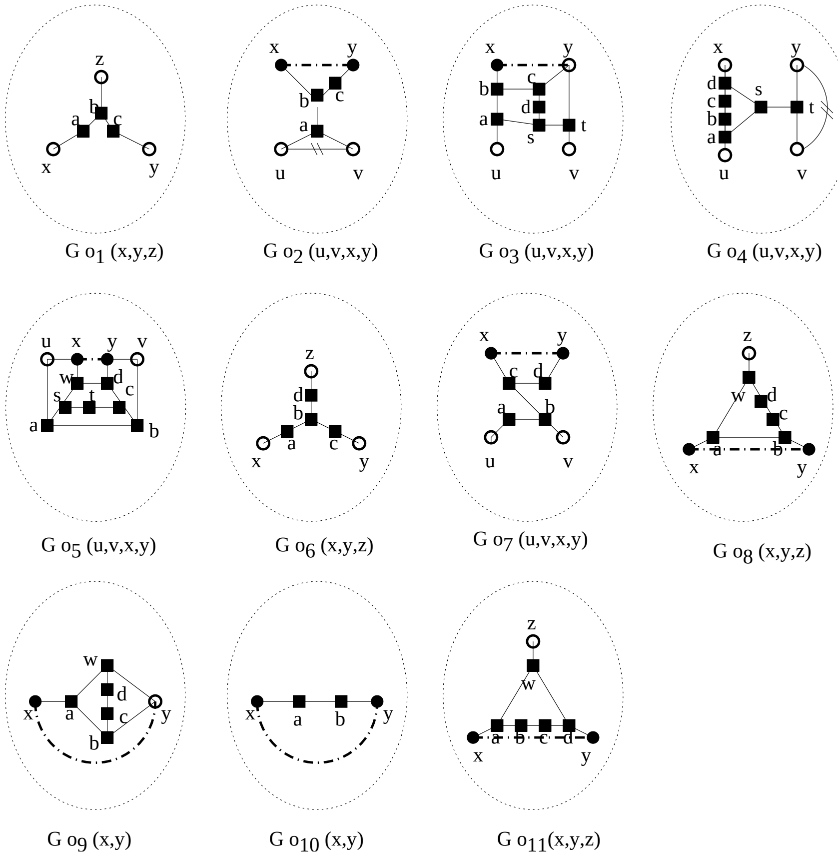


Figure 3: The graph operations $\circ_i$ for $i = 1, 2, \cdots, 11$.

be an edge of $G$ and it is deleted in the operation. In $G \circ_5 (u, v, x, y)$, $ux, yv$ are edges of $G$. For $i = 1, 2, \cdots, 11$, we shall denote by $G\circ_i$ any graph obtained from $G$ by applying the operation $\circ_i$.

For a graph $G$, $\mathcal{B}(G)$ denotes the family of all maximum induced bipartite subgraphs of $G$. Let $d_1 = d_2 = d_{10} = 2, d_6 = d_7 = 3, d_3 = d_4 = d_8 = d_9 = d_{11} = 4, d_5 = 5$.

**Lemma 2.2** *If $G'$ is a triangle-free subcubic graph and $G = G'\circ_i$ for some $i \in \{1, 2, \cdots, 11\}$, then $G$ is a triangle-free subcubic graph and $\alpha_2(G) \geq \alpha_2(G') + d_i$.*

*Proof.* It follows from the definition that the graphs $G$ are triangle-free subcubic. If $G = G'\circ_i$ and $H \in \mathcal{B}(G')$ then it is easy to verify that $H + A_i$ is an induced bipartite subgraph of $G$, where $A_1 = \{a, c\}, A_2 = \{a, c\}, A_3 = \{b, c, d, s\}, A_4 = \{s, a, b, c\}, A_5 = \{a, c, d, t, w\}, A_6 = \{a, c, d\}, A_7 = \{a, c, d\}, A_8 = \{a, b, c, d\}, A_9 = \{a, b, c, d\}, A_{10} = \{a, b\}, A_{11} = \{a, b, c, d\}$ (refer to Figure 3). As $|A_i| = d_i$, we conclude that $\alpha_2(G) \geq \alpha_2(G') + d_i$. ∎

If $G = G'\circ_i$ and $\alpha_2(G) = \alpha_2(G') + d_i$, then we write $G = G'\circ_i^*$. For example, $G = G'\circ_1^*$ means that $G = G' \circ_1 (x, y, z)$ for some 2-vertices $x, y, z$ of $G'$ and $\alpha_2(G) = \alpha_2(G') + 2$, and $G = G'\circ_5^*$ means that $G = G' \circ_5 (u, v, x, y)$ for some vertices $u, v, x, y$ of $G'$ and $\alpha_2(G) = \alpha_2(G') + 5$.

**Definition 2.3** *The graph class $\mathcal{G}_2$ is defined recursively as follows:*

- *If $G' \in \mathcal{G}_1$, $i \in \{5,6,7\}$ and $G = G' \circ_i^*$, then $G \in \mathcal{G}_2$.*
- *If $G' \in \mathcal{G}_2$, $i \in \{1,2,3,4\}$ and $G = G' \circ_i^*$, then $G \in \mathcal{G}_2$.*

**Definition 2.4** *The graph class $\mathcal{G}_3$ is defined recursively as follows:*

- *If $G' \in \mathcal{G}_1$, $i \in \{8,9,10,11\}$ and $G = G' \circ_i^*$, then $G \in \mathcal{G}_3$.*
- *If $G' \in \mathcal{G}_2$, $i \in \{5,6,7\}$ and $G = G' \circ_i^*$, then $G \in \mathcal{G}_3$.*
- *If $G' \in \mathcal{G}_3$, $i \in \{1,2,3,4\}$ and $G = G' \circ_i^*$, then $G \in \mathcal{G}_3$.*

Each of the classes $\mathcal{G}_2, \mathcal{G}_3$ is finite. To see this, it suffices to observe that the graph operations $\circ_1, \circ_2, \circ_3, \circ_4$ cannot be applied repeatedly infinitely many times. This is so because if $G = G' \circ_i$ for some $i \in \{1,2,3,4\}$, then $n_2(G) \leq n_2(G') - 1$. It is easy to show that $n_2(G') \leq 5$ for $G' \in \mathcal{G}_2$ and $n_2(G') \leq 7$ for $G' \in \mathcal{G}_3$. Hence $\mathcal{G}_2$ and $\mathcal{G}_3$ are finite.

Indeed, the graphs in $\mathcal{G}_i$ $(i = 1,2,3)$ have very nice structure. It can be verified easily that $\mathcal{G}_2$ contains only four graphs shown in Figure 4.

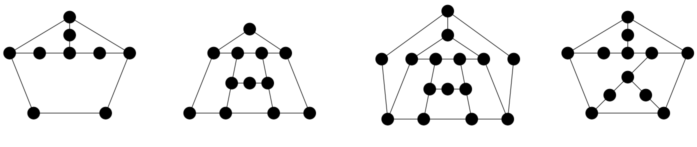


Figure 4: The graphs $F_i$ for $6 \leq i \leq 9$.

The class $\mathcal{G}_3$ is larger. It contains 61 graphs. This class of graphs can also be constructed manually. For a plane graph $G$, let $F(G)$ be the set of faces of $G$. For each face $f \in F(G)$, the *degree* $d(f)$ of $f$ is the number of edges on its boundary. Let $\mathcal{P}$ be the set of 2-connected triangle-free subcubic plane graphs $G$ with minimum degree 2. The computer verification shows that the following hold (this result is not needed for the proof of the other results in this paper).

$$\begin{aligned}
\mathcal{G}_1 &= \{G \in \mathcal{P} : \forall f \in F(G), d(f) = 5\}, \\
\mathcal{G}_2 &= \{G \in \mathcal{P} : \forall f \in F(G), d(f) = 5, \text{ except that one face } f \text{ has } d(f) = 7\}, \\
\mathcal{G}_3 &\subseteq \{G \in \mathcal{P} : \forall f \in F(G), d(f) = 5, \text{ except that one face } f \text{ has } d(f) = 9, \\
&\quad \text{or two faces } f_1, f_2 \text{ have } d(f_1) = d(f_2) = 7\}.
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{Q} &= \{G \in \mathcal{P} : \forall f \in F(G), d(f) = 5, \text{ except that one face } f \text{ has } d(f) = 9, \\
&\quad \text{or two faces } f_1, f_2 \text{ have } d(f_1) = d(f_2) = 7\}.
\end{aligned}$$

A few graphs in $\mathcal{Q}$ are not contained in $\mathcal{G}_3$. These graphs belong to the next class which we construct now.

**Definition 2.5** *Suppose $G_0, G_1, \cdots, G_{k-1} \in \mathcal{G}_1$, where $k \geq 1$. For $0 \leq i \leq k-1$, let $a_i, b_i$ be two distinct 2-vertices of $G_i$. We denote by $C(\{(G_i, a_i, b_i) : 0 \leq i \leq k-1\})$ the graph obtained from the disjoint union $G_0, G_1, \cdots, G_{k-1}$ by adding edges $b_i a_{i+1}$ for $i = 1, 2, \cdots, k$ (summation in indices is modulo $k$). The graph $G = C(\{(G_i, a_i, b_i) : 0 \leq i \leq k-1\})$ is called an $F$-cycle. The graphs $G_1, G_2, \cdots, G_k$ are called the $F$-*subgraphs *of $G$, and the vertices $a_i, b_i$ are called the* join vertices *of $G$. The edges $b_i a_{i+1}$ are called the* join edges *of $G$.*

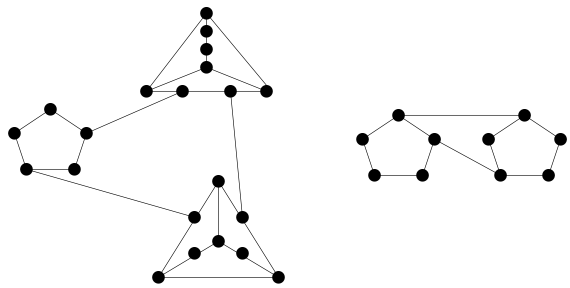

Figure 5: Two $F$-cycles.

Let $D$ be the graph obtained from the dodecahedron by subdividing one edge into a path of length 3. Let $\mathcal{F}$ be the family of $F$-cycles. The family of $F$-cycles is an infinite family and its members are not always planar graphs. Note that if $k = 1$, then the $F$-cycle $C(G_0)$ is obtained from $G_0$ by adding one edge connecting two distinct 2-vertices of $G_0$. Figure 5 above shows two examples of $F$-cycles. The second one belong to $\mathcal{Q}$.

Let

$$\begin{aligned} \mathcal{G}'_2 &= \mathcal{G}_2 \cup \{K_1\}, \\ \mathcal{G}'_3 &= \mathcal{G}_3 \cup \mathcal{F} \cup \{D\}. \end{aligned}$$

Suppose $G$ is a triangle-free subcubic graph. Let $G'$ be obtained from $G$ by deleting all cut-edges of $G$. Each connected component of $G'$ is called a *piece* of $G$. So each piece of $G$ is either a block of $G$ containing a cycle (and hence has at least 4 vertices) or a single vertex. If $P$ is a piece of $G$ and $x$ is a cut vertex of $G$ contained in $P$, then $x$ is called a *join vertex* of $P$.

