Bipartite density of triangle-free subcubic graphs

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Abstract

A graph is subcubic if its maximum degree is at most 3. The bipartite density of a graph G is defined as $b(G) = \max\{|E(B)|/|E(G)| : B$ is a bipartite subgraph of $G\}$. It was conjectured by Bondy and Locke that if G is a triangle-free subcubic graph, then $b(G) \geq \frac{4}{5}$ and equality holds only if G is in a list of seven small graphs. The conjecture has been confirmed recently by Xu and Yu. This note gives a shorter proof of this result.

Keywords: triangle-free; subcubic; bipartite density; max-cut.
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1 Introduction

The Max-Cut problem is to find a bipartite subgraph of G with maximum number of edges [1,6]. Given a graph G and an integer m, the problem to determine if G has a bipartite subgraph H with m edges is NP-complete even when restricted to triangle-free cubic graphs [9]. A natural question is to find lower bound for the number of edges in a maximum bipartite subgraph of G. The *bipartite density* $b(G)$ of G is defined as

$$b(G) = \max\{|E(B)|/|E(G)| : B \text{ is a bipartite subgraph of } G\}.$$

Erdős [3] proved that if G is $2m$-colourable then $b(G) \geq \frac{m}{2m-1}$. Stanton [7] and Locke [5] proved that if G is cubic and $G \neq K_4$, then $b(G) \geq \frac{7}{9}$. Hopkins and Stanton [4] proved that if G is cubic and triangle-free then $b(G) \geq \frac{4}{5}$. A graph is subcubic if its maximum degree is at most 3. Bondy and Locke [2] give a polynomial time algorithm that, for a given triangle-free subcubic graph G, finds a bipartite subgraph H of G with at least $4|E(G)|/5$ edges. They further proved that the Petersen graph and the dodecahedron are the only triangle-free cubic graphs with bipartite density $\frac{4}{5}$.

There are triangle-free subcubic graphs with minimum degree 2 that have bipartite density $\frac{4}{5}$. Graphs F_1, F_2, F_3, F_4, F_5 in Figure 2 are triangle-free, subcubic, and have bipartite density $\frac{4}{5}$. Bondy and Locke [2] conjectured that all other triangle-free subcubic graphs have bipartite density strictly larger than $\frac{4}{5}$.

This conjecture was confirmed by Xu and Yu [8]. Let F_6, F_7 be the Petersen graph and the dodecahedron, respectively.

Theorem 1.1 *[8] If G is a triangle-free subcubic graph with $b(G) = \frac{4}{5}$, then $G \in \{F_i : 1 \leq i \leq 7\}$.*

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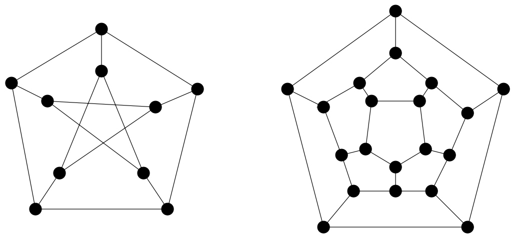

Figure 1: The Petersen graph and the dodecahedron.

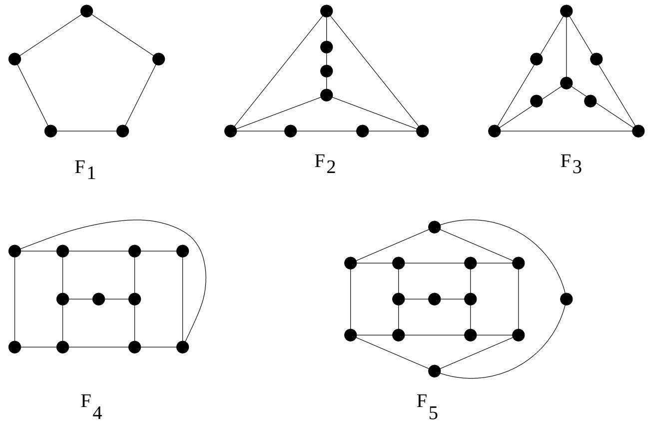

Figure 2: triangle-free subcubic graphs with bipartite density 4/5.

This note gives a shorter proof of Theorem 1.1, however the result in [2] on cubic graphs is used in our proof.

