Chromatic Ramsey Numbers

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Abstract

Suppose G is a graph. The chromatic Ramsey number $r_c(G)$ of G is the least integer m such that there exists a graph F of chromatic number m for which the following is true: For any 2-colouring of the edges of F there is a monochromatic subgraph isomorphic to G. Let $M_n = \min\{r_c(G) : \chi(G) = n\}$. It was conjectured by S. A. Burr, P. Erdös and L. Lovász that $M_n = (n-1)^2+1$. This conjecture has been confirmed previously for $n \leq 4$. In this paper, we shall prove that the conjecture is true for $n = 5$. We shall also improve the upper bounds for M_6 and M_7.

1 Introduction

Suppose F, G, H are finite graphs. We use $F \to (G, H)$ to mean the following statement:

For every colouring of the edges of F with red and blue, either the red subgraph of F contains a copy of G or the blue subgraph contains a copy of H.

A natural question is to characterize those F for which $F \to (G, H)$ for given graphs G and H. This question is in general extremely difficult. Simpler problems are to describe properties of such graphs. One such problem, discussed by S. A. Burr, P. Erdös and L. Lovász [1] is to find the smallest possible chromatic number of such a graph.

Let $\mathcal{G}, \mathcal{H}$ be sets of graphs. We write $F \to (\mathcal{G}, \mathcal{H})$ if for every colouring of the edges of F with red and blue, there is either a red subgraph of F

isomorphic to a member of $\mathcal{G}$, or there is a blue subgraph isomorphic to a member of $\mathcal{H}$. We define the chromatic Ramsey number $r_c(\mathcal{G},\mathcal{H})$ of $\mathcal{G},\mathcal{H}$ to be the least integer m such that there exists a graph F with $\chi(F)=m$ and $F\to(\mathcal{G},\mathcal{H})$. The Ramsey number $r(\mathcal{G},\mathcal{H})$ is the least integer n such that $K_n\to(\mathcal{G},\mathcal{H})$. We shall write $r_c(\mathcal{G})$ for $r_c(\mathcal{G},\mathcal{G})$ and $r(\mathcal{G})$ for $r(\mathcal{G},\mathcal{G})$. In case $\mathcal{G}=\{G\}$ or $\mathcal{H}=\{H\}$, we write G or H as an argument. Usually, $r(K_n,K_m)$ is written as $r(n,m)$, $r(K_n,K_n)$ is written as $r(n)$.

Let $\mathcal{K}_n$ be the class of all n-chromatic graphs. We shall be interested in the number

$$M_n=\min\{r_c(G):G\in\mathcal{K}_n\}.$$

This number was first studied by S. A. Burr, P. Erdös and L. Lovász [1]. They proposed the following conjecture:

Conjecture 1 *For any integer n,*

$$M_n=(n-1)^2+1.$$

It was proved in [1] that Conjecture 1 is true for $n\le 4$. We shall prove in this paper that the conjecture is true for $n=5$. We shall give better upper bounds for M_6 and M_7.

2 $M_5=17$

Suppose G,H are graphs. We call H a homomorphic image of G if there is a homomorphism (an edge preserving vertex mapping) from G to H. For any graph G, we denote by $\hom(G)$ the set of all homomorphic images of G. If $\mathcal{G}$ is a class of graphs, then $\hom(\mathcal{G})=\cup_{G\in\mathcal{G}}\hom(G)$. We shall need the following result proved in [1]:

Lemma 1 *Suppose $\mathcal{G},\mathcal{H}$ are classes of finite graphs. Then*

$$r_c(\mathcal{G},\mathcal{H})=r(\hom(\mathcal{G}),\hom(\mathcal{H})).$$

It is easy to see that $\hom(\mathcal{K}_n)=\mathcal{K}_n$. Therefore for any integers n,s we have $r_c(\mathcal{K}_n,\mathcal{K}_s)=r(\mathcal{K}_n,\mathcal{K}_s)$. It is easy to see ([1]) that $r(\mathcal{K}_n,\mathcal{K}_s)=(n-1)(s-1)+1$. Therefore for any graph G of chromatic number n, $r_c(G)\ge r_c(\mathcal{K}_n)=(n-1)^2+1$. Hence

$$M_n=\min\{r_c(G):G\in\mathcal{K}_n\}\ge(n-1)^2+1.$$

Conjecture 1 above asserts that the equality holds for all n. As observed in [1], this conjecture is weaker than the following conjecture of Hedetniemi [6].