Suppose $P$ is a piece of $G$. Let

$$\epsilon(P) = \begin{cases} -2/7, & \text{if } P \in \mathcal{G}_1, \\ -1/7, & \text{if } P \in \mathcal{G}'_2, \\ 0, & \text{if } P \in \mathcal{G}'_3, \\ 1/7, & \text{otherwise.} \end{cases}$$

Let $\beta(G)$ be the number of cut-edges of $G$. Now we are ready to define the error term $\epsilon(G)$:

$$\epsilon(G) = (2\beta(G) + n_0(G))/7 + \sum \epsilon(P),$$

where the summation is taken over all the pieces $P$ of $G$.

To get an intuition of the error term, we may assume $G$ is connected (otherwise, the error term is just the summation of the error terms of whose connected components). If all the pieces of $G$ are graphs from $\mathcal{G}_1$, then $\epsilon(G) = -2/7$, i.e., there is a deficit of 2/7. If all the pieces of $G$ are graphs from $\mathcal{G}_1$, except that one piece is a graph from $\mathcal{G}_2'$, then there is a deficit of 1/7. If all the pieces of $G$ are graphs from $\mathcal{G}_1$, except that one piece is a graph from $\mathcal{G}_3'$ or two pieces are graphs from $\mathcal{G}_2'$, then there is no deficit and also no surplus. For all other graphs $G$, there is a surplus of 1/7.

The reason that we need this surplus is that Theorem 2.1 applies only to those subcubic graphs each of its connected components has a vertex of degree at most 2. It does not apply to cubic graphs. To get the required results for cubic graphs, we shall consider subgraphs of cubic graphs. For that purpose, we need this surplus. Indeed, a lot of efforts is made to get this surplus. If we do not need this surplus, then we do not need to have the graph class $\mathcal{G}_3'$, and the proofs will be easier. But then the conclusion applies only to connected subcubic graphs with minimum degree at most 2.

# 3 Some preliminary lemmas

The proof of Theorem 2.1 is non-trivial. In this section, we list a few lemmas that will be needed in its proof.

Suppose $H$ is an induced subgraph of $G$ and $X$ is a subset of $V(G)$. Then $H + X$ and $H - X$ denote the subgraph of $G$ induced by $V(H) \cup X$ and $V(H) - X$, respectively. If $H_1, H_2$ are two induced subgraphs of $G$, then $H_1 + H_2$ denotes the subgraph of $G$ induced by $V(H_1) \cup V(H_2)$.

The following observation is trivial.

**Observation 3.1** *If $G_1, G_2, \cdots, G_k$ are the pieces of $G$ and $H_j \in \mathcal{B}(G_j)$, then $H_1+H_2+\cdots+H_k \in \mathcal{B}(G)$. Moreover, every $H \in \mathcal{B}(G)$ is of this form. Assume $G$ is an $F$-cycle, and $G_1, G_2, \cdots, G_k$ are the $F$-subgraphs of $G$, and $H_i \in \mathcal{B}(G_i)$ for $i = 1, 2, \cdots, k$. Let $H = H_1 + H_2 + \cdots + H_k$. If some join vertex of $G$ is not contained in $H$, then $H \in \mathcal{B}(G)$.*

**Lemma 3.2** *Suppose $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ and $e$ is an edge of $G$. Then $\alpha_2(G - e) = \alpha_2(G) + 1$. As a consequence, for any $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{F}$, for any vertex $w$ of $G$, there is an $H \in \mathcal{B}(G)$ such that $w \notin V(H)$.*

*Proof.* If $G \in \mathcal{G}_1$, this can be verified directly. Suppose the lemma is true for $G'$ and $G = G' \circ_i^*$ for some $1 \leq i \leq 11$. Let $e$ be an edge of $G$. We shall show that $\alpha_2(G-e) = \alpha_2(G)+1$. It is obvious that $\alpha_2(G - e) \leq \alpha_2(G) + 1$. In the following, we shall show that $\alpha_2(G - e) \geq \alpha_2(G) + 1$. If $e$ is an edge of $G'$, then $\alpha_2(G' - e) = \alpha_2(G') + 1$. Let $H \in \mathcal{B}(G' - e)$. Then $H + A_i$ induces a bipartite subgraph of $G - e$. Hence $\alpha_2(G - e) \geq \alpha_2(G' - e) + d_i = \alpha_2(G) + 1$. It remains to consider the case that $e$ is one of the added edges. We need to check separately for each $i \in \{1, 2, \cdots, 11\}$ and for each of the added edges. There are many cases to check, but each of the checks is easy. We shall consider two cases to show how the induction hypothesis is used in the proof. Consider the case that $i = 1$. Without loss of generality, we may assume that $e \in \{ax, ab\}$. By induction hypothesis, there exists $H \in \mathcal{B}(G')$ be a such that $y \notin V(H)$. Then $H + \{a, b, c\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G - e) = \alpha_2(G') + 3 = \alpha_2(G) + 1$. As another example, consider the case that $i = 5$. Let $e' = xy$. By induction hypothesis, $\alpha_2(G' - e') = \alpha_2(G') + 1$. Let $H \in \mathcal{B}(G' - e')$. If $e = wd$, then $H + \{w, d, c, t, a\}$ induces a bipartite subgraph of $G - e$. If $e = dc$, then $H + \{b, c, d, t, s\}$ induces a bipartite subgraph of $G - e$. If $e = ct$, then $H + \{w, s, t, c, b\}$ induces a bipartite subgraph of $G - e$. If $e = ab$, then $H + \{a, b, s, t, d\}$ induces a bipartite subgraph of $G - e$. In any case, $\alpha_2(G - e) \geq \alpha_2(G' - e) + 5 = \alpha_2(G') + 6 = \alpha_2(G) + 1$.

The conclusion for graphs in $\mathcal{F}$ follows from Observation 3.1 and the fact that the conclusion holds for its $F$-subgraphs. ∎

An *end-piece* of $G$ is a piece incident to at most one cut-edge. A *pseudo end-piece* is either an end-piece or the union of an end piece $P$ with a neighbouring piece $P'$ (i.e., $P'$ is connected to $P$ by a cut-edge of $G$) so that $P \cup P'$ is incident to at most one other cut-edge. If $P$ is a pseudo end-piece and $G$ has a cut edge $e$ which has exactly one end vertex in $P$, then that end vertex is referred to as the join vertex of $P$.

**Lemma 3.3** *Suppose $G$ is a 2-connected triangle-free subcubic graph $G$ and $G' = (G - X) + xy$, where $X \subseteq V(G)$ and $x, y \in V(G) - X$. If $N_G(X) - X \subseteq \{u, v, x, y\}$ ($N_G(X)$ is the set of neighbours of $X$), then at least one of the following holds:*

1. *Each of $u, v$ is contained in a distinct end-piece of $G'$.*
2. *At least one of $u, v$ is contained in an end-piece, and $x, y$ are contained in the same pseudo end-piece of $G'$.*

*Proof.* If $G'$ is 2-connected, then there is only one piece, so $x, y$ are in the same pseudo end-piece of $G'$. Otherwise, $G'$ has at least two distinct end pieces. As $G$ is 2-connected, each end-piece of $G'$ contains at least one vertex from the set $\{u, v, x, y\}$. If each of $u, v$ is contained in a distinct end-piece of $G'$, then we are done. Assume $u, v$ do not belong to distinct end-pieces. Then at least one of $x, y$ is in an end-piece. If $x, y$ are in the same end-piece, we are done. Otherwise, say $y$ is in an end-piece $P$ and $x$ is not, then $e = xy$ is the only cut-edge of $G'$ incident to $P$. Let $P'$ be the piece containing $x$. If the union $P \cup P'$ is incident to more than one cut-edge of $G'$, then $G'$ has at least two more end-pieces. This is a contradiction, as each other end-piece of $G'$ must contain $u$ or $v$. Therefore $P \cup P'$ is incident to at most one cut-edge of $G'$, and hence $P \cup P'$ is a pseudo end-piece. If $P \cup P' = G'$, then $u, v$ are contained in end-pieces. Otherwise, there is another end-piece, which contains at least one of $u, v$. ∎

Suppose $H$ is a bipartite subgraph of $G$. For two subsets $X, Y$ of $G$, we write $X \bowtie_H Y$ if $X \cap V(H)$ is a subset of one partite set of $H$ and $Y \cap V(H)$ is a subset of another partite set of $H$. Note that even if $X, Y$ are non-empty sets, it is possible that one or both of $X \cap V(H)$ and $Y \cap V(H)$ are empty sets. If $X = \{x\}$ and $Y = \{y\}$, then we write $x \bowtie_H y$ instead of $\{x\} \bowtie_H \{y\}$. If $Y = \emptyset$, then $X \bowtie_H Y$ simply means that $X \cap V(H)$ is contained in the same partite set of $H$.

## 4 Proof of Theorem 2.1

Assume Theorem 2.1 is not true and $G$ is a minimum counterexample. We shall derive a sequence of properties of $G$ that finally leads to a contradiction.

**Lemma 4.1** *The graph $G$ is 2-connected.*

*Proof.* If $G$ is not connected and $G_1, G_2, \cdots, G_k$ ($k = c(G) \geq 2$) are the connected components of $G$, then $\alpha_2(G) = \sum_{j=1}^{k} \alpha_2(G_j)$ and $\sigma(G) = \sum_{j=1}^{k} \sigma(G_j)$ and $\epsilon(G) = \sum_{j=1}^{k} \epsilon(G_j)$.