A degree i vertex of G is called an i-vertex of G. Let n_i be the number of i-vertices of G. For a subcubic graph G, let $\sigma(G) = 7n_3/10 + 4n_2/5 + n_1 + n_0$. Let $\alpha_2(G)$ be the maximum number of vertices in an induced bipartite subgraph of G. The *heart* of a graph G is obtained from G by repeatedly deleting 1-vertices and isolated vertices. In this paper, we prove the following result.

Theorem 1.2 *If G is a triangle-free subcubic graph, then $\alpha_2(G) \geq \sigma(G)$ and equality holds if and only if the heart of G belong to the set $\{F_i : 1 \leq i \leq 7\}$.*

To see that Theorem 1.1 follows from Theorem 1.2, it suffices to observe the following: Let H be an induced bipartite subgraph of G with partite sets A, B. Partition $V(G - H)$ into A', B' in such a way that the total number of edges in $G[A \cup A']$ and $G[B \cup B']$ is minimum. Then each vertex in A' has at most one neighbour in $A \cup A'$ and each vertex in B' has at most one neighbour in $B \cup B'$. Therefore the bipartite subgraph Q of G with partite sets $A \cup A'$ and $B \cup B'$ has $|E(Q)| \geq |E(G)| - |A' \cup B'| \geq |E(G)| - (3n_3 + 2n_2)/10 \geq 4|E(G)|/5$, with equality holds only if $G = F_i$ for some $1 \leq i \leq 7$.

2 Proof of Theorem 1.2

Assume Theorem 1.2 is not true and G is a minimum counterexample. I.e., G is a minimum triangle-free subcubic graph such that either $\alpha_2(G) < \sigma(G)$ or $\alpha_2(G) = \sigma(G)$ and $G \notin \{F_i : 1 \leq i \leq 7\}$. It is obvious that G has minimum degree at least 2.

Lemma 2.1 *Suppose G is a subcubic graph with minimum degree at least 2, X is a proper subset of $V(G)$. Let n'_i be the number of i-vertices of G contained in X, and m be the number of edges with exactly one end vertex in X. If G' is obtained from G by deleting X and adding m' edges and $d_{G'}(v) \leq d_G(v)$ for each vertex v, then $\sigma(G) - \sigma(G') \leq (7n'_3 + 8n'_2 + 2m' - m)/10$.*

Proof. Let $\phi(0) = \phi(1) = 1, \phi(2) = 8/10, \phi(3) = 7/10$. By definition,

$$\begin{aligned}\sigma(G) &= \sum_{x \in V(G-X)} \phi(d_G(x)) + \sum_{x \in X} \phi(d_G(x)) \\ \sigma(G') &= \sum_{x \in V(G-X)} \phi(d_{G'}(x)). \\ \sigma(G) - \sigma(G') &= \sum_{x \in V(G-X)} (\phi(d_G(x)) - \phi(d_{G'}(x))) + (7n'_3 + 8n'_2)/10.\end{aligned}$$

By Assumption, for each vertex $x \in V(G - X)$, $d_G(x) \geq d_{G'}(x)$. As $d_G(x) \geq 2$, it follows from the definition that

$$\phi(d_G(x)) - \phi(d_{G'}(x)) \leq (d_{G'}(x) - d_G(x))/10.$$

As

$$\sum_{x \in V(G-X)} (d_{G'}(x) - d_G(x)) = 2m' - m,$$

we have

$$\sum_{x \in V(G-X)} (\phi(d_G(x)) - \phi(d_{G'}(x))) \leq (2m' - m)/10.$$

Therefore $\sigma(G) - \sigma(G') \leq (7n'_3 + 8n'_2 + 2m' - m)/10$. ∎

For a graph Q, denote by $\mathcal{B}(Q)$ the family of all maximum induced bipartite subgraphs of Q. For an induced subgraph H of G, and for $X \subseteq V(G)$, we shall denote by $H + X$ (respectively, $H - X$) the subgraph of G induced by $V(H) \cup X$ (respectively, $V(H) - X$). We denote by $G + uv$ the graph obtained from G by adding the edge uv.