For graphs G, H, the categorical product $G \times H$ of G and H has vertex set $V(G) \times V(H)$ and (a, b) is adjacent to (c, b) if and only if a is adjacent to c and b is adjacent to d. Hedetniemi's made the following conjecture about thirty years ago:

Conjecture 2 *For any integer n, if G, H are graphs of chromatic number at least n, then their categorical product $G \times H$ also has chromatic number at least n.*

To see that Conjecture 1 follows from Conjecture 2, we assume that Conjecture 2 is true for integer n. It is easy to see that for each 2-colouring c of the edges of $K_{(n-1)^2+1}$, there is a monochromatic subgraph of chromatic number at least n. Let G_c be a monochromatic subgraph (with respect to a 2-edge colouring c of $K_{(n-1)^2+1}$) of K_n of chromatic number n. Let $c_1, c_2, \cdots, c_m$ be all the 2 edge colourings of $K_{(n-1)^2+1}$. Let $G = G_{c_1} \times G_{c_2} \times \cdots \times G_{c_m}$. Then the "projection" mappings are homomorphisms of G to the factors G_{c_i}. Therefore $\hom(G) \supset \{G_{c_i} : i = 1, 2, \cdots, m\}$. Hence $r_c(G) = r(\hom(G)) \leq r(\{G_{c_i} : i = 1, 2, \cdots, m\}) \leq (n-1)^2 + 1$. On the other hand, it follows from Hedetniemi's conjecture that the product G has chromatic number n. Therefore $M_n \leq (n-1)^2 + 1$, and hence $M_n = (n-1)^2 + 1$.

The above argument shows that if Conjecture 1 is true for an integer n, then Conjecture 2 is also true for n. Burr, Erdös and Lovász [1] confirmed Conjecture 1 for $n = 4$. Later, El-Zahar and Sauer [3] proved, with a genius argument, that Conjecture 2 is also true for $n = 4$. Conjecture 2 has attracted considerable attention [2, 3, 5, 8, 9, 12, 13]. However both Conjectures 1 and 2 remained open in general for $n \geq 5$. We shall prove that Conjecture 1 is true for $n = 5$.

Theorem 1 *There is a graph G of chromatic number 5 such that $r_c(G) = 17$. Therefore $M_5 = 17$.*

In order to construct such a graph G, we first prove two lemmas:

Lemma 2 *Suppose c is a colouring of the edges of K_{17} with two colours* blue *and* red. *If there is a monochromatic copy of K_4, then there is a connected monochromatic subgraph A of K_{17} of chromatic number 5 such that A contains a K_4.*

Proof. Suppose c is a colouring of the edges of K_{17} with two colours blue and red and that there is monochromatic copy of K_4. Let R, B be the red subgraph and blue subgraph of K_{17} respectively. Without loss of generality, suppose that there is a red copy of K_4. If R is 4-colourable, then given a 4-colouring of R, one of the colour class contains five vertices. These five vertices induces a blue K_5, and we may take A to be this copy of K_5. Thus we assume that R has chromatic number at least 5. For the same reason, we may also assume that the blue subgraph B has chromatic number at least 5.

Let X be a maximal connected subgraph of R of chromatic number at least 5. If X contains a copy of K_4 then we take $A = X$ and we are done. Assume now that X contains no K_4. Since R contains a copy of K_4, we conclude that $V(X) \neq V(K_{17})$. By the maximality of X, we know that all the edges between $V(X)$ and $V(K_{17}) - V(X)$ are coloured blue. Therefore the blue graph B is connected. If B contains a copy of K_4, then since $\chi(B) \geq 5$ we may take $A = B$ and we are done. Thus we may assume that B contains no K_4.