If $G$ is connected but not 2-connected, then since $G$ is subcubic, $G$ has a cut-edge $e = xy$. Let $G' = G - e$. It is obvious that $\alpha_2(G) = \alpha_2(G')$. Also $G$ and $G'$ have the same set of pieces. If $x$ is a 1-vertex of $G$, then compared to $G'$, $G$ has one more cut-edge, one less 0-vertex, and $d_G(y) = d_{G'}(y) + 1 \geq 2$. Therefore $\sigma(G) = \sigma(G') - 1/7$, $\epsilon(G) = \epsilon(G') + 1/7$. If none of $x, y$ is a 1-vertex, then $d_G(x) = d_{G'}(x) + 1 \geq 2$ and $d_G(y) = d_{G'}(y) + 1 \geq 2$. Thus $\sigma(G) = \sigma(G') - 2/7$ and $\epsilon(G) = \epsilon(G') + 2/7$. In any case, $\sigma(G) + \epsilon(G) = \sigma(G') + \epsilon(G')$. By our choice of $G$, we have $\alpha_2(G') \geq \sigma(G') + \epsilon(G')$. Therefore $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. ∎

**Lemma 4.2** *No two 2-vertices of $G$ are adjacent.*

*Proof.* Assume to the contrary that $u, v$ are two adjacent 2-vertices. Let $x, y$ be the other neighbour of $u$ and $v$, respectively. If $x$ and $y$ have no common neighbour and $xy$ is not an edge of $G$, then let $e = xy$ and let $G' = (G - \{u, v\}) + e$. Then $G = G' \circ_{10} (x, y)$. Straightforward counting shows that $\sigma(G) = \sigma(G') + 12/7$. By Lemma 2.2, $\alpha_2(G) \geq \alpha_2(G') + 2$. Hence

$$\alpha_2(G) \geq \sigma(G') + \epsilon(G') + 2 = \sigma(G) + \epsilon(G') + 2/7.$$

Since $G$ is 2-connected, it follows that $G'$ is 2-connected. If $G'$ is the dodecahedron, then $\epsilon(G) = 0$ and it can be verified directly that $\alpha_2(G) = \sigma(G) = \sigma(G) + \epsilon(G)$. If $G'$ is cubic but not the dodecahedron, then by checking each of the graphs in $\mathcal{G}_i$ for $i = 1, 2, 3$, it can be verified that there is a vertex $w \in G'$ such that $G' - w \notin \mathcal{G}_i$ for $i = 1, 2, 3$ and $G' - w$ is not an $F$-cycle. Therefore $\alpha_2(G' - w) \geq \sigma(G' - w) + \epsilon(G' - w) = \sigma(G' - w) + 1/7$. Since $\sigma(G') = \sigma(G' - w) + 2/7$, we conclude that $\alpha_2(G') \geq \alpha_2(G' - w) \geq \sigma(G') - 1/7$. Therefore $\alpha_2(G) = \alpha_2(G') + 2 \geq \sigma(G) + 1/7$. Assume $G'$ has minimum degree 2. If $G' \in \mathcal{G}_1$, then $G \in \mathcal{G}_3$. Hence $\epsilon(G) = \epsilon(G') + 2/7$. If $G' \notin \mathcal{G}_1$ then $\epsilon(G') \geq -1/7$ and $\epsilon(G) \leq 1/7$. In any case, we have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. If $e$ is already an edge of $G$, then let $G' = G - \{u, v\}$. As $\sigma(G) = \sigma(G') + 10/7$ and $\alpha_2(G) = \alpha_2(G') + 2$, we have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

If $x$ and $y$ have a common neighbour $w$, but one of $\{x, y, w\}$, say $w$, is a 2-vertex, then let $G' = G - \{x, u, v, y, w\}$. Since $G \neq F_1$, $G'$ is not empty. Then $\sigma(G) = \sigma(G') + 26/7$. For $H \in \mathcal{B}(G')$, $H + \{x, u, v, w\}$ induces a bipartite subgraph of $G$. So

$$\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + \epsilon(G') + 4 = \sigma(G) + \epsilon(G') + 2/7.$$

If $\epsilon(G') = -2/7$, then by definition, each piece of $G'$ is a graph in $\mathcal{G}_1$. Since $G$ is 2-connected, it follows that $G$ is an $F$-cycle, and hence $\epsilon(G) = 0$. Therefore $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. If $\epsilon(G') \geq -1/7$, then since $\epsilon(G) \leq 1/7$, we also have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

In the following, we assume that $x$ and $y$ have a common neighbour $w$ and $x, y, w$ are 3-vertices. We divide the argument into two cases.

**Case 1** $x, y$ have another common neighbour $a$, as shown in Figure 6.

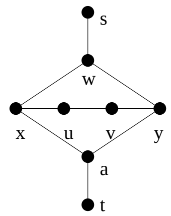


Figure 6: Two adjacent 2-vertices.

Let $s, t$ be the other neighbour of $w, a$, respectively. If $st$ is not an edge, then let $G' = (G - \{w, x, u, v, y\}) + sa$. Then $G = G' \circ_9 (a, s)$. Since $G$ is 2-connected, it follows that $G'$ is 2-connected. Straightforward counting shows that $\sigma(G) - \sigma(G') = 26/7$. By Lemma 2.2,

$$\alpha_2(G) = \alpha_2(G') + 4 \geq \sigma(G') + \epsilon(G') + 4 = \sigma(G) + \epsilon(G') + 2/7.$$

If $\epsilon(G') \geq -1/7$, then since $\epsilon(G) \leq 1/7$, we have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Assume $\epsilon(G') = -2/7$, then $G' \in \mathcal{G}_1$ (as $G'$ is 2-connected). Hence $G \in \mathcal{G}_3$. So $\epsilon(G) = 0$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

If $st$ is an edge, then since $G \neq F_2$, $G$ has other vertices. Let $G' = G - \{s, t, w, a, x, u, v, y\}$. Let $s', t'$ be the neighbour of $s, t$, respectively in $G'$. Then $G'$ is connected and $\sigma(G) = \sigma(G') + 40/7$. For $H \in \mathcal{B}(G')$, $H + \{s, w, x, u, v, a\}$ induces a bipartite subgraph of $G$. So $\alpha_2(G) \geq \alpha_2(G') + 6$. If $\epsilon(G') \geq -1/7$, then since $\epsilon(G) \leq 1/7$, we have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. If $\epsilon(G') = -2/7$, then by definition, each piece of $G'$ is in $\mathcal{G}_1$. Since $G$ is 2-connected, we conclude that $G$ is an $F$-cycle and $\epsilon(G) = 0$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

**Case 2** $x, y$ have no other common neighbour. Let $a, b, s$ be the other neighbour of $x, y, w$, respectively (as shown in Figure 7).

**Case 2(a)** Two of the vertices $a, b, s$ have no common neighbour.

Assume $a, b$ have no common neighbour (the cases $a, s$ have no common neighbour or $b, s$ have no common neighbour are proved similarly). Then let $G' = (G - \{x, u, v, y, w\}) + ab$. Then $G = G' \circ_{11} (a, b, s)$.

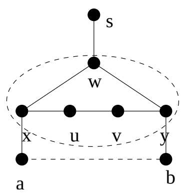


Figure 7: The graph $G' = (G - \{x, u, v, y, w\}) + ab$.

We have $\sigma(G) - \sigma(G') = 26/7$. By Lemma 2.2, $\alpha_2(G) \geq \alpha_2(G') + 4$. If $\epsilon(G') \geq -1/7$, then since $\epsilon(G) \leq 1/7$, we have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Assume $\epsilon(G') = -2/7$. By definition, each piece of $G'$ is a graph in $\mathcal{G}_1$. If $G'$ is 2-connected, then $G' \in \mathcal{G}_1$. So either $\alpha_2(G) = \alpha_2(G') + 5 \geq \sigma(G) + \epsilon(G)$, or $G \in \mathcal{G}_3$, $\epsilon(G) = 0$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Thus we assume that $G'$ is not 2-connected. Since $G$ is 2-connected, we conclude that $a, b, s$ do not belong to the same piece of $G'$. If $ab$ is a cut-edge of $G'$, then $a$ or $b$ is a cut vertex of $G$, in contrary to the assumption that $G$ is 2-connected. So $a, b$ belong to the same piece of $G'$. Let $Q$ be the piece of $G'$ containing $a, b$ and let $e = ab$. By Lemma 3.2, $\alpha_2(Q - e) = \alpha_2(Q) + 1$. This implies that $\alpha_2(G' - e) = \alpha_2(G') + 1$. Since $s$ is in another piece of $G'$, by Lemma 3.2, there is an $H \in \mathcal{B}(G' - e)$ not containing $s$. So $H + \{w, x, u, v\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) = \alpha_2(G' - e) + 4 = \alpha_2(G') + 5 \geq \sigma(G) + \epsilon(G)$.

**Case 2(b)** Every two vertices of $a, b, s$ have a common neighbour.

If $s$ is a 3-vertex, then let $s', s''$ be the other two neighbours of $s$ and let $G' = G - \{s, w, x, u, v, y\}$. Then $G = G' \circ_4 (a, s', b, s'')$, and $\sigma(G) = \sigma(G') + 28/7$, $\alpha_2(G) \geq \alpha_2(G') + 4$. If $\alpha_2(G) = \alpha_2(G') + 4$, then $\epsilon(G) = \epsilon(G')$ and hence $\alpha_2(G) = \alpha_2(G') + 4 \geq \sigma(G') + \epsilon(G') + 4 = \sigma(G) + \epsilon(G)$. Otherwise, $\alpha_2(G) = \alpha_2(G') + 5 \geq \sigma(G) + \epsilon(G)$.

Assume $s$ is a 2-vertex. Then $a, b, s$ has a common neighbour $z$, as depicted in Figure 8.

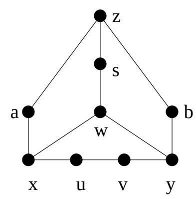


Figure 8: The case that $a, b, s$ have a common neighbour.

If $G$ has no other vertices, then $G \in \mathcal{G}_2$ and $\epsilon(G) = -1/7$, and $\alpha_2(G) = \sigma(G) + \epsilon(G)$. Assume $G$ has other vertices. Since $G$ is 2-connected and $s$ is a 2-vertex, we conclude that $a, b$ are 3-vertices. Let $G' = G - \{z, a, s, b, w, x, u, v, y\}$. Then $\sigma(G) - \sigma(G') = 46/7$. For $H \in \mathcal{B}(G')$, $H + \{a, z, s, w, y, v, u\}$ induces a bipartite subgraph of $G$. So

$$\alpha_2(G) \geq \alpha_2(G') + 7 \geq \sigma(G') + \epsilon(G') + 7 = \sigma(G) + \epsilon(G') + 3/7.$$

As $G$ is 2-connected, it follows that $G'$ is connected. By definition, $\epsilon(G') \geq -2/7$. As $\epsilon(G) \leq 1/7$, we have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. ∎

**Lemma 4.3** *No 3-vertex is adjacent to three 2-vertices.*

*Proof.* Assume a 3-vertex $x$ is adjacent to three 2-vertices $a, b, c$. Let $a', b', c'$ be the other neighbour of $a, b, c$, respectively. Let $G' = G - \{a, b, c, x\}$. Then $G = G' \circ_6 (a', b', x')$ and $\sigma(G) = \sigma(G') + 20/7$. By Lemma 2.2, $\alpha_2(G) \geq \alpha_2(G') + 3$. If $\epsilon(G') \geq 0$, then $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

Assume $\epsilon(G') \leq -1/7$. Then each piece of $G'$ is a graph in $\mathcal{G}_1 \cup \mathcal{G}_2$. If $G'$ is 2-connected, then $G' \in \mathcal{G}_i$ for some $1 \leq i \leq 2$. By definition, either $\alpha_2(G) = \alpha_2(G') + 4 \geq \sigma(G) + \epsilon(G)$ or $G \in \mathcal{G}_{i+1}$ and $\epsilon(G) = \epsilon(G') + 1/7$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Assume $G'$ is not 2-connected. Then $G'$ has at least two pieces, and hence at least two of the vertices $a', b', c'$ belong to distinct pieces. By Lemma 3.2, there is an $H \in \mathcal{B}(G')$ which contains at most one of $a', b', c'$. Therefore $H + \{x, a, b, c\}$ induces a bipartite subgraph of $G$ and hence $\alpha_2(G) = \alpha_2(G') + 4 \geq \sigma(G) + \epsilon(G)$. ∎

**Lemma 4.4** *No* 3*-vertex is adjacent to two* 2*-vertices.*

*Proof.* Assume a 3-vertex $x$ is adjacent to two 2-vertices $a, b$. Let $a', b', x'$ be the other neighbour of $a, b, x$, respectively. By Lemma 4.2 and Lemma 4.3, $a', b', x'$ are 3-vertices. Let $G' = G - \{a, b, x\}$. Then $G = G' \circ_1 (a', b', x')$ and $\sigma(G) = \sigma(G') + 2$. By Lemma 2.2, $\alpha_2(G) \geq \alpha_2(G') + 2 \geq \sigma(G') + \epsilon(G') + 2 = \sigma(G) + \epsilon(G')$.