Observation 2.2 *Each $H \in \mathcal{B}(F_1)$ equals $F_1 - v$ for some vertex v. For any two non-adjacent 2 vertices x, y, $F_2 - \{x, y\} \in \mathcal{B}(F_2)$. For any two 2-vertices x, y of distance 3, $F_3 - \{x, y\} \in \mathcal{B}(F_3)$. For $i = 4, 5$, there is an $H \in \mathcal{B}(F_i)$ such that H contains no 2-vertices.* ∎

Lemma 2.3 *The graph G is 2-connected.*

Proof. Otherwise G has a cut-edge uv. Let G_1, G_2 be two components of $G - uv$. Let $H_i \in \mathcal{B}(G_i)$ for $i = 1, 2$. Then $H_1 \cup H_2 \in \mathcal{B}(G)$. Hence $\alpha_2(G) \geq \alpha_2(G_1) + \alpha_2(G_2) \geq \sigma(G_1) + \sigma(G_2) > \sigma(G)$. ∎

Lemma 2.4 *No 3-vertex is adjacent to two 2-vertices.*

Proof. Assume a 3-vertex x has two 2-vertices a, b as its neighbours. Let a', b', x' be the other neighbour of a, b, x, respectively. Let $G' = G - \{x, a, b\}$.

Then $\sigma(G) - \sigma(G') \leq (7+16-3)/10 = 2$, and equality holds only if none of a', b', x' is a 2-vertex in G, which implies that the heart of G' is itself. For any $H \in \mathcal{B}(G')$, $H + \{a, b\}$ is a bipartite subgraph of G. So $\alpha_2(G) \geq \alpha_2(G') + 2 \geq \sigma(G)$. If equality holds, then $\alpha_2(G') = \sigma(G')$ and hence $G' = F_i$ for some $i \in \{1, 2, 3, 4\}$ (observe that G' has at least three 2-vertices). If $G' = F_i$ for

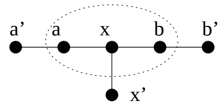

Figure 3: A 3-vertex adjacent to two 2-vertices.

some $i \in \{2,3,4\}$, then by Observation 2.2, there is an $H \in \mathcal{B}(G')$ such that H contains at most one of the vertices a', b', x'. Then $H + \{a, x, b\} \in \mathcal{B}(G)$, implying that $\alpha_2(G) > \sigma(G)$. If $G' = F_1$ then x' is adjacent to at least one of a', b', for otherwise $G = F_3$. Assume $x'b'$ is an edge, then $H = G' - \{a'\} \in \mathcal{B}(G')$, and $H + \{a, x, b\} \in \mathcal{B}(G)$, implying that $\alpha_2(G) > \sigma(G)$. ∎

Lemma 2.5 *No 4-cycle contains a 2-vertex.*

Proof. Assume (a, b, c, d) is a 4-cycle and a is a 2-vertex. By Lemma 2.4, c is a 3-vertex. Let $G' = G - \{a, b, c, d\}$. Then $\sigma(G) - \sigma(G') \leq (14+16-2)/10 < 3$ (as G is 2-connected, at least two of b, c, d are 3-vertices). If one of b, d is a 2-vertex, say b is a 2-vertex, then for $H \in \mathcal{B}(G')$, $H + \{a, b, c\}$ is a bipartite subgraph of G. Hence $\alpha_2(G) > \sigma(G)$. Assume b, d are 3-vertices, and b', c', d' are the other neighbour of b, c, d, respectively. Suppose $H \in \mathcal{B}(G')$. If one of $b', c', d' \notin V(H)$, say $b' \notin V(H)$, then $H + \{a, b, c\}$ is a bipartite subgraph of G. Assume $b', c', d' \in V(H)$. If b', d' are in the same partite set of H, then $H + \{d, a, b\}$ is a bipartite subgraph of G. Otherwise without loss of generality, we may assume c' and d' are in different partite sets. Then $H + \{a, c, d\}$ is a bipartite subgraph of G. In any case $\alpha_2(G) \geq \alpha_2(G') + 3$, hence $\alpha_2(G) > \sigma(G)$. ∎

Lemma 2.6 *No two 2-vertices of G are adjacent.*

Proof. Assume to the contrary that x, y are two adjacent 2-vertices of G. Let a, b be the other neighbour of x, y, respectively. Let $G' = (G - \{x, y\}) + ab$. Then $\sigma(G) - \sigma(G') = 16/10 < 2$ and for any $H \in \mathcal{B}(G')$, $H + \{x, y\}$ is a bipartite subgraph of G. If G' is not cubic, then by induction hypothesis, $\alpha_2(G') \geq \sigma(G')$. If G' is cubic, then by a result in [2], G' has bipartite density at least 4/5, implying that $\alpha_2(G') \geq \sigma(G')$. Hence $\alpha_2(G) \geq \alpha(G') + 2 > \sigma(G)$. ∎

In Lemmas 2.7, 2.8, 2.9, we assume x is a 2-vertex and u, v are the neighbours of x. By Lemma 2.6, both u and v are 3-vertices. Assume $N(u) = \{x, a, b\}$. By Lemma 2.5, $N(u) \cap N(v) = \{x\}$. Assume $N(v) = \{x, c, d\}$.