If $|V(X)| \geq 9$, then since $r(K_4, K_3) = 9$ and since X contains no K_4, we conclude that the blue subgraph on X, i.e., the subgraph of B induced by X, contains a copy of K_3. Then the union of this copy of K_3 and a vertex in $V(K_{17}) - V(X)$ induces a blue copy of K_4, contrary to our assumption. Therefore we must have $|V(X)| \leq 8$.

Since X has chromatic number 5 and contains no K_4, we conclude that X is not complete, and hence there are two vertices $a, b \in V(X)$ such that the edge ab is coloured blue. Since $|V(K_{17}) - V(X)| \geq 9$, it follows that either there are two vertices $c, d \in V(K_{17}) - V(X)$ such that the edge cd is coloured blue, or we have red copy of K_9. In the former case $\{a, b, c, d\}$ induces a blue copy of K_4, contrary to our assumption, and in the latter case, we may take A to be the red K_9. This completes the proof of Lemma 2. ∎

Since $r(K_4, K_4) = 18$, there is a colouring of the edges of K_{17} such that there is no monochromatic K_4. Actually, there is a unique such edge colouring [4]. Our next lemma explores the properties of the monochromatic subgraph of such an edge colouring of K_{17}.

For a graph G, the n-colouring graph K_n^G has vertices all the mappings $f : V(G) \to \{1, 2, \cdots, n\}$, and two such mappings f, g are adjacent in K_n^G if and only if for every edge xy of G, we have $f(x) \neq g(y)$.

Lemma 3 *Suppose the edges of K_{17} are coloured by red and blue, and that there is no monochromatic copy of K_4. Let R be the red subgraph of K_{17}. Then the 4-colouring graph K_4^R of R is 4-colourable.*

Proof. It is well-known that there is a unique edge colouring of K_{17} with two colours such that the resulting coloured graph contains no monochromatic K_4. In such an edge colouring, the red graph R, as well as the blue graph B, is isomorphic to the graph on Z_{17} in which ij is an edge if and only if $|i - j|$ is one of the numbers $1, 2, 4, 8, 9, 13, 15, 16$, [4]. For each vertex f of the 4-colouring graph K_4^R, i.e., f is a mapping of $V(R)$ to $\{1,2,3,4\}$, let i be any number in $\{1,2,3,4\}$ such that $|f^{-1}(i)| \geq 5$ (because $|V(R)| = 17$, such an integer exists). We set $\Delta(f) = i$, and we shall show that Δ is a proper colouring of K_4^R.

Assume to the contrary that Δ is not a proper colouring of K_4^R. Then there are two adjacent vertices f, g of K_4^R such that $\Delta(f) = \Delta(g) = i$ for some $i \in \{1,2,3,4\}$. By the definition of Δ, there exist vertices x_1, x_2, x_3, x_4, x_5 such that $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5) = i$, and there exist vertices y_1, y_2, y_3, y_4, y_5 such that $g(y_1) = g(y_2) = g(y_3) = g(y_4) = g(y_5) = i$.

Since f, g are adjacent in K_4^R, we conclude that $x_i y_j \notin E(R)$ for all $i, j \in \{1, 2, \cdots, 5\}$. In other words, if $x_i \neq y_j$ then the edge $x_i y_j$ is coloured blue.

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and let $Y = \{y_1, y_2, y_3, y_4, y_5\}$. If $|X \cap Y| \leq 1$, then we may assume that $x_i \neq y_j$ for $i, j \in \{1,2,3,4\}$. Since there is no monochromatic K_4, we conclude that there are indices $i, j \in \{1,2,3,4\}$ such that the edge $x_i x_j$ is coloured blue. Similarly there are $u, v \in \{1,2,3,4\}$ such that the edge $y_u y_v$ is coloured blue. Then $\{x_i, x_j, y_u, y_v\}$ induces a blue K_4, contrary to our assumption. If $2 \leq |X \cap Y| \leq 4$, then we may assume that $x_1 = y_1, x_2 = y_2$ and $x_5 \neq y_5$. Then $\{x_1, x_2, x_5, y_5\}$ induces a blue K_4, contrary to our assumption. If $X = Y$ then $\{x_1, x_2, x_3, x_4\}$ induces a blue K_4, contrary to our assumption. Therefore Δ is indeed a proper 4-colouring of K_4^R. This completes the proof of Lemma 3. ∎

With these two lemmas, we can construct the graph G of Theorem 1 as follows:

Let α_0 be the unique 2-colouring of the edges of K_{17} such that the resulting coloured graph contains no monochromatic K_4. Let G_0 be the red subgraph.

Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be all the 2-colourings of the edges of K_{17} such that the resulting coloured graph contains a monochromatic K_4. For each $1 \leq i \leq m$, let G_i be a connected monochromatic subgraph (with respect to the colouring α_i) such that $\chi(G_i) \geq 5$ and G_i contains a copy of K_4. (By Lemma 2, such graphs G_i exist.) Finally let $G = G_0 \times G_1 \times G_2 \times \cdots \times G_m$.

We shall show that $r_c(G) \leq 17$. It suffices to show that $r(\hom(G)) \leq 17$. Since it is obvious that $G_i \in \hom(G)$ for $i = 0, 1, \cdots, m$, it suffices to show that $r(\{G_0, G_1, \cdots, G_m\}) \leq 17$. However this follows trivially from the definition of G_i (i.e., for each 2-colouring of the edges of K_{17} there is a monochromatic subgraph isomorphic to one of the graphs G_i).

To complete the proof of Theorem 1, it suffices to show that G has chromatic number at least 5. This follows easily from the following two lemmas, which are proved in [2] and [3], respectively:

Lemma 4 *Suppose A, B are connected graphs of chromatic number at least n and each of A, B contains a K_{n-1}. Then $A \times B$ has chromatic number at least n.*

Lemma 5 *Suppose A, B are graphs of chromatic number at least n. If the $(n-1)$-colouring graph K_{n-1}^A of A is $(n-1)$-colourable, then the product $A \times B$ has chromatic number at least n.*

Indeed, by inductively applying Lemma 4, we conclude that $G_1 \times \cdots \times G_m$ has chromatic number at least 5. Since G_0 has chromatic number 5 and $K_4^{G_0}$ is 4-colourable, by Lemma 5, $G_0 \times G_1 \times \cdots G_m$ has chromatic number 5. This completes the proof of Theorem 1.

3 Better upper bounds for M_6 and M_7

It follows from the definition that $M_n \le r(K_n, K_n)$. Therefore any upper bound for $r(K_n, K_n)$ is an upper bound for M_n. However this upper bound is much too big. In this section, we shall give better upper bounds for M_6 and M_7.

Theorem 2 *There is a graph G of chromatic number 6 such that $r_c(G) \le 41$. Therefore $26 \le M_6 \le 41$.*

We shall first prove the following lemma:

Lemma 6 *For any 2-colouring of the edges of K_{41}, either there is a monochromatic connected subgraph A which has chromatic number 6 and which contains a K_5, or there is a monochromatic subgraph A' which has chromatic number 6 and $K_5^{A'}$ is 5-colourable.*

Proof. Let c be a 2-colouring of the edges of K_{41} by red and blue, and let R, B be the red and blue subgraphs of K_{41} respectively. If R is 5-colourable, then for a 5-colouring of R, there is a colour class of order at least 6. Therefore B contains a K_6, and we are done. Thus we may assume that $\chi(R) \ge 6$. Similarly, we may assume that $\chi(B) \ge 6$. We now consider two cases.

Case 1. There is a monochromatic K_5.

Without loss of generality, we may assume that R contains a K_5. Let H be a maximal connected subgraph of R which has chromatic number at least 6. If H contains a K_5, then we may choose $A = H$ and we are done. If H does not contain a K_5, then $|V(H)| \le 36$ and all the edges between

$V(H)$ and $V(K_{41}) - V(H)$ are coloured blue. Therefore B is connected. If B contains a K_5, then we may choose $A = B$ and we are done. Thus we assume that B contains no K_5.

If $|V(H)| = 36$, we shall show that K_5^H is 5-colourable, and hence we may choose $A' = H$.