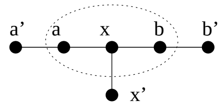


Figure 9: A 3-vertex adjacent to two 2-vertices.

If $\epsilon(G') \geq 1/7$ then $\epsilon(G) \leq \epsilon(G')$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Assume $\epsilon(G') \leq 0$. Then each piece of $G'$ is a graph in $\mathcal{G}_1$, except that at most one piece which is in $\mathcal{G}'_3$, or at most two pieces that are in $\mathcal{G}'_2$.

If $G'$ has more than one piece, then since $G$ is 2-connected, at least two of $a', b', x'$ belong to two distinct end-pieces. Since $a', b', x'$ are 3-vertices, the end-pieces of $G'$ are not singletons. Therefore, the end-pieces are graphs in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ or is an $F$-cycle. By applying Lemma 3.2 to the end-pieces, we conclude that there is an $H \in \mathcal{B}(G')$ which contains at most one of $a', b', x'$. Hence $H + \{a, b, x\} \in \mathcal{B}(G)$ and $\alpha_2(G) = \alpha_2(G') + 3 \geq \sigma(G) + \epsilon(G)$.

Assume $G'$ has only one piece, i.e., $G'$ is 2-connected. Then $G' \in \mathcal{G}_i$ for some $1 \leq i \leq 3$, or $G'$ is an $F$-cycle. If $G' \in \mathcal{G}_i$ for some $1 \leq i \leq 3$, then either $\alpha_2(G) = \alpha_2(G') + 3 \geq \sigma(G) + \epsilon(G)$ or $G \in \mathcal{G}_i$, $\epsilon(G) = \epsilon(G')$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Assume $G'$ is an $F$-cycle. If $a', b', x'$ do not belong to the same $F$-subgraph of $G'$, then by Lemma 3.2 and Observation 3.1, there is an $H \in \mathcal{B}(G')$ which contains at most one of $a', b', x'$. Hence $H + \{a, b, x\} \in \mathcal{B}(G)$ and $\alpha_2(G) = \alpha_2(G') + 3 \geq \sigma(G) + \epsilon(G)$. If $a', b', x'$ belong to the same $F$-subgraph of $G'$, then apply the argument in the previous paragraph to this $F$-subgraph, we conclude that $G$ is an $F$-cycle. Hence $\epsilon(G) = \epsilon(G')$ and $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. ∎

**Lemma 4.5** *No* 4*-cycle contains a* 2*-vertex.*

*Proof.* Assume $(a, b, c, d)$ is a 4-cycle and $a$ is a 2-vertex. By Lemmas 4.2 and 4.4, $b, c, d$ are 3-vertices. Let $G' = G - \{a, b, c, d\}$. Then $\sigma(G) = \sigma(G') + 18/7$. Let $b', c', d'$ be the other neighbour of $b, c, d$, respectively. Suppose $H \in \mathcal{B}(G')$. If one of $b', c', d' \notin V(H)$, say $b' \notin V(H)$, then $H + \{a, b, c\}$ induces a bipartite subgraph of $G$. Assume $b', c', d' \in V(H)$. If $b', d'$ are in the same partite set of $H$, then $H + \{d, a, b\}$ induces a bipartite subgraph of $G$. Otherwise without loss of generality, we may assume $c'$ and $d'$ are in different partite sets. Then $H + \{a, c, d\}$ induces a bipartite subgraph of $G$. In any case $\alpha_2(G) \geq \alpha_2(G') + 3 \geq \sigma(G') + \epsilon(G') + 3 \geq \sigma(G) + \epsilon(G') + 3/7 \geq \sigma(G) + \epsilon(G)$. ∎

In Lemmas 4.6, 4.7, 4.8, we assume $x$ is a 2-vertex and $u, v$ are the neighbours of $x$. By Lemma 4.2, both $u, v$ are 3-vertices. Let $a, b$ be the other two neighbours of $u$, and $c, d$ be the other two neighbours of $v$. By Lemma 4.5, $a, b, c, d$ are distinct vertices.

**Lemma 4.6** *Each of* $a, b$ *is adjacent to at least one vertex of* $c, d$.

*Proof.* Assume $b$ is not adjacent to any of $c, d$. Let $a', a''$ be the other two neighbours of $a$. By Lemma 4.4, $a$ is a 3-vertex and $a', a''$ cannot be both 2-vertices.

**Case 1** One of $a', a''$, say $a'$, is a 2-vertex.

Let $w$ be the other neighbour of $a'$. By Lemma 4.4 and Lemma 4.5, $w$ is distinct from $v, b$ and of course $w \neq a''$. Let $G' = (G - \{a, a', u, x\}) + bv$. Then $G = G' \circ_7 (w, a'', b, v)$ and$\sigma(G) = \sigma(G') + 20/7$.

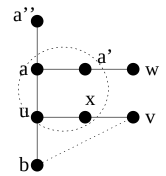


Figure 10: $G' = (G - \{a, a', u, x\}) + bv$

By Lemma 2.2,

$$\alpha_2(G) \geq \alpha_2(G') + 3 \geq \sigma(G') + \epsilon(G') + 3 = \sigma(G) + \epsilon(G') + 1/7.$$

If $\epsilon(G') \geq 0$, then since $\epsilon(G) \leq 1/7$, we have

$$\alpha_2(G) \geq \sigma(G) + \epsilon(G).$$

Otherwise, each piece of $G'$ is in $\mathcal{G}_1$, except at most one piece which is in $\mathcal{G}_2$ or is a singleton.

If $G'$ is 2-connected, then $G' \in \mathcal{G}_i$ for some $i = 1, 2$. By definition, either $\alpha_2(G) = \alpha_2(G') + 4$ or $G \in \mathcal{G}_{i+1}$ and $\epsilon(G) = \epsilon(G') + 1/7$. In any case, $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

Assume $G'$ has at least two pieces. If $a'', w$ are in different end-pieces, then since $a'', w$ are 3-vertices, these end-pieces are not singletons. By applying Lemma 3.2 to the end-pieces, there is an $H \in \mathcal{B}(G')$ which does not contain $w, a''$. Therefore $H + \{a, a', u, x\}$ induces a bipartite subgraph of $G$. In any case, $\alpha_2(G) = \alpha_2(G') + 4 \geq \sigma(G) + \epsilon(G)$. Assume $w, a''$ do not belong to distinct end-pieces. By Lemma 3.3, $bv$ is contained in a pseudo end-piece of $G'$, and $w$ and/or $a''$ is contained in another end-piece. By applying Lemma 3.2, one can find an $H \in \mathcal{B}(G')$ such that $H$ contains at most one of $w, a''$ and $H \cap \{w, a''\}$ is disconnected to $b$ and $v$. Therefore $H + \{a', a, u, x\}$ induces a bipartite subgraph of $G$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

**Case 2** $a', a''$ are 3-vertices of $G$.

Let $G' = (G - \{a, u, x\}) + bv$. Then $G = G' \circ_2 (a', a'', b, v)$ and $\sigma(G) = \sigma(G') + 14/7$.

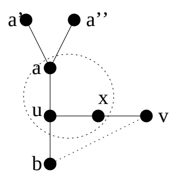


Figure 11: $G' = (G - \{a, u, x\}) + bv$

By Lemma 2.2,

$$\alpha_2(G) \geq \alpha_2(G') + 2 \geq \sigma(G') + \epsilon(G') + 2 = \sigma(G) + \epsilon(G').$$

If $\epsilon(G') \geq 1/7$, then $\epsilon(G) \leq \epsilon(G')$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Thus we may assume that $\epsilon(G') \leq 0$.

Since $G$ is 2-connected, so $G'$ is connected. If $G'$ is 2-connected, then $G' \in \mathcal{G}_i$ for some $1 \leq i \leq 3$. By definition, either $\alpha_2(G) = \alpha_2(G') + 3 \geq \sigma(G) + \epsilon(G)$ or $G \in \mathcal{G}_i$ and $\epsilon(G) = \epsilon(G')$. In any case, $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

Assume $G'$ has at least two pieces. As $\epsilon(G') \leq 0$, each piece of $G'$ is either a singleton or a graph in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. If the piece $P$ containing $a'$ is a singleton, then since $a'$ has degree 2 in $G'$, it follows that $P$ is not an end piece. By Lemma 3.3, the piece containing $a''$ is an end piece (and not a singleton, as $a''$ has degree 2 in $G'$), and $b, v$ is contained in a pseudo end-piece. By applying Lemma 3.2, one can find an $H \in \mathcal{B}(G')$ which does not contain the join vertex of the pseudo end-piece containing $bv$, and does not contain the join vertex of the end piece containing $a''$. So $a'', a'$ and $b, v$ are contained in three components of $H$. Therefore $H + \{a, u, x\}$ induces a bipartite subgraph of $G$ and $\alpha_2(G) = \alpha_2(G') + 3 \geq \sigma(G) + \epsilon(G)$. Assume none of $a', a''$ is contained in a singleton piece. If $a', a''$ belong to distinct pieces of $G'$, then by Lemma 3.2, there is an $H \in \mathcal{B}(G')$ which does not contain $a', a''$. Then $H + \{a, u, x\}$ induces a bipartite subgraph of $G$ and hence $\alpha_2(G) = \alpha_2(G') + 3 \geq \sigma(G) + \epsilon(G)$. Assume $a', a''$ are contained in the same piece $P$ of $G'$. By Lemma 3.3, $P$ is an end piece and $b, v$ are contained in another pseudo end-piece. By applying Lemma 3.2, one can find an $H \in \mathcal{B}(G')$ which does not contain $a'$ and does not contain the join vertex of the pseudo end-piece containing $b, v$. Hence $H + \{a, u, x\}$ induces a bipartite subgraph of $G$ and $\alpha_2(G) = \alpha_2(G') + 3 \geq \sigma(G) + \epsilon(G)$. ∎

Assume $ac$ is an edge of $G$.

**Lemma 4.7** *Vertices $b$ and $c$ are not adjacent.*

*Proof.* Assume $bc$ is an edge. Let $N(v) = \{x, c, d\}$. If none of $ad, bd$ is an edge, then let $G' = (G - \{a, c, u, x, v, b\})$.