Lemma 2.7 *Vertices a, v have a common neighbour and vertices b, v have a common neighbour.*

Proof. Assume b, v have no common neighbour. Let a', a'' be the other two neighbours of a (by Lemma 2.4, a is a 3-vertex). Let $G' = (G - \{a, u, x\}) + bv$.

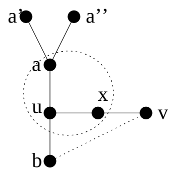

Figure 4: Vertices x, u, v and some of their neighbours

Then $\sigma(G) - \sigma(G') \leq (14 + 8 - 2)/10 = 2$ and equality holds only if none of a', a'' is a 2-vertex in G, which implies that the heart of G' is itself. For $H \in \mathcal{B}(G')$, $H + \{u, x\}$ is a bipartite subgraph of G. So $\alpha_2(G) \geq \alpha_2(G') + 2 \geq \sigma(G') + 2 \geq \sigma(G)$. If equality holds, then $\alpha_2(G') = \sigma(G')$. Since G' is not a counterexample to Theorem 1.2, and G' has at least two non-adjacent 2-vertices, namely a', a'', at least two adjacent 3-vertices, namely b, v, we conclude that $G' = F_i$ for some $i = 2, 3, 4, 5$. By Observation 2.2, if $i \neq 3$ or a', a'' have no common neighbour in G', then there is an $H \in \mathcal{B}(G')$ such that $a', a'' \notin V(H)$. Then $H + \{a, u, x\} \in \mathcal{B}(G)$, implying that $\alpha_2(G) > \sigma(G)$. In case $G' = F_3$ and a', a'' have a common neighbour in G', it is easy to find an $H \in \mathcal{B}(G')$ in which a', a'' are in the same partite set and b is in the other partite set. Again, $H + \{a, u, x\} \in \mathcal{B}(G)$, implying that $\alpha_2(G) > \sigma(G)$. ∎

Assume c is a common neighbour of a, v.

Lemma 2.8 *Vertices b and c are not adjacent.*

Proof. Assume bc is an edge. If none of ad, bd is an edge, then let $G' = (G - \{a, c, u, x, v, b\})$.

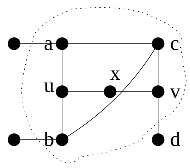

Figure 5: v is a 3-vertex and none of ad, bd is an edge.

Then $\sigma(G) - \sigma(G') \leq (35 + 8 - 3)/10 = 4$ and equality holds only if none of a', b', d is a 2-vertex in G (note that by Lemma 2.4, a, b are 3-vertices). For $H \in \mathcal{B}(G')$, $H + \{a, c, u, x\}$ is a bipartite subgraph of G. Hence $\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + 4 \geq \sigma(G)$. If equality holds, then $G' = F_i$ for some $1 \leq i \leq 4$ (as G' has at least three 2-vertices d and a', b', which are the other neighbours of a, b, respectively). By Observation 2.2, there is an $H \in \mathcal{B}(G')$ that does not contain b'. So $H + \{a, c, u, x, b\}$ is a bipartite subgraph, and hence $\alpha_2(G) > \sigma(G)$.

If ad is an edge, then let $G' = (G - \{a, c, u, x, v, b, d\})$.

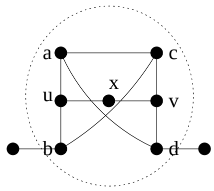

Figure 6: ad is an edge.

Then $\sigma(G) - \sigma(G') \leq (42 + 8 - 2)/10 = 24/5$. For $H \in \mathcal{B}(G')$, $H + \{a, c, u, x, b\}$ is a bipartite subgraph of G. Hence $\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + 4 > \sigma(G)$. ∎

Since bc is not an edge, and b, v have a common neighbour, we conclude that ac, bd are edges of G, and ad, bc are not edges of G. The graph induced by $\{a, b, u, x, v, c, d\}$ is depicted in Figure 7 (i).

Let a', b', c', d' be the other neighbour of a, b, c, d, respectively.