Let $f \in V(K_5^H)$ be a mapping of $V(H) \to \{1, 2, \cdots, 5\}$. Then there is an index i such that $|f^{-1}(i)| \geq 8$. We colour f with colour $\Delta(f) = i$. We now prove that Δ is a proper colouring of K_5^H.

Assume to the contrary that there are two adjacent vertices f, g of $V(K_5^H)$ such that $\Delta(f) = \Delta(g) = i$. By the definition of Δ, there exist vertices $x_1, x_2, \cdots, x_8$ such that $f(x_1) = f(x_2) = \cdots = f(x_8) = i$, and there exist vertices $y_1, y_2, \cdots, y_8$ such that $g(y_1) = g(y_2) = \cdots = g(y_8) = i$.

Let $X = \{x_1, x_2, \cdots, x_8\}$ and let $Y = \{y_1, y_2, \cdots, y_8\}$. Since f, g are adjacent, if follows that if $x_i \neq y_j$, the $x_i y_j$ is not an edge of H. Hence the edge $x_i y_j$ is coloured blue.

If $|X \cap Y| \leq 3$, then we may assume that $\{x_1, x_2, \cdots, x_5\} \cap \{y_1, y_2, \cdots, y_5\} = \emptyset$. Since H contains no K_5, there are indices $i, j \in \{1, 2, \cdots, 5\}$ such that $x_i x_j$ is coloured blue, and there are indices $u, v \in \{1, 2, \cdots, 5\}$ such that $y_u y_v$ is coloured blue. Let z be a vertex in $V(K_{41}) - V(H)$, the five vertices $\{z, x_i, x_j, y_u, y_v\}$ induces a blue K_5, contrary to our assumption.

If $|X \cap Y| \geq 4$, then we may assume that $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4$. Similarly, let z be a vertex in $V(K_{41}) - V(H)$, the five vertices $\{z, x_1, x_2, x_3, x_4\}$ induces a blue K_5, contrary to our assumption.

Thus we may assume that $|V(H)| \leq 35$. It follows that there are two vertices $x, y \in V(K_{41}) - V(H)$ such that xy is a blue edge. (For otherwise, we would have a red K_6, and we may take A to be the red K_6 and we are done.) If there is a blue K_3 induced by three vertices of H, then this K_3 and the blue edge in $V(K_{41}) - V(H)$ would form a blue K_5, contrary to our assumption. Therefore we may assume that the coloured graph restricted to $V(H)$ does not contain a blue K_3. Since $r(K_5, K_3) = 14$, and H contains no K_5, i.e., the coloured graph restricted to $V(H)$ contains no red K_5, and no blue K_3. Therefore $|V(H)| \leq 13$. It follows that $|V(K_{41}) - V(H)| \geq 28$.

If there is red K_6 contained in $V(K_{41}) - V(H)$, we can choose A to be this red K_6 and we are done. Thus we may assume that the coloured graph restricted to $V(K_{41}) - V(H)$ contains no red K_6. Since $r(K_6, K_3) = 18$, we conclude that there is a blue K_3 contained in $V(K_{41}) - V(H)$. Take a blue edge from $V(H)$ (which obviously exists), we obtain a blue K_5, contrary to our assumption. This completes the proof of Case 1.

Case 2. There is no monochromatic K_5.

Similarly to the argument in the previous case, we have $\chi(R) \geq 6$ and $\chi(B) \geq 6$. First we consider the 5-colouring graph of the red subgraph.

Let f be a vertex of K_5^R, i.e., f is a mapping of $V(R) \to \{1, 2, \cdots, 5\}$. Since $|V(R)| = 41$, there is an integer i such that $|f^{-1}(i)| geq 9$. Let $\Delta(f) = i$. If Δ is a proper 5-colouring of K_5^R, then we may take $A' = R$ and we are done. Thus we assume that Δ is not a proper colouring of K_5^R. Therefore there are two adjacent vertices f, g of K_5^R such that $\Delta(f) = \Delta(g)$.