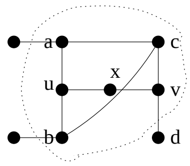


Figure 12: None of $ad, bd$ is an edge.

Then $\sigma(G) = \sigma(G') + 28/7$ (note that by Lemma 4.4, $a, b$ are 3-vertices). For $H \in \mathcal{B}(G')$, $H + \{a, c, u, x\}$ induces a bipartite subgraph of $G$. So $\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + \epsilon(G') + 4 \geq \sigma(G) + \epsilon(G')$. If $\epsilon(G') \geq 1/7$, then $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Otherwise each piece of $G'$ is in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. Let $a', b'$ be the other neighbour of $a, b$, respectively. By Lemma 3.2, there is an $H \in \mathcal{B}(G')$ which does not contain $b'$. Then $H + \{a, c, u, x, b\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) = aa(G') + 5 \geq \sigma(G) + \epsilon(G)$.

If $ad$ is an edge, then let $G' = (G - \{a, c, u, x, v, b, d\})$.

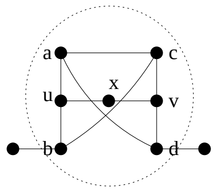


Figure 13: $ad$ is an edge.

Then $\sigma(G) = \sigma(G') + 34/7$. If $\epsilon(G') \geq 0$, then for $H \in \mathcal{B}(G')$, $H + \{a, c, u, x, b\}$ induces a bipartite subgraph of $G$, hence

$$\alpha_2(G) \geq \alpha_2(G') + 5 \geq \sigma(G') + \epsilon(G') + 5 = \sigma(G) + \epsilon(G') + 1/7 \geq \sigma(G) + \epsilon(G).$$

If $\gamma(G') \leq -1/7$, then each piece of $G'$ is in $\mathcal{G}_1$, except that at most one piece which is in $\mathcal{G}_2$ or is a singleton. We may assume the piece containing $b'$ is in $\mathcal{G}_1$. Then by Lemma 3.2, there is an $H \in \mathcal{B}(G')$ which does not contain $b'$. Then $H + \{b, u, a, x, c, d\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) = \alpha_2(G') + 6 \geq \sigma(G) + \epsilon(G)$. ∎

Since $bc$ is not an edge, and $b, v$ have a common neighbour, we may assume that $ac, bd$ are edges of $G$, and $ad, bc$ are not edges of $G$. The graph induced by $\{a, b, u, x, v, c, d\}$ is depicted in Figure 14 (i).

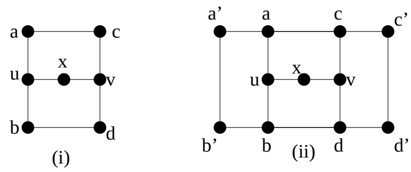


Figure 14: The graphs induced by $\{a, b, u, x, v, c, d\}$ and by $\{a, b, u, x, v, c, d, a', b', c', d'\}$.

Let $a', b', c', d'$ be the other neighbour of $a, b, c, d$, respectively.

**Lemma 4.8** *Vertices $a', b'$ are distinct and adjacent, and vertices $c', d'$ are distinct and adjacent.*

*Proof.* If $a' = b'$ or $a' \neq b'$ and $a'b'$ is not an edge, then let $G' = (G - \{u, v, x, b, c, d\}) + ab'$. Then $G = G' \circ_3 (d', c', b', a)$ and $\sigma(G) = \sigma(G') + 28/7$. By Lemma 2.2,

$$\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + \epsilon(G') + 4 \geq \sigma(G) + \epsilon(G').$$

If $\epsilon(G') \geq 1/7$, then $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Assume $\epsilon(G') \leq 0$. Then each piece of $G'$ is in $\mathcal{G}_1$, except that there may be one piece in $\mathcal{G}_3$, or there at most two pieces that are either in $\mathcal{G}_2$ or are singletons.

If $G'$ is 2-connected, then $G' \in \mathcal{G}_i$ for some $i \leq 3$. By definition, either $\alpha_2(G) = \alpha_2(G') + 5 \geq \sigma(G) + \epsilon(G)$ or $G \in \mathcal{G}_i$ and hence $\epsilon(G) = \epsilon(G')$ and $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

Assume $G'$ has at least two pieces. If $c', d'$ are in different pieces of $G'$. By Lemma 3.2, there is an $H \in \mathcal{B}(G')$ which does not contain $c', d'$. Then $H + \{c, v, d, x, b\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) = \alpha_2(G') + 5 \geq \sigma(G) + \epsilon(G)$. Assume $c', d'$ are in the same piece of $G'$. Then $ab'$ is not a cut-edge of $G'$, for otherwise either $a$ or $b'$ is a cut-vertex of $G$, which is in contrary to our assumption. As $G$ is 2-connected, the piece of $G'$ containing $c', d'$ and the piece containing $a, b'$ are the two end pieces of $G'$. By Lemma 3.2, there is an $H \in \mathcal{B}(G)$ such that $H$ contains at most one of $c', d'$ and does not contain the join vertex of the piece containing $a, b'$. So there is no path between $H \cap \{c', d'\}$ and $H \cap \{a, b'\}$. Therefore $H + \{c, v, d, x, b\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) = \alpha_2(G') + 5 \geq \sigma(G) + \epsilon(G)$. ∎

Let $X = \{a', b', c', d', a, b, c, d, u, x, v\}$. The subgraph of $G$ induced by $X$ is depicted in Figure 14 (ii), possibly with edges between vertices $a', b'$ and $c', d'$.

**Lemma 4.9** *There is no edge connecting $a'$ and $d'$, and no edge connecting $b'$ and $c'$. Moreover, $a', c'$ have a common neighbour $s$ and $b', d'$ have a common neighbour $t$, and $s, t$ are 3-vertices and have no common neighbour.*

*Proof.* If $a'd'$ is an edge, then since $G \neq F_4$, we have either $b'c'$ is an edge or $G - X$ is nonempty. In the former case, $G$ is an $F$-cycle, and $\alpha_2(G) = \sigma(G) + \epsilon(G)$. In the latter case, let $G' = G - X$. We have $\sigma(G) = \sigma(G') + 54/7$. For $H \in \mathcal{B}(G')$, $H + \{a', a, c, v, x, b, d, d'\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) \geq \alpha_2(G') + 8 \geq \sigma(G') + \epsilon(G') + 8 = \sigma(G) + \epsilon(G') + 2/7$. If $\epsilon(G') \geq -1/7$, then we have $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Otherwise, each piece of $G'$ is a graph in $\mathcal{G}_1$. Since $G$ is 2-connected, we conclude that $G$ is an $F$-cycle, and hence $\epsilon(G) = 0$ and $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$. Thus we assume that there is no edge connecting $a'$ and $d'$. By symmetry, we also assume that there is no edge connecting $b'$ and $c'$.

If $a', c'$ have no common neighbour, then let $G' = (G - \{a, c, u, x, v, b, d\}) + a'c'$. Then $G = G' \circ_5 (b', d', a', c')$ and $\sigma(G) = \sigma(G') + 34/7$.

If $\epsilon(G') \geq 0$, then for $H \in \mathcal{B}(G')$, $H + \{a, c, x, v, b\}$ induces a bipartite subgraph of $G$, hence

$$\alpha_2(G) \geq \alpha_2(G') + 5 \geq \sigma(G') + \epsilon(G') + 5 = \sigma(G) + \epsilon(G') + 1/7 \geq \sigma(G) + \epsilon(G).$$

Thus we may assume $\epsilon(G') \leq -1/7$. Then each piece is a graph in $\mathcal{G}_1$, except that one piece which is either in $\mathcal{G}_2$ or a singleton.

**Case 1** $b', d'$ are in different pieces of $G'$.

Since $G$ is 2-connected, none of the pieces containing $b'$ and $d'$ is a singleton. By Lemma 3.2, there is an $H \in \mathcal{B}(G')$ which does not contain $b', d'$. Then $H + \{a, c, u, v, b, d\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) = \alpha_2(G') + 6 \geq \sigma(G) + \epsilon(G)$.

**Case 2** $b', d'$ are in the same piece of $G'$.

Then $a', c', b', d'$ are in the same piece. Since $G$ is 2-connected, $G'$ has only one piece, i.e., $G'$ is 2-connected. Hence $G' \in \mathcal{G}_i$ for some $1 \leq i \leq 2$. By definition, either $\alpha_2(G) \geq \alpha_2(G') + 6 \geq \sigma(G) + \epsilon(G)$, or $G \in \mathcal{G}_{i+1}$, and $\epsilon(G) = \epsilon(G') + 1/7$. Hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

Let $s$ be the common neighbour of $a', c'$ and let $t$ be the common neighbour of $b', d'$. The conclusion that $s, t$ are 3-vertices and have no common neighbour follows from the fact that $G$ is 2-connected and $G \neq F_5$. ∎

Now we are ready to derive the final contradiction.

Let $w, r$ be the other neighbour of $s$ and $t$, respectively.

**Lemma 4.10** *The vertices $w, r$ have a common neighbour $z$.*

*Proof.* If $w, r$ have no common neighbour, then let $G' = (G - (X \cup \{s, t\})) + wr$. Then $\sigma(G) = \sigma(G') + 66/7$. For $H \in \mathcal{B}(G')$, $H + \{s, c', c, a, v, x, d, b, b', t\}$ induces a bipartite subgraph of $G$. So

$$\alpha_2(G) \geq \alpha_2(G') + 10 \geq \sigma(G') + \epsilon(G') + 10 = \sigma(G) + \epsilon(G') + 4/7 \geq \sigma(G) + \epsilon(G).$$

∎

Let $Y = X \cup \{s, t, w, z, r\}$. The subgraph of $G$ induced by $Y$ is depicted in Figure 15.

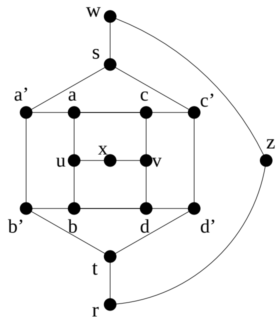


Figure 15: The subgraph induced by $Y$.

This subgraph contains two adjacent 2-vertices. By Lemma 4.2, $G$ has other vertices. If one of $w, z, r$ is a 2-vertex, say $w$ is a 2-vertex, then let $G' = G - Y$. We have $\sigma(G) = \sigma(G') + 80/7$. For $H \in \mathcal{B}(G')$, $H + \{z, w, s, a', a, c, u, v, b, d, d', t\}$ induces a bipartite subgraph of $G$. So

$$\alpha_2(G) \geq \alpha_2(G') + 12 \geq \sigma(G') + \epsilon(G') + 12 = \sigma(G) + \epsilon(G') + 4/7 \geq \sigma(G) + \epsilon(G).$$

Thus we may assume that $w, z, r$ are 3-vertices. Let $w', z', r'$ be the other neighbour of $w, z, r$, respectively.