Lemma 2.9 *Vertices a', b' are distinct and adjacent, and vertices c', d' are distinct and adjacent.*

Proof. If $a' = b'$ or $a' \neq b'$ and $a'b'$ is not an edge, then let $G' = (G - \{u, v, x, b, c, d\}) + ab'$. Then $\sigma(G) - \sigma(G') \leq (35 + 8 - 3)/10 = 4$ and equality holds only if none of c', d' is a 2-vertex. For

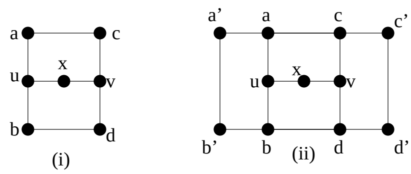

Figure 7: The graphs induced by $\{a, b, u, x, v, c, d\}$ and by $\{a, b, u, x, v, c, d, a', b', c', d'\}$.

$H \in \mathcal{B}(G')$, $H+\{u, x, v, b\}$ is a bipartite subgraph of G. So $\alpha_2(G) \geq \alpha_2(G')+4 \geq \sigma(G')+4 \geq \sigma(G)$. If equality holds, then $\alpha_2(G') = \sigma(G')$ and $G' \in \{F_i : i = 2, 3, 4\}$. By Observation 2.2, there is an $H \in \mathcal{B}(G')$ such that $a, c' \notin V(H)$. Then $H + \{c, v, d, x, u\}$ is a bipartite subgraph of G, showing that $\alpha_2(G) > \sigma(G)$. ∎

Let $X = \{a', b', c', d', a, b, c, d, u, x, v\}$. The subgraph of G induced by X is depicted in Figure 7 (ii). Observe that this is a subgraph of F_4 and F_5.

Lemma 2.10 *There is no* 2*-vertex.*

Proof. Assume x is a 2-vertex of G. Then G contains the graph depicted in Figure 7 (ii) as a subgraph. If $a'd'$ is an edge, then since $G \neq F_4$ and $\alpha_2(G) \leq \sigma(G)$, $G - X$ is nonempty. Let $G' = G-X$. Then $\sigma(G)-\sigma(G') \leq (70+8-2)/10 = 38/5$. For $H \in \mathcal{B}(G')$, $H+\{a', a, c, v, x, b, d, d'\}$ is a bipartite subgraph of G. Hence $\alpha_2(G) > \sigma(G)$.

If a', c' have no common neighbour, then let $G' = (G - \{a, c, u, x, v, b, d\}) + a'c'$. Then $\sigma(G) - \sigma(G') \leq (42 + 8 - 2)/10 = 24/5$. For $H \in \mathcal{B}(G')$, $H + \{a, c, x, v, b\}$ is a bipartite subgraph of G. Hence $\alpha_2(G) > \sigma(G)$.

Thus we assume that a', c' have a common neighbour s. Similarly, b', d' have a common neighbour t. If s, t have a common neighbour w, then since $G \neq F_5$, we conclude that G has a cut edge ww' But then $\alpha_2(G - ww') = \alpha_2(G) \geq \sigma(G - ww') > \sigma(G)$. So s, t have no common neighbour. If G has no other vertex, then $\alpha_2(G) = 10 > \sigma(G) = 46/5$. Otherwise, let $G' = G - (X \cup \{s, t\})$. Then $\sigma(G)-\sigma(G') \leq (84+8-2)/10 = 9$. For $H \in \mathcal{B}(G')$, $H+\{s, c', c, a, v, x, d, b, b'\}$ is a bipartite subgraph of G. So $\alpha_2(G) \geq \alpha_2(G') + 9 \geq \sigma(G') + 9 \geq \sigma(G)$. If equality holds, then $\alpha_2(G') = \sigma(G')$ and hence $G' \in \{F_i : i = 1, 2, 3, 4, 5\}$. By Observation 2.2, there is an $H \in \mathcal{B}(G')$ such that $s' \notin V(H)$, where s' is the other neighbour of s in G. Then $H + \{s, c', c, a, v, x, d, b, b', t\}$ is a bipartite subgraph of G, showing that $\alpha_2(G) > \sigma(G)$. ∎

By Lemma 2.10, G is a cubic graph. It is proved in [2] that if G is cubic and triangle-free, then $b(G) \geq 4/5$ and equality holds if and only if $G = F_6, F_7$. Therefore G has an induced bipartite graph with at least $\sigma(G) = 7|V(G)|/10$ vertices, and equality holds if and only if $G = F_6, F_7$. This completes the proof of Theorem 1.2.

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