Let $X = \{x_1, x_2, \cdots, x_9\}$ be the 9-set such that $f(x_j) = \Delta(f)$ and let $Y = \{y_1, y_2, \cdots, y_9\}$ be the 9-set such that $g(y_j) = \Delta(g)$. If $X \cap Y \neq \emptyset$, then we may assume that $x_1 = y_1$. There are two vertices $x_i, x_j \in X - \{x_1\}$ such that the edge $x_i x_j$ is coloured blue, for other wise we would have a red K_8, and we may take $A = K_8$ and we done. For the same reason, there are also vertices $y_u, y_v \in Y - \{y_1, x_i, x_j\}$ such that the edge $y_u y_v$ is coloured blue. Now $\{x_1, x_i, x_j, y_u, y_v\}$ is a blue K_5, contrary to our assumption.

Thus we may assume that $X \cap Y = \emptyset$. Consider the blue graph induced by X. If the blue graph has chromatic number 2, then there would be a red K_5, contrary to our assumption. Therefore there is an odd blue cycle, say X', contains in X. Similarly there is an odd blue cycle, say Y', contained in Y. Let H be the blue graph induced by $X' \cup Y'$. We shall show that K_5^H is 5-colourable.

Suppose f is a vertex of K_5^H, i.e., f is a mapping of $V(H)$ to $\{1, 2, 3, 4, 5\}$. If there are vertices $x \in X', y \in Y'$ such that $f(x) = f(y) = i$, then we let $\Delta(f) = i$. Otherwise $f(X') \cap f(Y') = \emptyset$. In case $|f(X')| \leq |f(Y')|$, then $|f(X')| \leq 2$. Since X' contains an odd cycle, there is an edge xx' of H such that $f(x) = f(x')$. In this case we let $\Delta(f) = f(x) = f(x')$. Otherwise $|f(X')| > |f(Y')|$, and $|f(Y')| \leq 2$. Similarly there is an edge yy' of H such that $f(y) = f(y')$. In this case we let $\Delta(f) = f(y) = f(y')$. In the following we shall prove that Δ is a proper colouring of K_5^H.

Assume to the contrary that there are two adjacent vertices f, g of K_5^H such that $\Delta(f) = \Delta(g) = i$. Since for all $x \in X'$ and $y \in Y'$ and f, g are adjacent in K_5^H, we conclude that for all $x \in X'$ and $y \in Y'$ we have $f(x) \neq g(y)$ and $g(x) \neq f(y)$. In other words, $f(X') \cap g(Y') = \emptyset$ and $f(Y') \cap g(X') = \emptyset$. If $f(X') \cap f(Y') \neq \emptyset$ then by the definition of Δ, we have $\Delta(f) \in f(X') \cap f(Y')$. Hence $\Delta(f) \notin g(X') \cup g(Y')$. However, it follows from the definition that $\Delta(g) \in g(X') \cup g(Y')$, contrary to the assumption that $\Delta(g) = \Delta(f)$. Thus we assume that $f(X') \cap f(Y') = \emptyset$. Similarly we may assume that $g(X') \cap g(Y') = \emptyset$. This implies that $(f(X') \cup g(X')) \cap (f(Y') \cup g(Y')) = \emptyset$.

First we consider the case that $|f(X') \cup g(X')| \leq |f(Y') \cup g(Y')|$. Then $|f(X') \cup g(X')| \leq 2$. Since f is adjacent to g in K_5^H, we know that $|f(X') \cup g(X')| \neq 1$. Therefore $|f(X') \cup g(X')| = 2$. Without loss of generality, we assume that $f(X') \cup g(X') = \{1, 2\}$. Suppose $X' = \{x_1, x_2, \cdots, x_{2k+1}\}$

and $x_i x_{i+1}$ is an edge of H for $i = 1, 2, \cdots, 2k+1$ (where $2k+2 = 1$). Without loss of generality, we assume that $f(x_1) = 1$. Then $g(x_2) = 2$, because f, g are adjacent vertices in K_5^H. Inductively we can show that $f(x_3) = 1, g(x_4) = 2, \cdots, f(x_{2k+1}) = 1, g(x_1) = 2, \cdots, g(x_{2k+1}) = 2$. In other words, we have $f(X') = \{1\}$ and $g(X') = \{2\}$. By the definition of Δ, we should have $\Delta(f) = 1$ and $\Delta(g) = 2$, contrary to the assumption that $\Delta(f) = \Delta(g)$. The case that $|f(X') \cup g(X')| > |f(Y') \cup g(Y')|$ can be treated similarly (although not exactly the same), and we omit the details. This completes the proof of Lemma 6 ∎

Theorem 2 follows easily from Lemma 6 and Lemmas 4 and 5. The argument is the same as that in the proof of Theorem 1, and we omit the details.