**Case 1** Two of the vertices $w', z', r'$ have no common neighbour, say $w', r'$ have no common neighbour.

Let $G' = (G - Y) + w'r'$. Then $\sigma(G) = \sigma(G') + 80/7$. For $H \in \mathcal{B}(G')$, $H + \{w, z, s, a', a, c, u, v, b, d, d', t\}$ induces a bipartite subgraph of $G$. So

$$\alpha_2(G) \geq \alpha_2(G') + 12 \geq \sigma(G) + \epsilon(G') + 4/7 \geq \sigma(G) + \epsilon(G).$$

**Case 2** Any two of the vertices $w', z', r'$ have a common neighbour, but one of $w', z', r'$ is a 2-vertex.

Let $G' = G - Y$. Then the same calculation as above shows that $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

**Case 3** Any two of the vertices $w', z', r'$ have a common neighbour, and $w', z', r'$ are 3-vertices.

Let $f$ be the common neighbour of $w', r'$, $g$ be the common neighbour of $w', z'$ and $h$ be the common neighbour of $z', r'$. Let $Z = Y \cup \{w', z', r', f, g, h\}$. The subgraph of $G$ induced by $Z$ is depicted in Figure 16 (we allow the possibility that $f = g = h$).

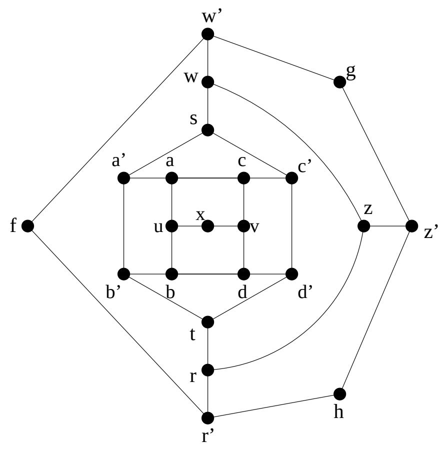


Figure 16: The subgraph induced by $Z$.

If $G$ has no other vertices, then $\alpha_2(G) = 17$ and $\sigma(G) = 114/7$ and hence $\alpha_2(G) \geq \sigma(G)+\epsilon(G)$. If $f = g = h$, then $\alpha_2(G) = 15$ and $\sigma(G) = 103/7$ and hence $\alpha_2(G) \geq \sigma(G) + \epsilon(G)$.

Assume $G$ has other vertices. Then $f, g, h$ are distinct. Let $G' = G - Z$. Then $\sigma(G) = \sigma(G') + 108/7$. For $H \in \mathcal{B}(G')$, $H + \{r', f, w', w, z, z', s, a', a, c, u, v, b, d, d', t\}$ induces a bipartite subgraph of $G$. So $\alpha_2(G) \geq \alpha_2(G') + 16 \geq \sigma(G) + \epsilon(G)$. This completes the proof of Theorem 2.1. ∎

## 5 Proof of Theorem 1.3 and Theorem 1.4

Before proving Theorem 1.3 and Theorem 1.4, we need one more lemma that takes care of subcubic graphs of odd girth at least 7.

**Lemma 5.1** *Suppose $G$ is a triangle-free subcubic $n$-vertex graph, each connected component has at least two vertices and has a vertex of degree at most 2. If $G$ has no 5-cycle and has at most $n/2 - 1$ components, then $\alpha_2(G) \geq \sigma(G) + 2/7$.*

*Proof.* Assume the lemma is not true, and $G$ is a minimum counterexample. It is easy to verify that $G$ is 2-connected and non-bipartite. If $G$ has a 4-cycle $C = (v_0, v_1, v_2, v_3)$ which contains a 2-vertex $v_0$, then let $G' = G - C$. As $G$ is 2-connected and non-bipartite and has no 5-cycle, $G'$ is connected and has more than four vertices. Since $C$ has at least two 3-vertices, we have $\sigma(G) \leq \sigma(G') + 20/7$. On the other hand, it is easy to verify that $\alpha_2(G) \geq \alpha_2(G') + 3$. By the minimality of $G$, we know that $\alpha_2(G') \geq \sigma(G') + 2/7$. Hence $\alpha_2(G) \geq \sigma(G) + 2/7$. In the following, we assume that no 4-cycle of $G$ contains a 2-vertex. We divide the discussion into two cases.

**Case 1** There is a 7-cycle $C = (v_0, v_1, \cdots, v_6)$ which contains a 2-vertex $v_0$.

Assume among all 7-cycles, $C$ contains the maximum number of 2-vertices. As $G$ is 2-connected, $C$ contains at most five 2-vertices. If $v_i$ is a 3-vertex, then let $u_i$ be the other neighbour of $v_i$.

**Case 1(i)** The vertex $v_0$ is the only 2-vertex contained in $C$.

Let $e_1 = u_1u_2, e_2 = u_4u_5$, and let $G' = (G - C) + \{e_1, e_2\}$. We have $\sigma(G) = \sigma(G') + 34/7$. By Theorem 2.1, $\alpha_2(G') \geq \sigma(G') + \epsilon(G')$. For any $H \in \mathcal{B}(G')$, $H + \{v_0, v_1, v_2, v_4, v_5\}$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) \geq \alpha_2(G') + 5 \geq \sigma(G) + \epsilon(G') + 1/7$. It remains to show that $\epsilon(G') \geq 1/7$. For this purpose, it suffices to prove that each component $Q$ of $G'$ has $\epsilon(Q) \geq 0$ and one component $Q$ of $G'$ has $\epsilon(Q') \geq 1/7$. By noting that the component of $G'$ containing the edge $e_1$ contains at least four vertices (as each of $u_1, u_2$ has degree at least 2 in $G'$), we only need to prove the following claim.

**Claim** Suppose $Q$ is a connected component of $G'$. Then

$$\epsilon(Q') \begin{cases} = 0, & \text{if } Q = K_2, \\ \geq 1/7, & \text{otherwise.} \end{cases}$$

Let $Q$ be a connected component of $G'$. If $Q$ contains a single vertex $u$, then $u$ is adjacent to at least two vertices of $C$ (as $G$ is 2-connected). As $G$ has no 5-cycle, it follows that $u$ is adjacent to exactly two vertices of $C$ that are distance two apart. Thus $u$ is a 2-vertex contained in a 4-cycle, in contrary to the previous conclusion. So $Q$ contains at least two vertices.

If $Q$ contains two pieces $P_1, P_2$ such that each $P_i$ is a 5-cycle, then each $P_i$ contains exactly one of the edge $e_1, e_2$ (because $G$ itself has no 5-cycles). Assume $e_i$ is an edge of $P_i$ for $i = 1, 2$. Then $C' = (P_1 - e_1) + \{v_1, v_2\}$ and $C'' = (P_2 - e_2) + \{v_4, v_5\}$ are 7-cycles. By the choice of $C$, each of $C', C''$ has at most one 2-vertex. Therefore each of $P_1, P_2$ is incident to at least 4 cut-edges of $Q$, which implies that $Q$ contains at least 6 end-pieces. As $G$ is 2-connected, each end-piece of $Q$ is connected to $C$ by an edge. This is impossible, because there are only 6 edges between $C$ and $G'$, and 4 of these edges connect $C$ and $P_1 \cup P_2$.

If $Q$ contains one piece $P$ which is a 5-cycle, then $(P + \{v_1, v_2, v_3, v_4, v_5\}) - \{e_1, e_2\}$ contains a 7-cycle $C'$. Indeed, if $P$ contains one of $e_1, e_2$, say $e_1 \in P$ and $e_2 \notin P$, then $(P - e_1) + \{v_1, v_2\}$ is a 7-cycle. If $P$ contains both $e_1, e_2$, then $P - \{e_1, e_2\}$ consists of two paths, one has length 1 and the other has length 2. So $(P - \{e_1, e_2\}) + \{v_iu_i : i = 1, 2, 4, 5\}$ consists of two paths, one has length 3, the other has length 4. Add the path $(v_1, v_2, v_3, v_4, v_5)$ to each of the two paths, each of the resulting graph contains a cycle, one of length 7, the other of length 6 (here we used the fact that $G$ has no 5-cycle). By our choice of $C$, the 7-cycle $C'$ has at most one 2-vertex. This implies that $P$ is incident to at least 2 cut-edges of $Q$. So $Q$ has at least 3 pieces. By the previous paragraph, no other piece of $Q$ is a 5-cycle. Moreover, each other piece $P'$ contains at most one 5-cycle. This implies that $P'$ is not an $F$-cycle, and not a graph in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. Thus $P'$ is either a singleton, or has $\epsilon(P') = 1/7$. If $Q$ has more than three pieces or at least one of the other piece is not a singleton, then we have $\epsilon(Q) \geq 1/7$. Assume $Q$ has exactly three pieces and each of the other two pieces is a singleton. If $G' = Q$, then it is easy to verify that $\alpha_2(G) \geq \sigma(G) + 2/7$. Otherwise, let $G'' = G' - Q = G - (C + Q)$. Then $\sigma(G) = \sigma(G'') + 73/7$ and $\alpha_2(G) \geq \alpha_2(G'') + 10 \geq \sigma(G) + 2/7$, in contrary to the assumption that $G$ is a counterexample.

Assume each piece of $Q$ is not a 5-cycle. If $Q$ has a piece $P$ which is a copy of $F_2$ or $F_3$, then $P$ contains both edges $e_1, e_2$ (as deleting one edge from $F_2$, $F_3$, the resulting graph still contains a 5-cycle). By using the fact that each 7-cycle of $G$ contains at most one 2-vertex, and using the fact that $(P - \{e_1, e_2\}) + \{v_1, v_2, v_4, v_5\}$ contains two 7-cycles, it is easy to verify that $P$ is incident to at least 2 cut-edges of $Q$. Hence $Q$ has at least three pieces. If $Q$ has more than three pieces, or one of the other piece is not a singleton, then $\epsilon(Q) \geq 1/7$. If $Q$ has exactly three pieces and each of the other two pieces is a singleton, then $G' = Q$ and in this case it is easy to verify that $\alpha_2(G) \geq \sigma(G) + 2/7$.

Assume each piece $P$ of $Q$ is not a 5-cycle and not $F_2, F_3$. Then $\epsilon(P) \geq -1/7$ and equality holds only if $P$ is a singleton. If $Q$ has only one piece, then this piece $P$ is not a singleton, not a graph in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, hence $\epsilon(Q) = 1/7$. If $Q$ has at least two pieces, and $Q \neq K_2$, then at least one of the piece $P$ has $\epsilon(P) \geq 0$, hence $\epsilon(Q) \geq 1/7$. This completes the proof of the claim, and the proof of Case 1(i).

**Case 1(ii)** $C$ has two 2-vertices $v_0, v_i$.

By symmetry, we may assume that $1 \leq i \leq 3$. Let

$$e = \begin{cases} u_2u_3, & \text{if } i = 1, \\ u_3u_4, & \text{if } i = 2, \\ u_1u_2, & \text{if } i = 3. \end{cases}$$

Let $G' = (G - C) + e$. Then $\sigma(G) = \sigma(G') + 34/7$. If $i = 1$, then for any $H \in \mathcal{B}(G')$, $H + \{v_0, v_1, v_2, v_3, v_5\}$ induces a bipartite subgraph of $G$. If $i = 2$, then for any $H \in \mathcal{B}(G')$, $H + \{v_6, v_0, v_2, v_3, v_4\}$ induces a bipartite subgraph of $G$. If $i = 3$, then for any $H \in \mathcal{B}(G')$, $H + \{v_0, v_1, v_2, v_3, v_5\}$ induces a bipartite subgraph of $G$. In any case, $\alpha_2(G) \geq \alpha_2(G') + 5$. By Theorem 2.1, $\alpha_2(G') \geq \sigma(G') + \epsilon(G')$. So $\alpha_2(G) \geq \sigma(G) + \epsilon(G') + 1/7$. It remains to show that $\epsilon(G') \geq 1/7$. Similarly as in the previous case, it suffices to show that for each connected component $Q$ of $G$, either $Q = K_2$ or $\epsilon(Q) \geq 1/7$. Let $Q$ be a connected component of $G'$. The same argument shows that $Q$ has at least 2 vertices. If no piece of $Q$ is a 5-cycle, then every piece $P$ of $Q$ is not a graph in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ and hence has $\epsilon(P) \geq 1/7$ or is a singleton. So either $Q = K_2$ or $\epsilon(Q) \geq 1/7$. Assume $Q$ has piece $P$ which is a 5-cycle. Then $(P - e) + \{v_0, v_i\}$ is a 7-cycle. By the choice of $C$, $P$ is incident to at least 3 cut edges of $Q$. Hence $Q$ has at least 3 end pieces. Each end piece $P'$ is either a singleton, or has $\epsilon(P') \geq 1/7$. If $Q$ has only four pieces and each piece other than $P$ is a singleton, then since each end piece of $Q$ contains one of $u_j$'s, we conclude that $G' = Q$. In this case, it is easy to verify that $\alpha_2(G) \geq \sigma(G) + 2/7$. Assume either $Q$ has more than four pieces or $Q$ has one piece other than $P$ which is not a singleton. Then it follows from definition that $\epsilon(Q) \geq 1/7$.

**Case 1(iii)** $C$ contains three 2-vertices.

Let $G' = G - C$. We have $\sigma(G) \leq \sigma(G') + 34/7$. No matter how the three 2-vertices are distributed on the cycle $C$, for any $H \in \mathcal{B}(G')$, it is easy to find a set $X$ of five vertices of $C$ such that $H + X$ induces a bipartite subgraph of $G$. As $G$ is 2-connected, $G'$ has at most two connected components. If $G'$ has two components and has at most 5 vertices, then it is straightforward to verify that $\alpha_2(G) \geq \sigma(G) + 2/7$. Otherwise, by the choice of $G$, we have $\alpha_2(G') \geq \sigma(G') + 2/7$. Hence $\alpha_2(G) \geq \sigma(G) + 2/7$.

**Case 1(iv)** $C$ contains four 2-vertices.

Then there is an index $i$ such that either $v_i, v_{i+1}$ are 3-vertices, or $v_i, v_{i+3}$ are 3-vertices. Let $u_iu_{i+1}$ or $e = u_iu_{i+3}$, respectively. Let $G' = (G - C) + e$, and let $v_k$ be the other 3-vertex contained in $C$. Then for any $H \in \mathcal{B}(G')$, $H + C - v_k$ induces a bipartite subgraph of $G$. Hence $\alpha_2(G) = \alpha_2(G') + 6 \geq \sigma(G) + 2/7$.

**Case 1(v)** $C$ contains five 2-vertices.

For $G' = G - C$, $\alpha_2(G) = \alpha_2(G') + 6$, and $\sigma(G) = \sigma(G') + 38/7$. By Theorem 2.1, $\alpha_2(G') \geq \sigma(G')$ (as $G'$ has no 5-cycles, so $G' \notin \mathcal{G}_1 \cup \mathcal{G}_2$), and hence $\alpha_2(G) \geq \sigma(G) + 2/7$.

This completes the proof of Case 1.

**Case 2** No 7-cycle of $G$ contains a 2-vertex.

If $G$ has two adjacent 2-vertices, $x, y$, then let $u, v$ be the other neighbour of $x, y$, respectively. Without loss of generality, we may assume that $u$ is a 3-vertex. Let $G' = (G - \{x, y, u\})$. Then $\sigma(G) = \sigma(G') + 2$ and $\alpha_2(G) = \alpha_2(G') + 2$. As $\alpha_2(G') \geq \sigma(G') + 2/7$, we have $\alpha_2(G) \geq \sigma(G) + 2/7$. Thus we assume that $G$ contains no adjacent 2-vertices. Let $x$ be a 2-vertex of $G$, and let $u, v$ be its neighbours. Let $a, b$ be the other two neighbours of $u$, $c, d$ be the other two neighbours of $v$. If $a$ is a 2-vertex, then let $G' = G - \{x, u, a\}$. We have $\sigma(G) = \sigma(G') + 2$ and $\alpha_2(G) = \alpha_2(G') + 2$. By the minimality of $G$, we have $\alpha_2(G') \geq \sigma(G') + 2/7$, hence $\alpha_2(G) \geq \sigma(G) + 2/7$. Assume $a$ is a 3-vertex. Let $G' = (G - \{u, x, a\}) + bv$. Then $\sigma(G) = \sigma(G') + 2$ and $\alpha_2(G) \geq \alpha_2(G') + 2$. Since $x$ is not contained in a 7-cycle, we conclude that $G'$ has no 5-cycle. Hence $\alpha_2(G') \geq \sigma(G') + 2/7$. So $\alpha_2(G) \geq \sigma(G) + 2/7$. This completes the proof of Lemma 5.1 ∎

**Proof of Theorem 1.4**: Assume $G$ is an $n$-vertex 2-connected triangle-free subcubic graph. If $G$ has a vertex of degree at most 2, then by Theorem 2.1, $\alpha_2(G) \geq \sigma(G) + \epsilon(G) = (5n + n_2)/7 + \epsilon(G) \geq 5n/7$ (because $n_2/7 + \epsilon(G) \geq 0$) and equality holds only if $G = F_5$.

Assume $G$ is cubic. If $G$ has no 5-cycle, then let $v$ be a vertex of $G$, and let $G' = G - v$. By Lemma 5.1, $\alpha_2(G') \geq \sigma(G') + 2/7 = 5n/7$. Hence $\alpha_2(G) \geq 5n/7$. Assume $G$ has a 5-cycle $C = (v_0, v_1, v_2, v_3, v_4)$. For $0 \leq i \leq 4$, let $u_i$ be the other neighbour of $v_i$.

First we consider the case that $G - C$ is disconnected. Let $G' = G - C$. Since $G$ is 2-connected, we conclude that $G'$ has two components say $G_1, G_2$. We may assume that $G_1$ contains three of the $u_i$'s, $G_2$ contains two of the $u_i$'s. Then $\sigma(G) = \sigma(G') + 20/7$ and $\alpha_2(G) \geq \alpha_2(G') + 3$. As each component of $G'$ contains 2-vertices, we have $\alpha_2(G') \geq \sigma(G') + \epsilon(G')$. If $\epsilon(G') \geq -1/7$, then it follows that $\alpha_2(G) \geq \sigma(G)$. Assume $\epsilon(G') \leq -2/7$. If $\epsilon(G_i) = -2/7$ for some $i \in \{1, 2\}$, then each piece of $G_i$ is a graph in $\mathcal{G}_1$. By Lemma 3.2 (if $G_i$ has more than one piece) and by checking the graphs in $\mathcal{G}_1$ (if $G_i \in \mathcal{G}_1$), we conclude that there is an $H \in \mathcal{B}(G_i)$ such that two of the $u_i$'s contained in $G_i$ are not contained in $H$. As $\epsilon(G_{3-i}) \leq 0$, it follows that $G_{3-i} \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ or is an $F$-cycle. By Lemma 3.2, there is an $H' \in \mathcal{B}(G_{3-i})$ such that one of the $u_i$'s contained in $G_{3-i}$ is not contained in $H'$. Now $H + H' \in \mathcal{B}(G')$ and three of $u_i$'s are not contained in $H + H'$. This implies that $\alpha_2(G) \geq \alpha_2(G') + 4$. As $\epsilon(G') \geq -4/7$, we conclude that $\alpha_2(G) \geq \sigma(G)$.

Now we assume that $G - C$ is connected. If $G - C$ has at least three end-pieces, and at least one of the pieces, say $P$, is in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ or is an $F$-cycle, then let $G' = G - C$. Similarly as above, $\sigma(G) = \sigma(G') + 20/7$ and $\alpha_2(G) \geq \alpha_2(G') + 3$. If $\epsilon(G') \geq -1/7$, then since $\alpha_2(G') \geq \sigma(G') + \epsilon(G')$, we have $\alpha_2(G) \geq \sigma(G)$. If $\epsilon(G') = -2/7$, then all the pieces are in $\mathcal{G}_1$. By Lemma 3.2, there is an $H \in \mathcal{B}(G')$ such that three of the $u_i$'s are not contained in $H$ (each end pieces has an $u_i$ not contained in $H$). Therefore $\alpha_2(G) = \alpha_2(G') + 4 \geq \sigma(G)$. If $G - C$ has two end-pieces, then let $u_i$ be contained in one end-piece, and $u_j$ be contained in the other end-piece. If $G - C$ is 2-connected, then choose $u_i, u_j$ such that they have no common neighbour (as $G$ is cubic, such $u_i, u_j$ exist). Let $G' = (G - C) + u_i u_j$ ($u_i u_j$ could be an edge of $G$). In any case case $G'$ is

2-connected. Now

$$\sigma(G) = \begin{cases} \sigma(G') + 22/7, & \text{if } u_i u_j \text{ is not an edge of } G, \\ \sigma(G') + 20/7, & \text{if } u_i u_j \text{ is an edge of } G. \end{cases}$$