Theorem 3 *There is a graph G of chromatic number 7 such that $r_c(G) \leq 102$. Therefore $M_7 = 102$.*

Proof. It suffices to show that for any 2-colouring of the edges of K_{102}, there is a monochromatic subgraph H which is connected and which contains a copy of K_6.

Let c be a 2-colouring of the edges of K_{102} and let R, B be the red subgraph and the blue subgraph, respectively. Since $r(K_6, K_6) \leq 102$ ([4]), there is a monochromatic copy of K_6. Without loss of generality, we may assume that R contains a copy of K_6. Similarly to the arguments in the previous proofs, we can assume that $\chi(R) \geq 7$ and $\chi(B) \geq 7$. Let H be a maximal connected subgraph of R with $\chi(H) \geq 7$. If H contains a copy of K_6 then we are done. Thus we may assume that H contains no K_6. Since R contains a K_6, we conclude that $|V(H)| \leq 102 - 6 = 96$. It follows that B is connected. If B contains a copy of K_6 then we are done. Thus we may assume that B contains no K_6. In following we shall use V to denote the vertex set of H, and denote by $\overline{V}$ the set $V(K_{102}) - V$.

If $|V| \geq 94$ then V contains a blue K_5 (i.e., the restriction of B to V contains a copy of K_5), because $r(K_6, K_5) \leq 94$ and V contains no red K_6. Now take another vertex from $\overline{V}$, we get a blue K_6, contrary to our assumption. Thus $|V| < 94$, and $|\overline{V}| \geq 9$. If $\overline{V}$ contains no blue edge, then we are done, as it is a red K_9. Assume that $\overline{V}$ contains a blue edge. If V contains a blue K_4, then the union of this blue K_4 and the blue edge in $\overline{V}$ is a blue K_6, contrary to our assumption. Therefore V contains no blue K_4. This implies that $|V| \leq 43$, because $r(K_6, K_4) = 44$. Hence $|\overline{V}| \geq 59$.

We now consider the restriction of R to $\overline{V}$. We denote this graph by R'. If R' is 6-colourable, then one of the colour classes has cardinality at least 10, which implies that there is blue copy of K_{10}, contrary to our assumption. Thus we may assume that $\chi(R') \geq 7$. Let H' be a maximal connected subgraph of R' with $\chi(H') \geq 7$. If H' contains a copy of K_6 then we are done. If H' does not contain a copy K_6, then since R' contains a copy of K_6, we have $|V(H')| \leq |V(R')| - 6$. Let U be the vertex of H' and let $\overline{U} = \overline{V} - U$. Then the 102 vertices of K_{102} is partitioned into three sets $V, U, \overline{U}$. All the edges between the three parts are blue edges. Moreover the restriction of the red subgraph R to V, U are connected, contains no K_6 and has chromatic numbers at least 7. Hence each of the sets U, V contains a blue edge. If the set $\overline{U}$ also contain a blue edge, then the union of the vertices of the three blue edges induces a blue K_6, contrary to our assumption. Assume that the set $\overline{U}$ contains no blue edge. Then it is complete red graph. If $|\overline{U}| \geq 7$, then we are done. If $|\overline{U}| = 6$, then $|U| \geq 53$ and hence U contains a blue K_3, because $r(K_6, K_3) = 18$ and U contains no red K_6. Now the union of the vertices of this blue K_3, the vertices of the blue edge in V and any vertex in $\overline{U}$ induces a blue K_6, contrary to our assumption.

Thus we have proved that for any 2-colouring of the edges of K_{102}, there is a monochromatic subgraph H which is connected and which contains a copy of K_6. Now Theorem 3 follows easily from Lemma 5 (by using the corresponding arguments in the proof of Theorem 1). ∎

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