For any $H \in \mathcal{B}(G')$, since $u_i \bowtie_H u_j$, it follows that there is an index $t$ such that $u_t \bowtie_H u_{t+1}$. Hence $H + \{v_t, v_{t+1}, v_{t+3}\}$ induces a bipartite subgraph of $G$ (summation in the indices modulo 5). So $\alpha_2(G) \geq \alpha_2(G') + 3$. If $\alpha_2(G') \geq \sigma(G') + 1/7$, then $\alpha_2(G) \geq \sigma(G) = 5n/7$. Otherwise, $G' \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ or is an $F$-cycle. If $u_i u_j$ is an edge of $G$, then $G'$ has exactly five 2-vertices. Thus $G' = C_5$ or $G' \in \mathcal{G}_3'$. In the former case, unless $G$ is the Petersen graph, we have $\alpha_2(G) = 8 \geq 5n/7$. In the latter case, $\alpha_2(G) \geq \alpha_2(G') + 3 = \sigma(G') + 3 \geq \sigma(G) = 5n/7$. Assume $u_i u_j$ is not an edge of $G$. Then $G'$ has three 2-vertices. This implies that $G' = F_4$ or $G' = F_8$ or $G' \in \mathcal{G}_3'$. If $G' = F_4$, then there is an $H \in \mathcal{B}(G')$ which does not contain any 2-vertices of $G'$, and hence $\alpha_2(G) = \alpha_2(G') + 4 \geq 5n/7$. If $G' = F_8$, then case by case check shows that $\alpha_2(G) = \alpha_2(G') + 4 \geq 5n/7$, unless $G$ is the dodecahedron. Assume $G' \in \mathcal{G}_3$. By checking Figure 17, there are 13 graphs in $\mathcal{G}_3$ each of which contains three 2-vertices. A computer check shows that if $G'$ is any of these 13 graphs, no matter how the $u_i$'s are distributed, $\alpha_2(G) = \alpha_2(G') + 4 \geq 5n/7$. Assume $G'$ is an $F$-cycle, and $G_1, G_2, \cdots, G_k$ are the $F$-subgraphs of $G'$. If $e = u_i u_j$ is not a join edge of the $F$-cycle $G'$, then $\alpha_2(G' - e) = \alpha_2(G') + 1$ and it is easy to verify that $\alpha_2(G) \geq \alpha_2(G' - e) + 3$. So $\alpha_2(G) = \alpha_2(G') + 4 \geq 5n/7$. Assume $e$ is a join edge of $G'$. If none of the $F$-subgraphs of $G'$ is $F_1$, then it is not difficult to verify that there is an $H \in \mathcal{B}(G')$ such that at least three of the $u_t$'s are not contained in $H$, and hence $\alpha_2(G) = \alpha_2(G') + 4 \geq 5n/7$. Assume one of the $F$-subgraphs, say $G_1$, of $G'$ is $F_1$. Observe that $G' - e$ has exactly five 2-vertices. A case by case check shows that the subgraph of $G$ induced by $C \cup G_1$ (which is the disjoint union of two 5-cycles plus four edges between vertices of these two 5-cycles) contains an induced bipartite subgraph on 8 vertices. So there is an $H \in \mathcal{B}(G')$ such that $H$ can be extended to an induced bipartite subgraph $G$ by adding four vertices. I.e., $\alpha_2(G) = \alpha_2(G') + 4 \geq 5n/7$. ∎

**Proof of Theorem 1.3**: Assume $G$ is an $n$-vertex 2-connected triangle-free subcubic graph. Let $\tau(G) = |V(G)| - \alpha_2(G)$ be the minimum number of vertices to be deleted from $G$ so that the resulting induced subgraph is bipartite. It is well-known that $\tau(G)$ is equal to the minimum number of edges to be deleted so that the resulting subgraph is bipartite. Indeed, if $H$ is an induced bipartite subgraph of $G$ with $|V(H)| = |V(G)| - \tau(G)$, and $H'$ is a spanning bipartite subgraph of $G$ which contains $H$ as a subgraph and which has maximum number of edges, then $|E(G)| - |E(H')| = \tau(G)$. This is so because each vertex $x \in V(G) - V(H)$ is adjacent to at most one vertex (in $G$) that is in the same partite set of $H'$ as $x$ (if there are two or more such vertices, then by moving $x$ to the other partite set we obtain a spanning bipartite subgraph with more edges than $H'$).

Observe that $|E(G)| = (3n_3 + 2n_2)/2$. If $G$ is cubic and is not the Petersen graph and not the dodecahedron, then by Theorem 1.4, $\tau(G) \leq 2n/7 = 4|E(G)|/21$. So $b(G) \geq 17/21$. If $G$ has a vertex of degree 2 and $G \notin \mathcal{G}_1 \cup \mathcal{G}_2$, then by Theorem 2.1, $\tau(G) \leq (2n_3 + n_2)/7 < 4|E(G)|/21$. Hence $b(G) > 17/21$. If $G \in \mathcal{G}_2$, then it is easy to check that $b(G) > 17/21$, except that if $G = F_8$ then $b(G) = 17/21$. For $G \in \mathcal{G}_1$ we have $b(G) = 4/5$. ∎

If $G$ is an $F$-cycle in which each $F$-subgraph is a copy of $F_5$, then $b(G) = 17/21$ and $b^*(G) = 5/7$. So the bounds in Theorem 1.3 and Theorem 1.4 are tight.

## Acknowledgement

I thank Jiarong Chang for his help in writing a computer algorithm used to help constructing explicitly the graph class $\mathcal{G}_3$, and thank Baogang Xu and an anonymous referee for careful readings of the manuscript and for their valuable comments.

## References

[1] M. Albertson, B. Bollobas and S. Ticker, The independence ratio and the maximum degree of a graph, *Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State University, Baton Rouge, La., 1976),* pp. 43-50. *Congressus Numerantium*, No. XVII, Utilitas Math., Winnipeg, Man., 1976.

[2] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Random Structure and Algorithm*, **21** (2002) 414-430.

[3] J. A. Bondy and S. C. Locke, Largest bipartite subgraphs in triangle-free graphs with maximum degree three, *J. Graph Theory* **10** (1986) 477-504.

[4] P. Erdős, Problems and results in graph theory and combinatorial analysis, *Graph Theory and Related Topics* (Proceedings of Waterloo Conference, dedicated to W. T. Tutte on his sixties birthday, Waterloo, 1977), Academic, New York (1979) 153–163.

[5] S. Fajtlowicz, On the size of independent sets in graphs, *Cong. Numerantium* **21** (1978), 269-274.

[6] J. Griggs and O. Murphy, Edge density and independence ratio in triangle-free graphs with maximum degree three, *Discr. Math.* **152** (1996), 157-170.

[7] H. Hatami and X. Zhu, The fractional chromatic number of graphs of maximum degree at most three, *Preprint*, (2006).

[8] C. Heckman and R. Thomas, A new proof of the independence ratio of triangle-free cubic graphs, *Discrete Math.* **233** (2001), no. 1-3, 233-237.

[9] G. Hopkins and W. Staton, Extremal bipartite subgraphs of cubic triangle-free graphs, *J. Graph Theory* **6** (1982) 115–121.

[10] K. F. Jones, Size and independence in triangle-free graphs with maximum degree three, *J. Graph Theory* **14** (1990), 525-535.

[11] S. C. Locke, Maximum $k$-colourable subgraphs, *J. Graph Theory* **6** (1982) 123–132.

[12] G. I. Orlova and Y. G. Dorfman, Finding the maximal cut in a graph, *Eng. Cyber.* **10** (1972) 502–506.

[13] W. Staton, Edge deletions and the chromatic number, *Ars Combinatoria* **10** (1980) 103–106.

[14] W. Staton, Some Ramsey-type numbers and the independence ratio, *Trans. Amer. Math. Soc.* **256** (1979), 353-370.

[15] B. Xu and X. Yu, Triangle-free subcubic graphs with minimum bipartite density, *Journal of Combinatorial Theory Ser. B*, to appear.

[16] M. Yannakakis, Node- and Edge-Deletion NP-Complete Problems, *STOC* (1978) 253-264.

[17] X. Zhu, Bipartite density of triangle-free subcubic graphs, *Preprint* (2007).

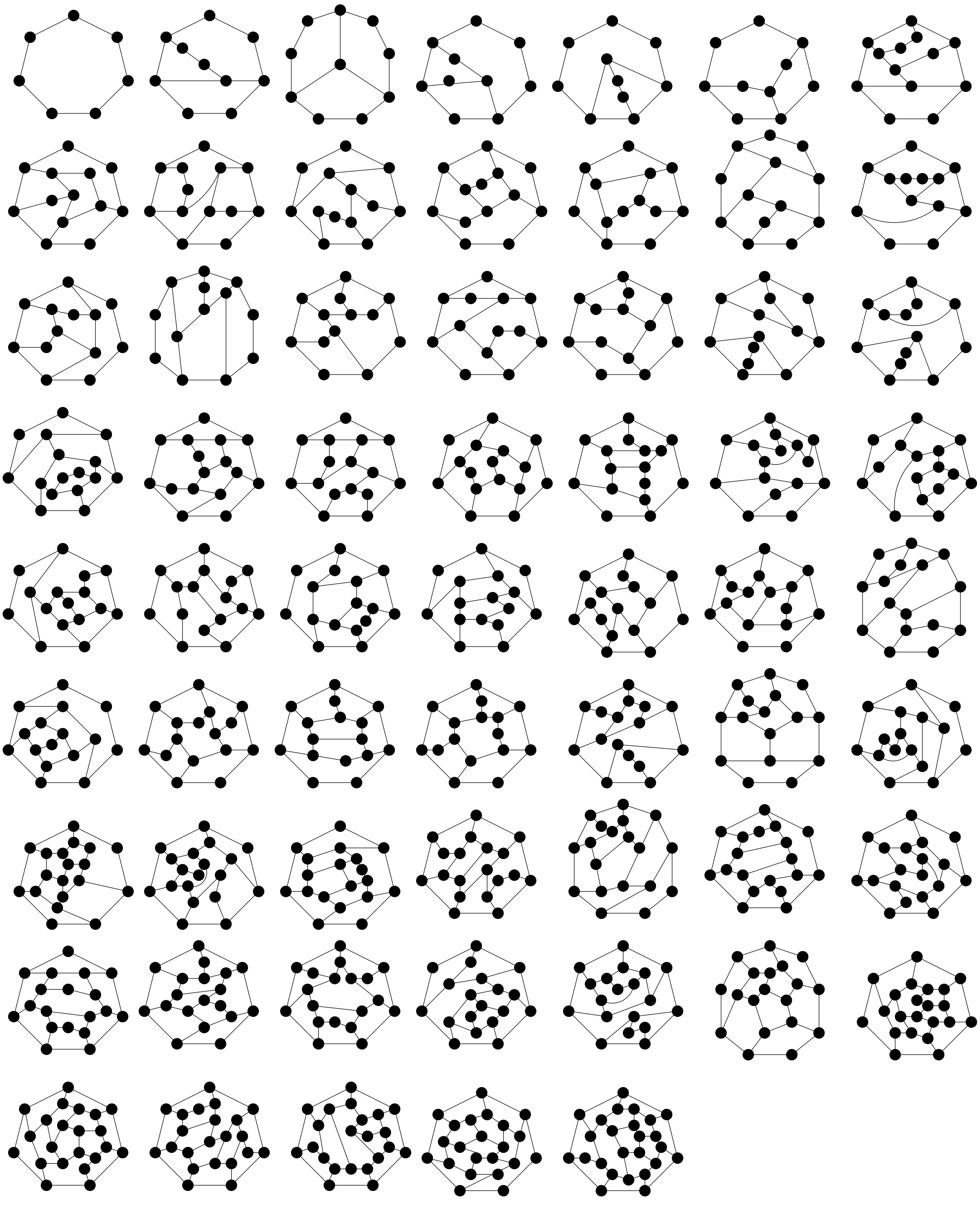

Figure 17: The graphs in $\mathcal{G}_3$